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Lesson 6: Curves in the Complex Plane

Student Outcomes

* Students convert between the real and complex forms of equations for ellipses.
* Students write equations of ellipses and represent them graphically.

Lesson Notes

Initially, the students will review how to represent numbers in the complex plane using the modulus and argument. They will review the characteristics of the graphs of the numbers , recognizing that they represent circles centered at the origin with the radius equal to the modulus . They will then explore sets of complex numbers written in the form , identifying the graphs as ellipses. Students will convert between the complex and real forms of equations for ellipses, including those whose center is not the origin. They will also be introduced to some of the components of ellipses, such as the vertices, foci, and axes. This will prepare them to explore ellipses more formally in Lesson 7, where they will derive the equation of an ellipse using its foci.

Classwork

Opening Exercise (5 minutes)

This exercise should be completed in pairs or small groups. After a few minutes, students should discuss their responses to Exercises 1–2 with another pair or group before completing Exercise 5. If the students are struggling with how to convert between the rectangular and polar form, the exercises could be completed as part of a teacher-led discussion. Early finishers could display their conjectures and plots for Problem 3, which could be used in a teacher-led discussion of the characteristics of the graph.

*Scaffolding:*

* For students below grade level, consider a concrete approach using , or provide a graphical representation of .
* Advanced students could explore the properties of the graph of and compare it to the graph of   
   to form conjectures about the properties of graphs represented by   
  .

Opening Exercise

* 1. Consider the complex number .
     1. Write in polar form. What do the variables represent?

*, where is the modulus of the complex number and is the argument.*

* + 1. If and , where would be plotted in the complex plane?

*The point is located above the origin on the imaginary axis.*

* + 1. Use the conditions in part (ii) to write in rectangular form. Explain how this representation corresponds to the location of that you found in part (ii).

*, where and*

*;*

*Then , which is located three units above the origin on the imaginary axis.*

* 1. Recall the set of points defined by for , where is measured in degrees.
     1. What does represent graphically? Why?

*It is the set of points that are from the origin in the complex plane. This is because the modulus is , which indicates that for any given value of , will be located a distance of from the origin.*

* + 1. What does represent geometrically?

*A circle with radius centered at the origin.*

* 1. Consider the set of points defined by .
     1. Plot for , , , , . Based on your plot, form a conjecture about the graph of the set of complex numbers.

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*For , .*

*For , .*

*For , .*

*For , .*

*This set of points seems to form an oval shape centered at the origin.*

* + 1. Compare this graph to the graph of . Form a conjecture about what accounts for the differences between the graphs.

*The coefficients of and are equal for   
, which results in a circle, which has a constant radius, while the coefficients are different for , which seems to stretch the circle.*

**Example 1 (5 minutes)**

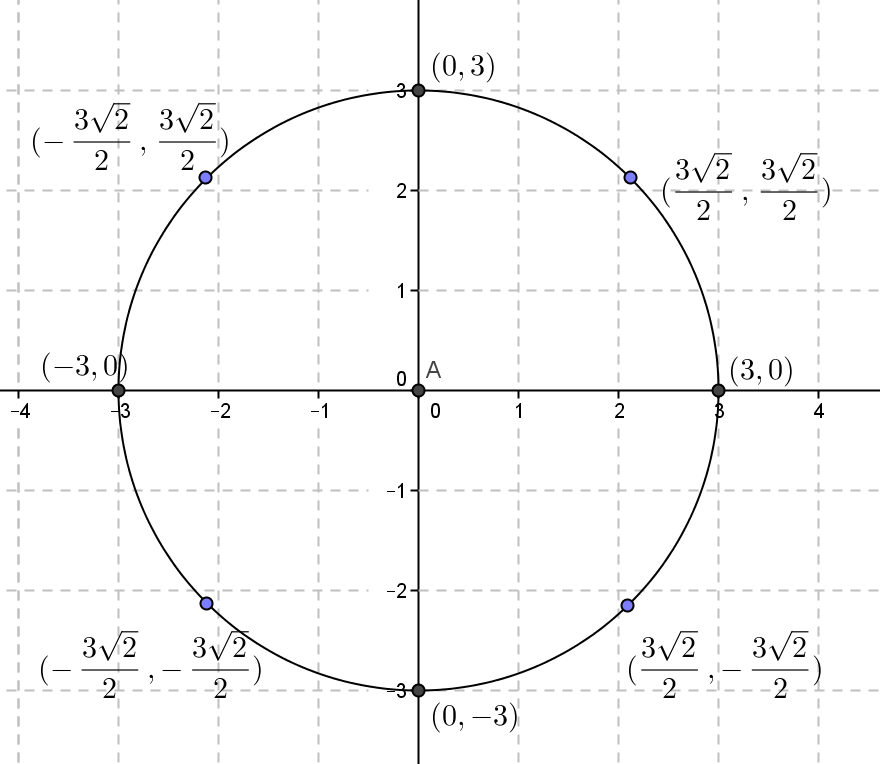
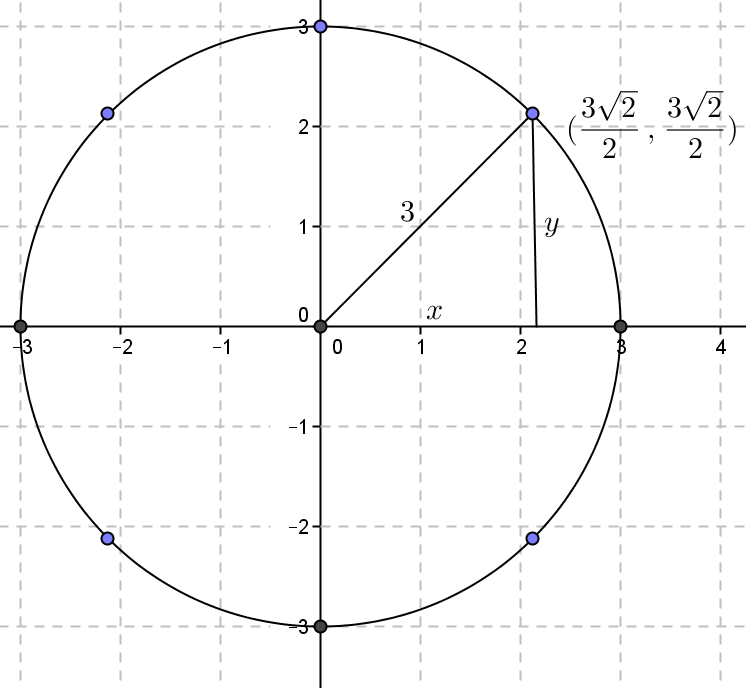
The students will be led through an example to demonstrate how to convert the equation of a circle from its complex form to real form, i.e., an equation in and . This will prepare them to convert the equations of ellipses from complex form to real form.

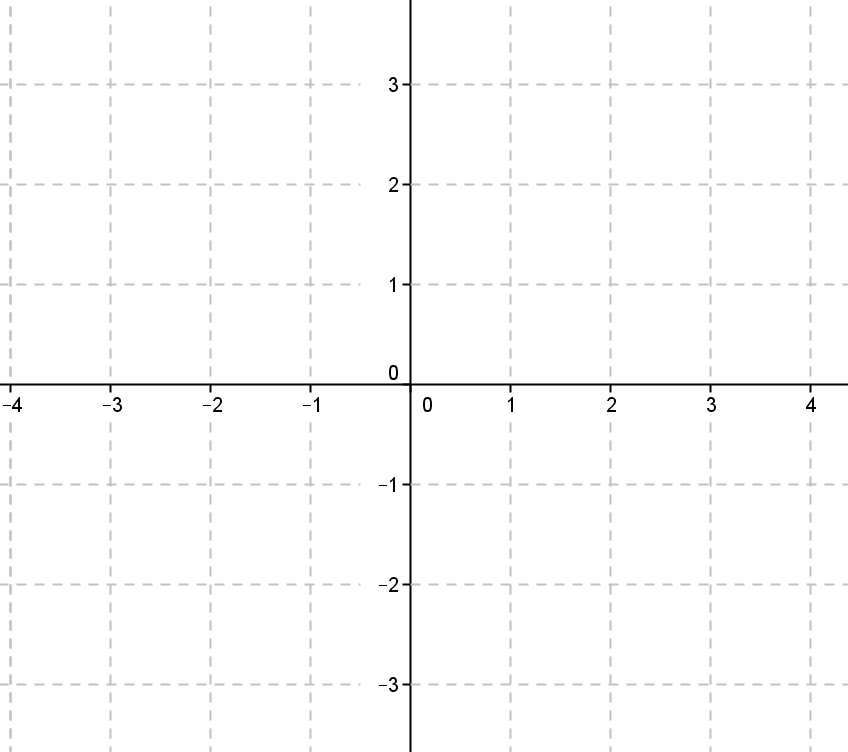
* We have seen that in the complex plane, the graph of the set of complex numbers defined by   
   for is a circle centered at the origin with radius . How could we represent each point on the circle using an ordered pair?
* Let’s say we wanted to represent in the real coordinate plane. We’d need to represent the points on the circle using an ordered pair . How can we write any complex number in terms of and ?
* So for , which expressions represent and ?
  + , *so* *and*
* What is the resulting ordered pair?
* Let’s graph some points and verify that this will give us a circle. Complete the table for the given values of .

**Example 1**

Consider again the set of complex numbers represented by for .

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* Plot the points in the table, and determine the type of curve created.
  + *The curve is a circle with center and radius .*
* If and , write an equation that relates and .
* Now, simplify the right side of that equation.
* Do you know a trigonometric identity that relates and ?
* Substitute into the previous equation.
* How does the graph of this equation compare with the graph of our equation in complex form?
  + Both graphs are circles centered at the origin with radius .

**

* 1. Use an ordered pair to write a representation for the points defined by as they would be represented in the coordinate plane.
  2. Write an equation that is true for all the points represented by the ordered pair you wrote in part (a).

*Since and :*

*We know that , so .*

* 1. What does the graph of this equation look like in the coordinate plane?

*The graph is a circle centered at the origin with radius .*

*Scaffolding:*

If students are struggling with converting the equations into real form, suggest that they square and and then isolate or in the equations. Alternatively, Exercise 1 could be completed as guided practice, and the students could then complete Exercise 2 independently.

Exercises 1–2 (6 minutes)

The students should complete the exercises independently. After a few minutes, they could verify their responses with a partner. The solutions should be reviewed in a whole-class setting after students have had a sufficient amount of time to complete the exercises.

Exercises 1–2

1. **Recall the set of points defined by .** 
   1. Use an ordered pair to write a representation for the points defined by as they would be represented in the coordinate plane.
   2. Write an equation in the coordinate plane that is true for all the points represented by the ordered pair you wrote in part (a).

*We have and , so and . We know .*

*Since , then . Since , then . By substitution, .*

1. Find an algebraic equation for all the points in the coordinate plane traced by the complex numbers   
   .

*All the complex numbers represented by can be written using the ordered pair in the coordinate plane.*

*We have and , so and . We know .*

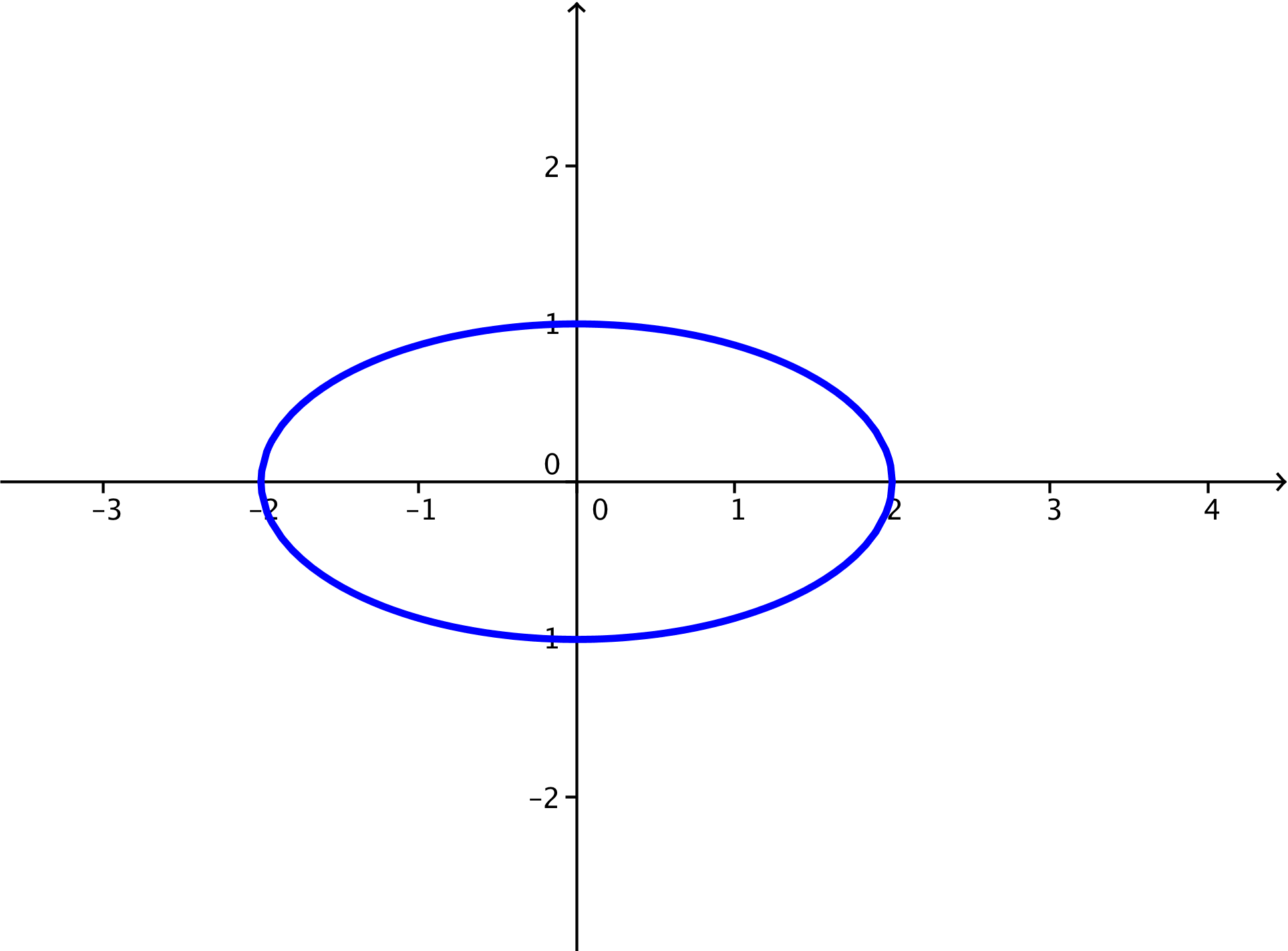
*Since , then . Then by substitution, .*

Discussion (5 minutes): Describing an Ellipse

* At the outset of the lesson, we determined that the graph of the complex numbers defined by  
   was an oval shape that was centered about the origin and intersected the axes at the points , , and .
* Make a conjecture about the graph of the complex numbers defined by . Sketch a rough graph of the points to test your conjecture. Share and discuss your conjecture with a neighbor. (Pull the class back together to debrief.)

**MP.3**

* + *Answers will vary, but students should recognize that the graph would be a closed curve centered at the origin that intersects the axes at the points , , and .*



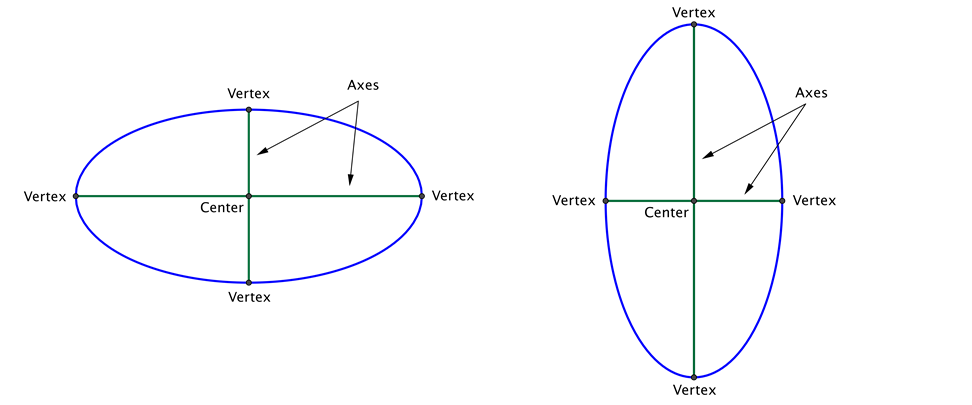
* What patterns do you notice between the graphs we have sketched and the structure of their equations in real form?
  + *Answers will vary but might include that the equations are written in the form of , where is the distance from the origin to the -intercepts and is the distance from the -intercepts*.
* And what patterns do you notice between the graphs we have sketched and the structure of their equations in complex form?

**MP.7**

* + *Answers will vary but might include that is the coefficient of and is the coefficient of .*
* Let’s formalize these observations. The shape that arises from the curve given by an equation in the form is called an *ellipse centered at the origin*. An ellipse can be stretched horizontally or vertically, as shown in the figure below.

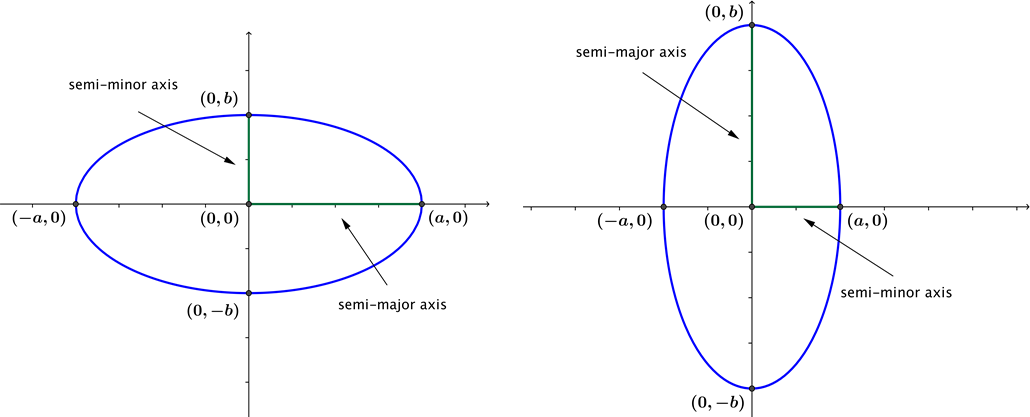
*Scaffolding:*

Have students complete a Frayer diagram for an ellipse. An example can be found in Module 1 Lesson 5.



The *vertices* of the ellipse are the four points on the ellipse that are the closest to and farthest from the center. The *axes* of the ellipse are the segments connecting opposite vertices. The *major axis* is the longer of the two axes, and the *minor axis* is the shorter of the two axes. In the ellipse shown to the left above, the major axis is horizontal; in the ellipse shown to the right, the major axis is vertical.

The *semi-major axis* is defined as a segment between the center of the ellipse and a vertex along the major axis, and the *semi-minor axis* is a segment between the center of the ellipse and a vertex along the minor axis.



* If the ellipse is centered at the origin, then the vertices of the ellipse are the intercepts , , and . In this case, what is the length of the semi-major axis?
  + *Either or , whichever is larger.*
* Good. For an ellipse centered at the origin, what is the length of the semi-minor axis?
* *Either or , whichever is smaller.*
* And what happens when ? Explain how you know.
* *We get a circle because the distances from the center to and are the same.*

**Example 2 (5 minutes)**

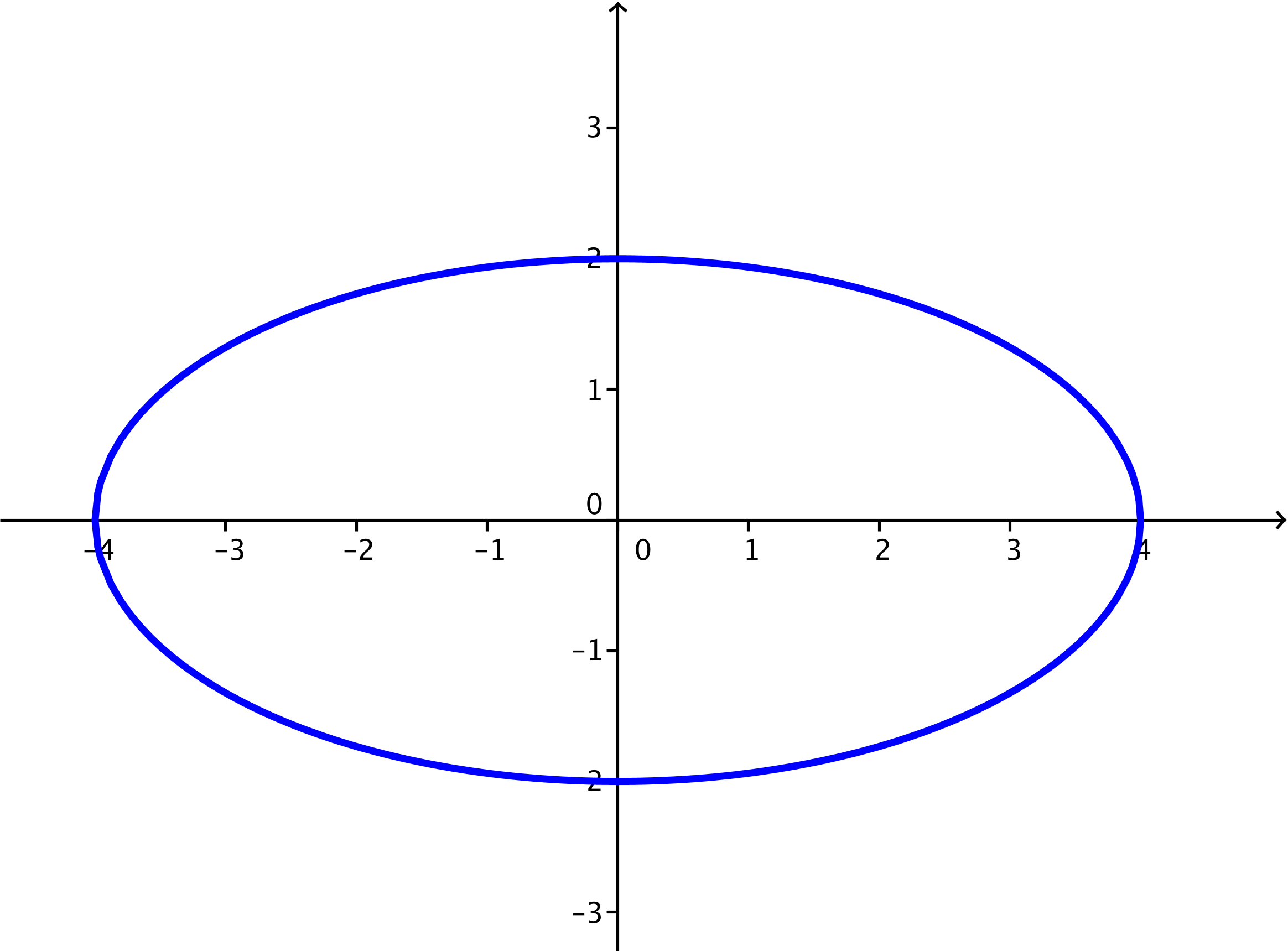
This example will prepare students to sketch the graphs of ellipses from their equations written in real form. It will also demonstrate how to convert equations of ellipses from real to complex form.

* How can we tell that an algebraic equation represents an ellipse without being told?
  + *It can be written in the form .*
* And how can we use what we know about and to plot the ellipse?
  + *Answers may vary but should address that represents the distance from the center of the ellipse to the points to the right and left of its center, so the graph of our equation will intersect the -axis at and . The number represents the distance from the center of the ellipse to the points above and below its center, so the graph will intersect the -axis at and .*
* Now, let’s write the equation in complex form. What is the structure of a general equation for an ellipse in complex form?
* What do we need to find, then, to write the equation in complex form?
  + Values of and .
* How could we find values of and ?
  + *Since , we have and , which means and .*
* What is the complex form of the equation of the ellipse given by ?

Example 2

The equation of an ellipse is given by .

* 1. Sketch the graph of the ellipse.



* 1. Rewrite the equation in complex form.

*Since , we have and .*

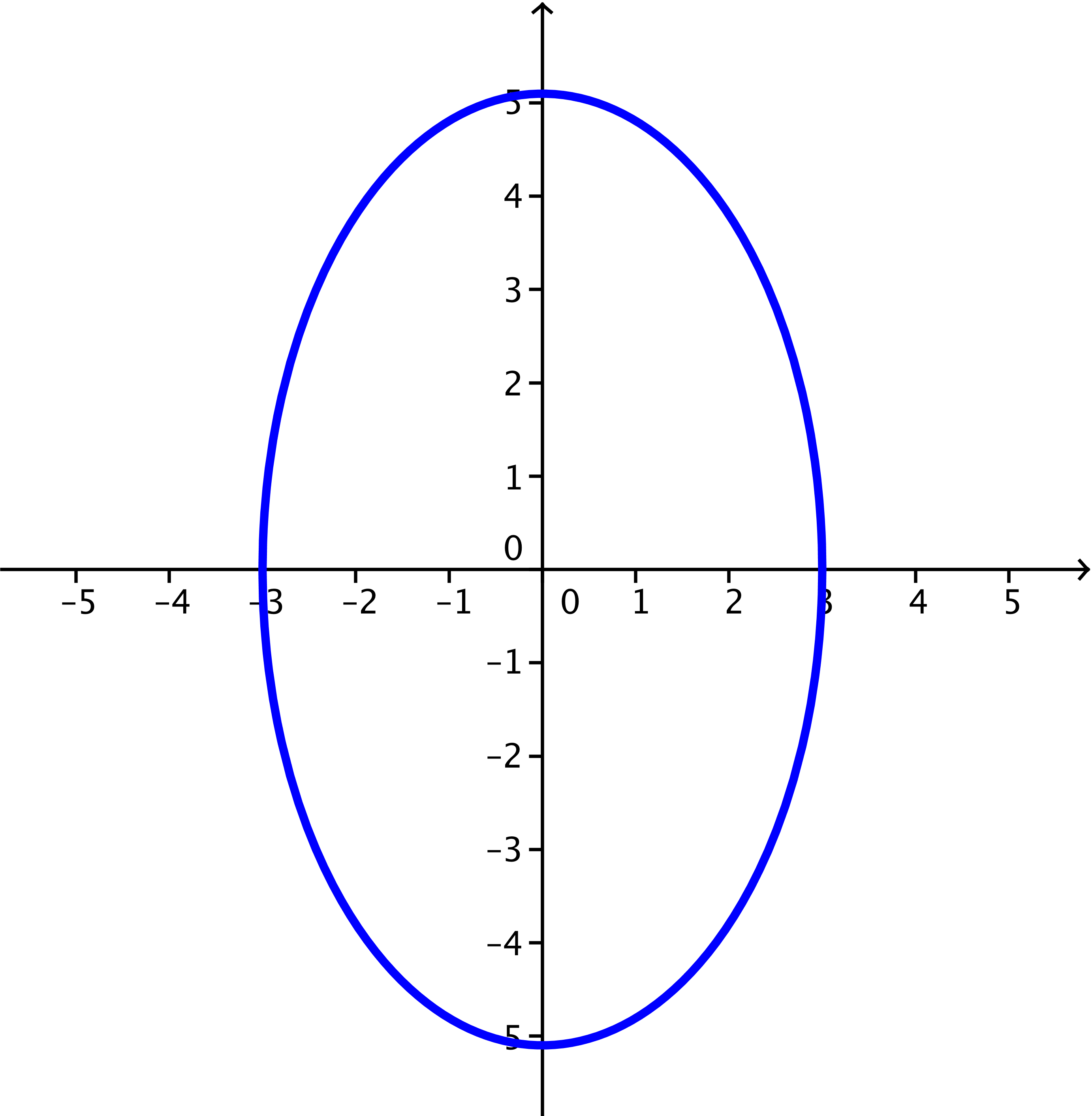
*The complex form of the ellipse is .*

Exercise 3 (4 minutes)

Students should complete the exercise in pairs. They should solve the problem independently and, after a few minutes, verify their solutions with a partner. At an appropriate time, pairs of selected students should share their sketches or the complex form for the equation. Make sure the students recognize the change in the orientation of the ellipse, i.e., that it is elongated vertically because .

Exercise 3

1. The equation of an ellipse is given by .
   1. Sketch the graph of the ellipse.



* 1. Rewrite the equation of the ellipse in complex form.

*The complex form of the ellipse is .*

**Example 3 (5 minutes)**

This example will introduce students to ellipses that are not centered at the origin and will prepare them to convert translated ellipses from complex to real form and to sketch their graphs.

* How does this equation look different from others we have seen in this lesson?
  + *There are constants included that were not in the other equations.*
* How can we represent the complex numbers in rectangular form?
* And what are the values of and for this equation?
  + *and*
* What has been our procedure for converting equations of ellipses from complex to real form?
  + *Isolate and* , *and then substitute the equivalent expressions into the equation .*
* What is the resulting equation?
* How do we know this equation represents an ellipse?
  + *It is written in the form .*
* What are the values of and ?
  + *and*
* How do the constants subtracted from and affect the graph of the ellipse?
  + *They represent a translation of the center from the origin to the right and up.*
* Describe the graph of the ellipse.
  + *Answers will vary but should address that the ellipse is centered at and is elongated horizontally, so the semi-major axis has length , and the semi-minor axis has length .*

Example 3

A set of points in the complex plane can be represented in the complex plane as as varies.

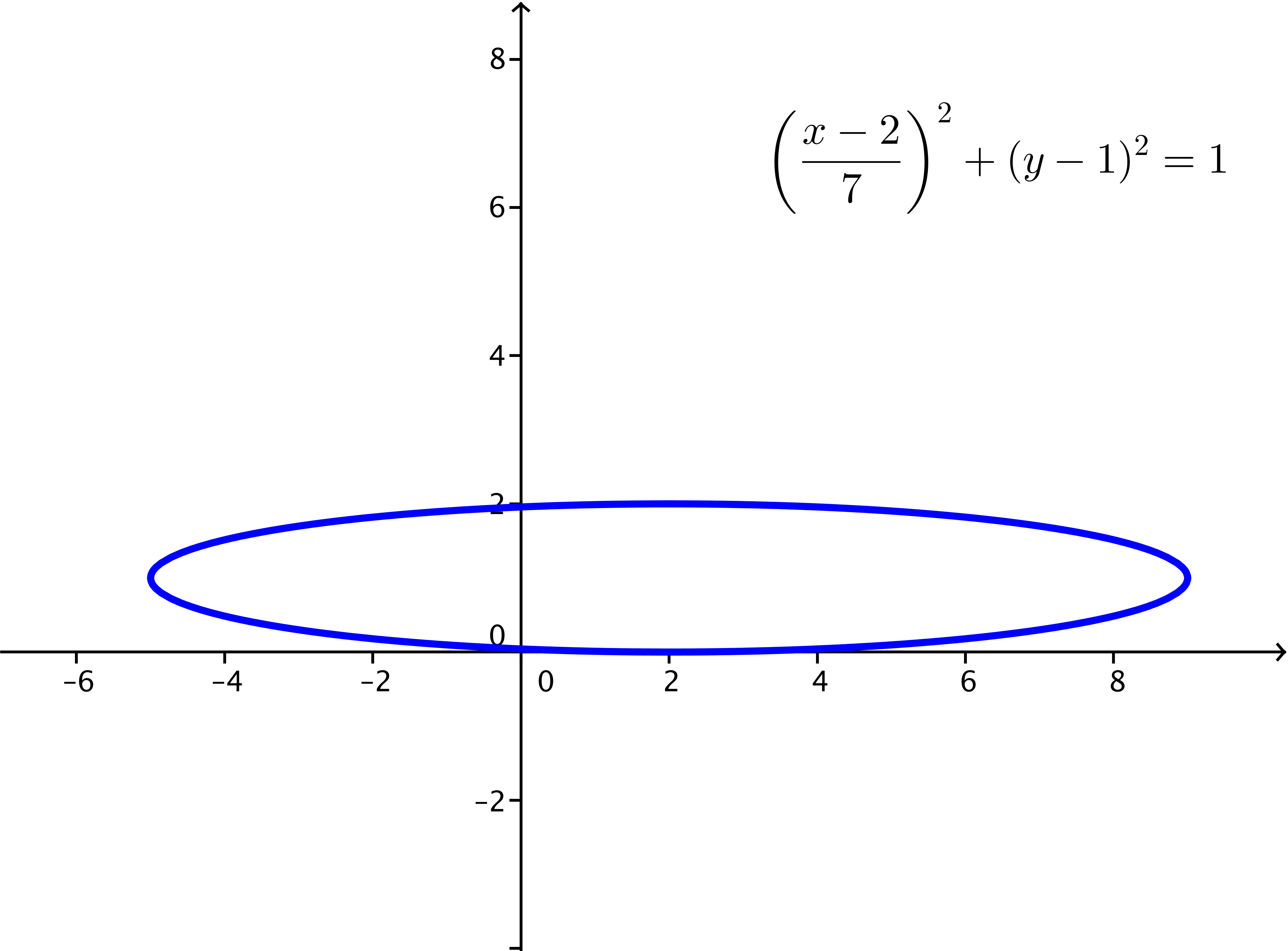
* 1. Find an algebraic equation for the points described.

*Since , then and .*

*So and*

*Since , we have , which is equivalent to   
.*

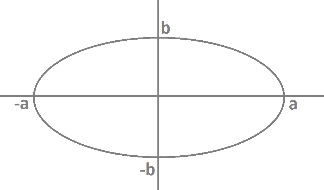
* 1. Sketch the graph of the ellipse.



Closing (5 minutes)

Have the students summarize the information on ellipses. As a class, a list of the key features of an ellipse can be compiled and displayed. A list of key features should address:

* An ellipse is a curve that represents the set of complex numbers that satisfy the equation   
   for .
* If , the curve is a circle with radius , and the equation can be simplified to .
* In the coordinate plane, ellipses centered at the origin can be represented by the equation , where is the half of the length of the horizontal axis and is the half of the length of the vertical axis.
* The sketch of a general ellipse is:



* An ellipse is elongated horizontally if and elongated vertically when .
* The points on an ellipse can be written in polar form as .
* An ellipse with center can be represented by the equation .

Exit Ticket (5 minutes)

Name Date

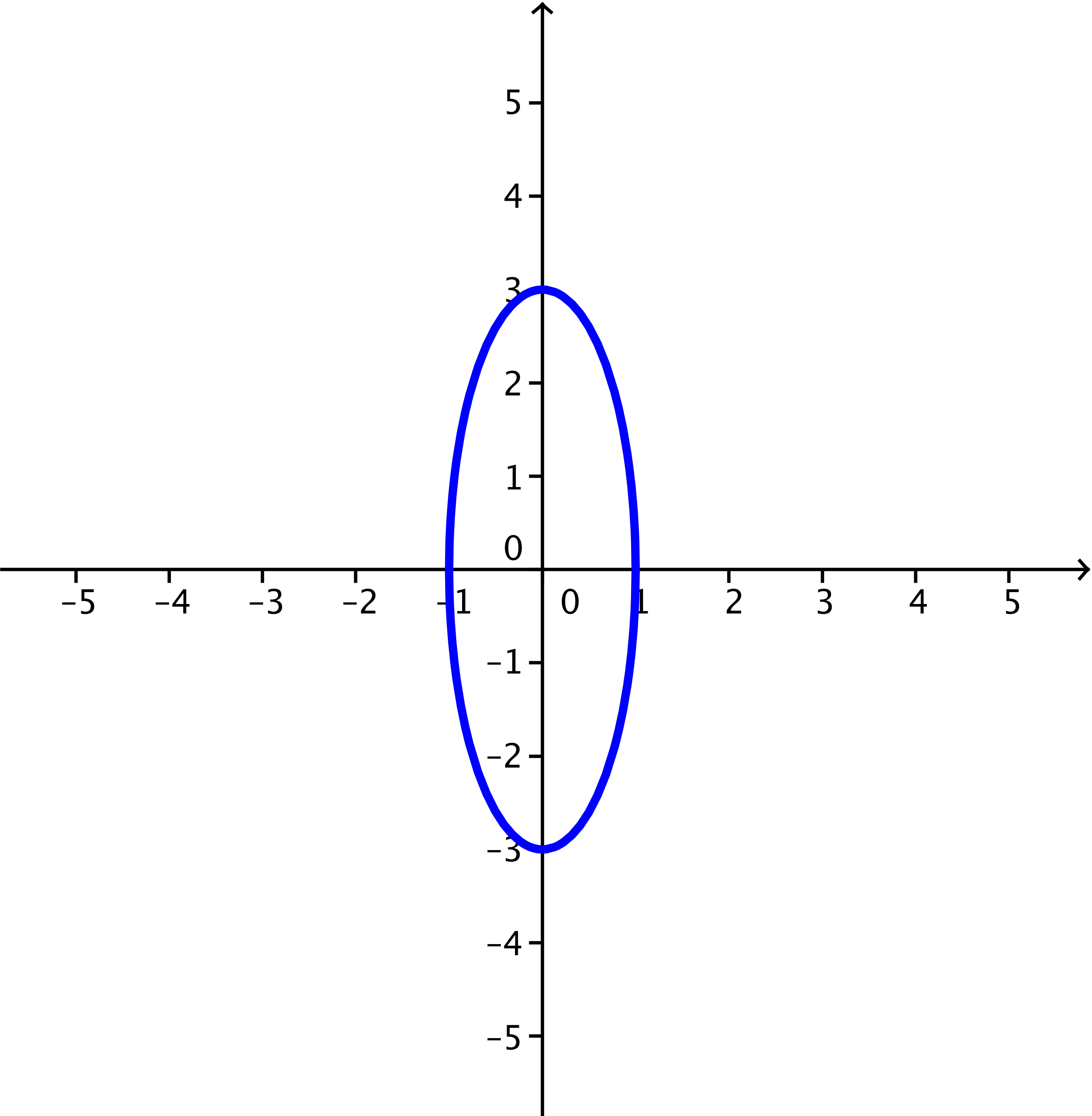
Lesson 6: Curves in the Complex Plane

Exit Ticket

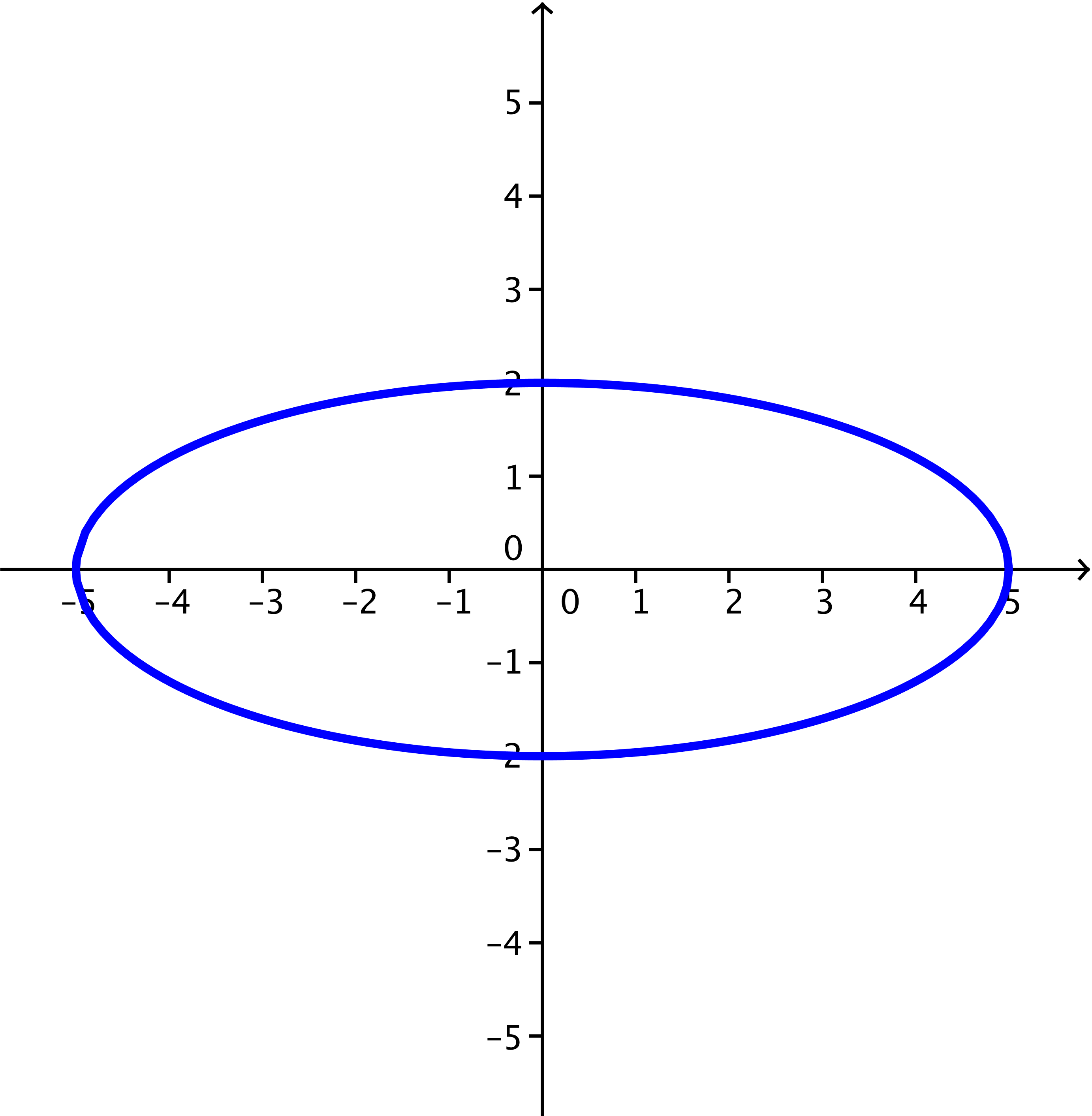
1. Write the real form of the complex equation . Sketch the graph of the equation.
2. Write the complex form of the equation . Sketch the graph of the equation.

Exit Ticket Sample Solutions

1. Write the real form of the complex equation . Sketch the graph of the equation.



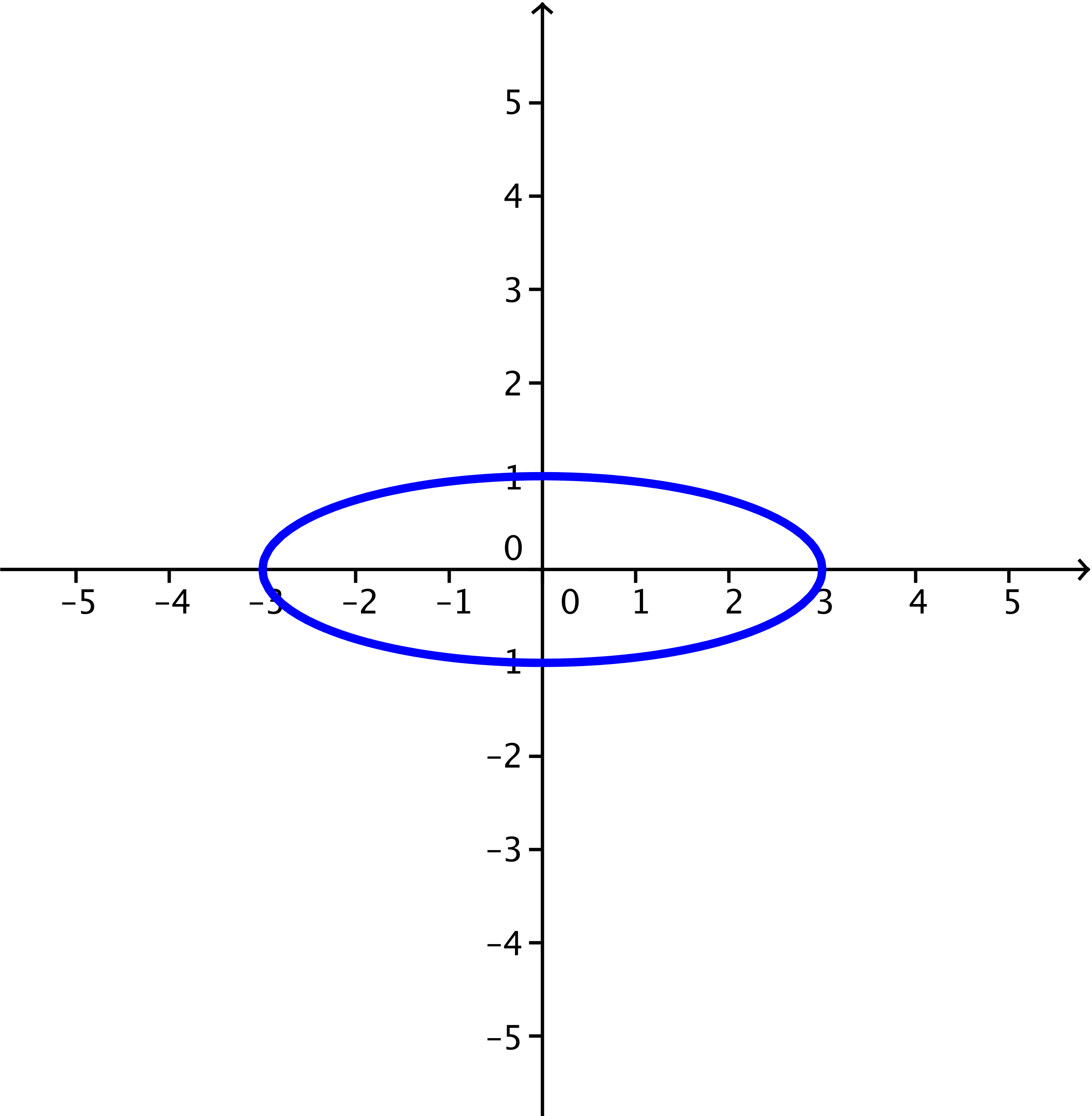
1. Write the complex form of the equation . Sketch the graph of the equation.

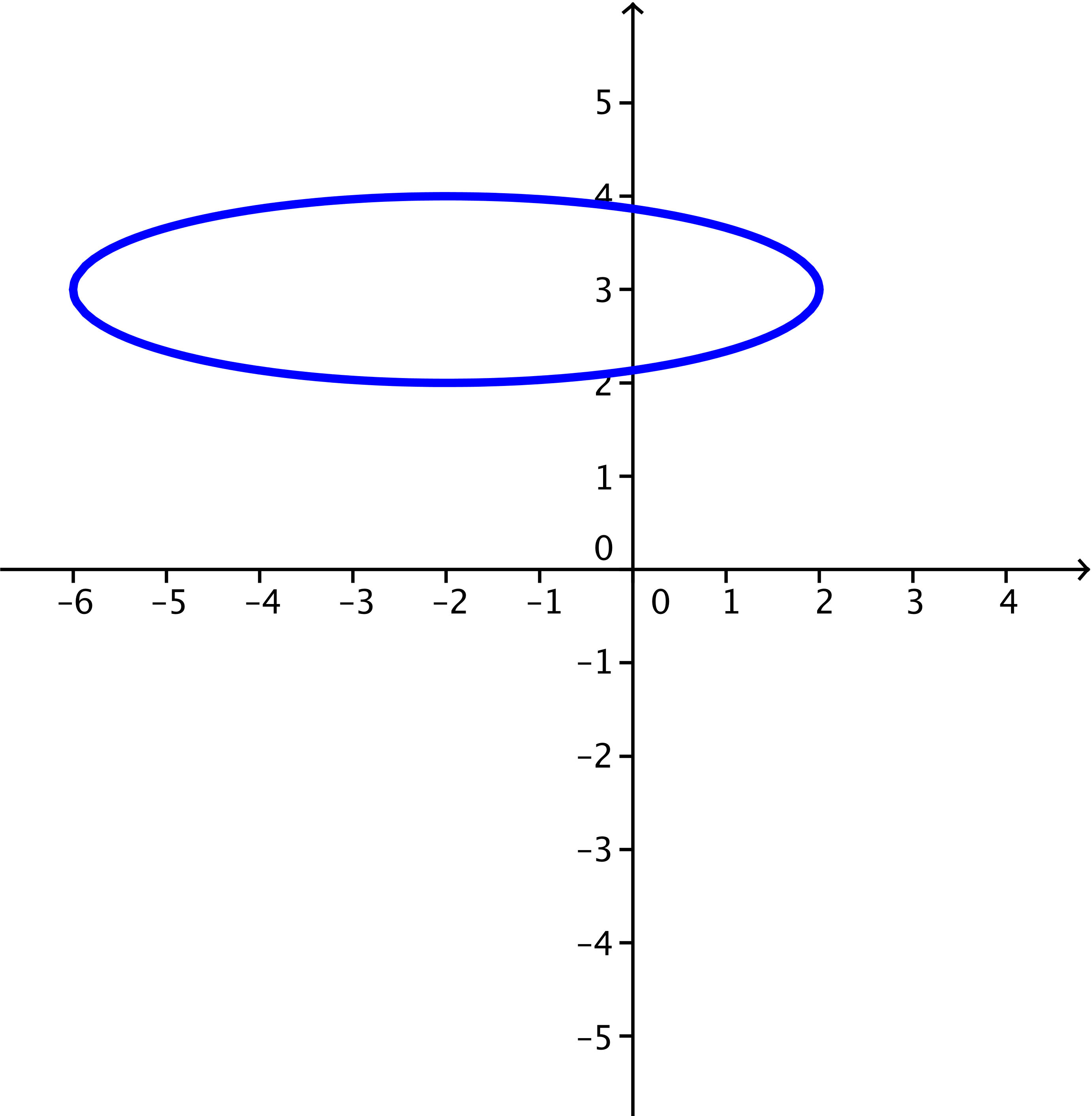


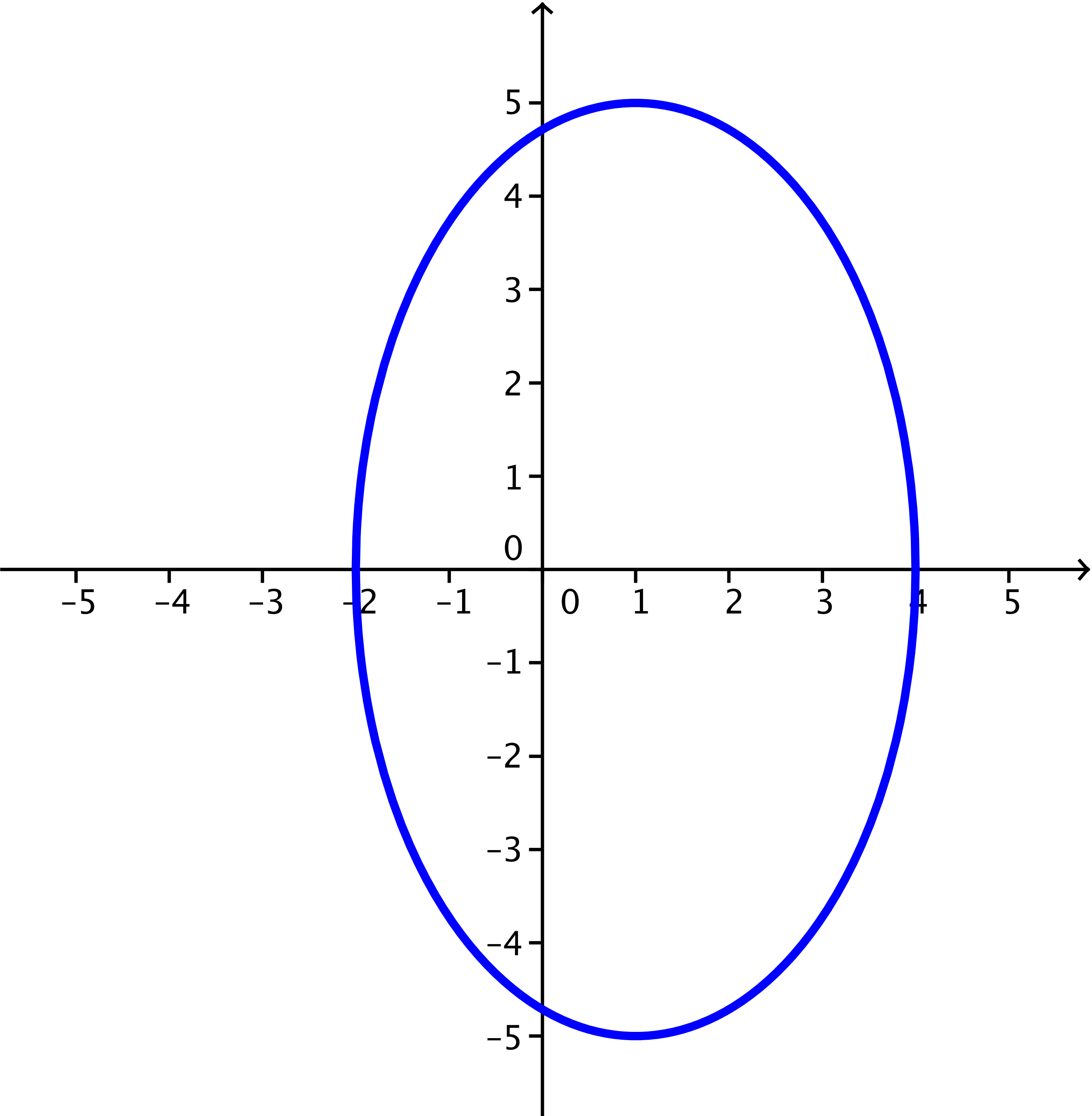
Problem Set Sample Solutions

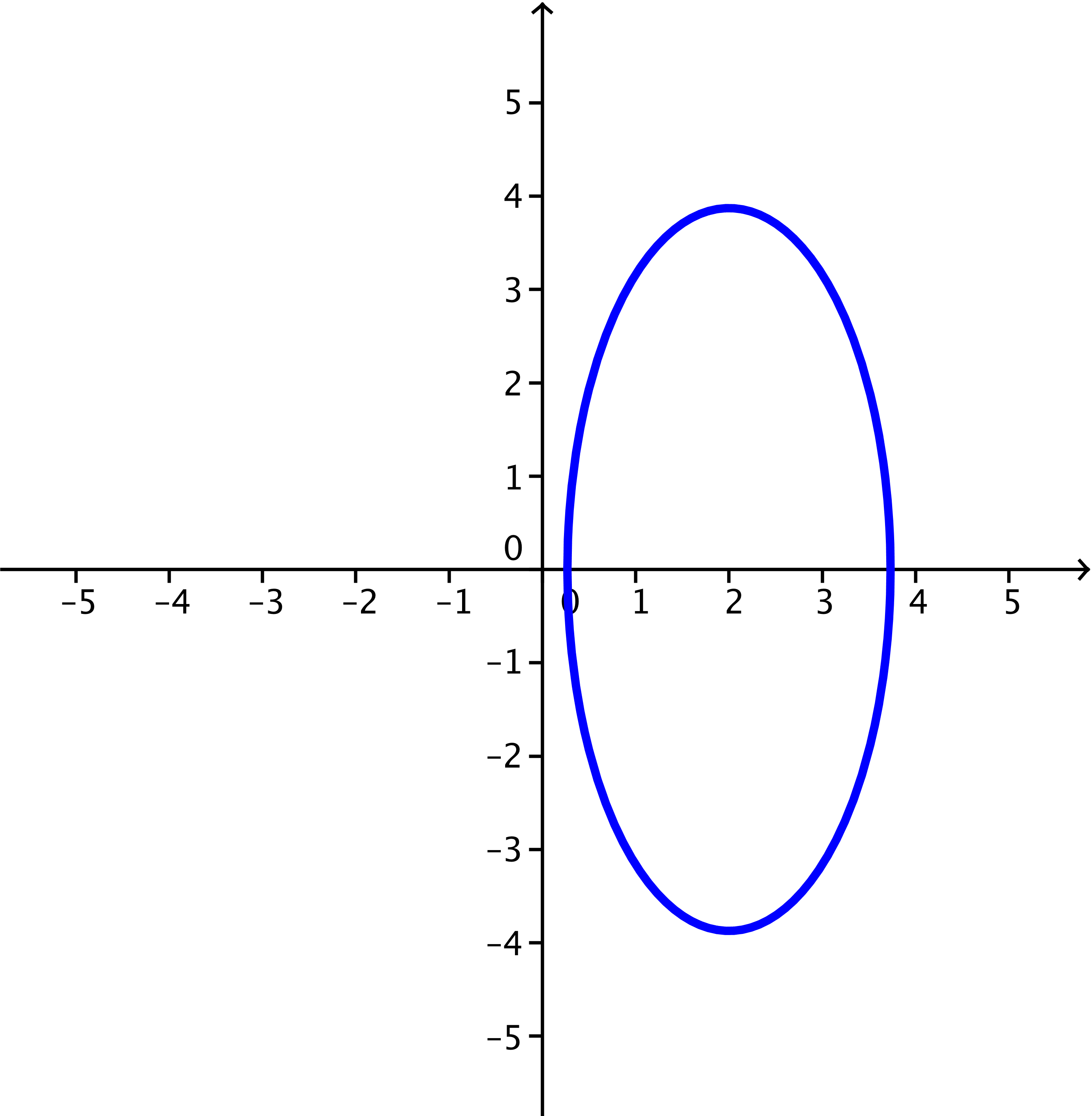
Problem 6 is an extension that requires students to convert an algebraic equation for an ellipse to standard form. The problem could be presented using the standard form of the equation (the answer for part (a)) to provide students with additional practice converting the equations of ellipses between complex and real forms.

1. Write the real form of each complex equation.
2. Sketch the graphs of each equation.









1. Write the complex form of each equation.
2. Carrie converted the equation to the real form . Her partner Ginger said that the ellipse must pass through the point and this point does not satisfy Carrie’s equation, so the equation must be wrong. Who made the mistake, and what was the error? Explain how you know.

Ginger is correct. Carrie set and which is correct but then made an error in converting to the real form of the equation by dividing by and instead of and .

1. Cody says that the center of the ellipse with complex equation is , while his partner Jarrett says that the center of this ellipse is . Which student is correct? Explain how you know.

*Cody is correct. This ellipse is the translation of the ellipse with equation by the vector   
, which moves the center of the ellipse from the origin to the point .*

Extension:

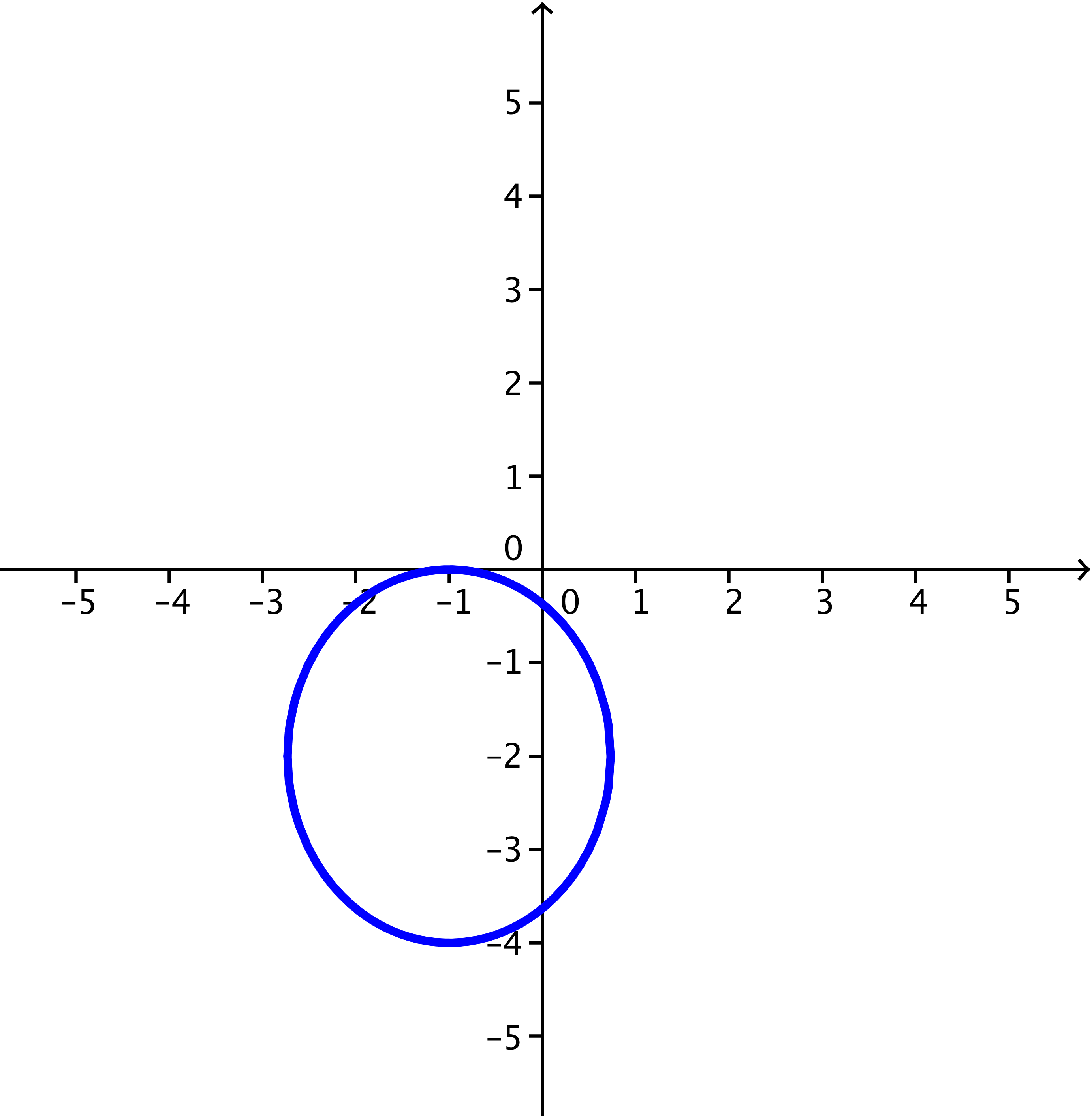
1. Any equation of the form with and might represent an ellipse. The equation is such an equation of an ellipse.
   1. Rewrite the equation in standard form to locate the center of the ellipse .

The center of the ellipse is the point .

* 1. Describe the graph of the ellipse, and then sketch the graph.

*The graph of the ellipse is centered at . It is elongated vertically*

*with a semi-major axis of length and a semi-minor axis of length .*



* 1. Write the complex form of the equation for this ellipse.

, *so* *and*

, *so*

, *so*