Lesson 5: The Binomial Theorem

Student Outcomes

- Students observe patterns in the coefficients of the terms in binomial expansions. They formalize their observations and explore the mathematical basis for them.
- Students use the binomial theorem to solve problems in a geometric context.

Lesson Notes

This lesson will provide the students with opportunities to explore additional patterns formed by the coefficients of binomial expansions. They will apply the binomial theorem to find a mathematical basis for the patterns observed. They will also apply the theorem to explore average rates of change for the volume of a sphere with a changing radius.

Classwork

Opening Exercise (3 minutes)

The students should complete the problems independently or in pairs. They could write the solutions on paper or display them on small white boards for quick checks. Remind students that the top row in Pascal's triangle is Row 0.

MP.7

Opening Exercise

Write the first six rows of Pascal's triangle. Then, use the triangle to find the coefficients of the terms with the powers of u and v shown, assuming that all expansions are in the form $(u+v)^n$. Explain how Pascal's triangle allows you to determine the coefficient.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

a. u^2v^4

15; this is in Row 6 (sum of exponents is 6) and the fifth term since the power of u started at 6 and has decreased to 2, which takes 5 steps.

b. u^3v^2

10; this is the third term in Row 5.

Scaffolding:

- For students struggling with the opening exercise, show an "amended" version of the first three rows of Pascal's triangle that includes the associated power of $(u+v)^n$:

  0 1
  1 1 1
  2 1 2 1
  3 1 3 3 1

  Ask questions such as "What patterns do you see?" and "What does the number in the leftmost column represent?" Expand $(u+v)^0$, $(u+v)^1$, $(u+v)^2$, and describe how the results are related to Pascal's triangle.
- Show $(u+v)^2 = u^2 + 2uv + v^2$ and $(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$. Ask, "What patterns do you notice in the powers of u and v?"

c. u^2v^2

6; this is the third term in Row 4.

d. v^{10}

1; this is the last term in Row 10 since there is no u.

Discussion (3 minutes)

- In the previous exercise, how did you determine which row of Pascal's triangle to use to find the requested coefficient? Provide an example.
  - *The sum of the powers of u and v equals n. This is the row of Pascal's triangle that contains the appropriate coefficients. For example, I used Row 6 of Pascal's triangle to find the coefficient of the term that contains u^2v^4 because the sum of the powers of u and v is 6.*
- Once you found the appropriate row of Pascal's triangle, how did you determine which coefficient to use?
  - *Answers will vary. An example of an appropriate response might be that the power of v is the same as the term in the expansion, e.g., the coefficient for u^2v^4 is the fourth term after the initial 1 in the sixth row.*
- And how could you use Pascal's triangle to find the coefficient of v^{10} in part (d) without expanding the triangle to Row 10?
  - *Answers will vary but should indicate that this term is the last term in the expansion of $(u+v)^{10}$, and the last coefficient in any binomial expansion is 1.*
- Good. And how could you use Pascal's triangle to find the coefficient of uv^6 in part (e) without writing out the seventh row?
  - *Answers will vary and might either indicate that the coefficient of the uv^{n-1} term in the expansion of $(u+v)^n$ is n or that the coefficient can be found by adding the coefficients of the last two numbers in Row 6 of the triangle.*

Example 1 (7 minutes)

The example should be completed as a teacher-led discussion. It will build upon initial patterns in binomial expansions explored in Lesson 4. The students will recognize a pattern in the alternating sums of each row of Pascal's triangle and will see how the binomial theorem can be used to provide a mathematical explanation for the observed pattern.

- Let's look again at the first six rows of Pascal's triangle. What patterns do you remember in the coefficients of the terms within each row?
  - *Answers will vary but should address that each row begins and ends with 1 and that the rows are symmetric, e.g., the value of the first term is the same as the $(n-1)^{st}$ term, the second term is the same as the $(n-2)^{nd}$ term, etc.*
- And what is the relationship of the values between rows?
  - *If we disregard the $1's$, the value of any entry in the triangle is the sum of the two values immediately above it.*

- So, if we were going to write out Row 7 for the triangle, how could we find the entries?
  - *The row would start and end with* 1*, and the rest of the terms would be found by adding adjacent terms from Row 6.*
- What would this look like?
  - $1\ (1+6)\ (6+15)\ (15+20)\ (20+15)\ (15+6)\ (6+1)\ 1 \leftrightarrow 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1$
- Good. Let's look at another interesting pattern in Pascal's triangle. First, let's compute the alternating sums of the rows of the triangle. For an alternating sum, we alternate adding and subtracting numbers. For Row 1 of the triangle, which has entries 1 and 1, the alternating sum is $1-1=0$. For Row 2 of the triangle, which has entries $1\ \ 2\ \ 1$, the alternating sum would be $1-2+1=0$. What is the value of the alternating sum of Row 3?
  - $1-3+3-1=0$
- Good. Now if we look at the alternating sums, what pattern do you see?
  - With the exception of the top row, the alternating sums are all 0.
- Why do you think the sums are all 0? Share your ideas with a partner.
  - *Answers will vary. Encourage the students to form conjectures and share them, e.g., perhaps the symmetry of the triangle affects the alternating sums.*
- Let's try to use the binomial theorem to explore this pattern by rewriting 0 as the sum $1+(-1)$. Now what is the value of $(1+(-1))^n$? How do you know?
  - It is 0 because the expression $1+(-1)$ is equal to 0 and $0^n=0$.
- So, $0=(1+(-1))^n$. Now let's use the binomial theorem to expand $(1+(-1))^n$. How can we apply the binomial theorem to this expansion?
  - Substitute $u=1$ and $v=1$ into the formula $(u+v)^n=u^n+A_1u^{n-1}v+A_2u^{n-2}v^2 \ldots + A_{n-1}uv^{n-1}+v^n$.
  - $0=1^n+A_1(1^{n-1})(-1)+A_2(1^{n-2})(-1)^2 \ldots + A_{n-1}(1)(-1)^{n-1}+(-1)^n$
- How else can we write this equation?
  - $0=1-A_1+A_2-\cdots+A_{n-1}(1)(-1)^{n-1}+(-1)^n$
- Why did we not simplify the last two terms in the expansion?
  - *The signs will vary based on the value of* n*. If* n *is even, the last terms will be* $-A_{n-1}$ *and* 1*. If* n *is odd, the last terms will be* A_{n-1} *and* -1*.*
- In both cases, what does our expansion represent?
  - *The coefficients of the* n^{th} *row of Pascal's triangle with alternating signs.*
- And what can we conclude from this?
  - *The alternating sum of the* n*th row of Pascal's triangle is always* 0*.*

Example 1

Look at the alternating sums of the rows of Pascal's triangle. An alternating sum alternately subtracts and then adds values. For example, the alternating sum of Row 2 would be $1-2+1$, and the alternating sum of Row 3 would be $1-3+3-1$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

a. **Compute the alternating sum for each row of the triangle shown.**

The sums are all zero.

b. **Use the binomial theorem to explain why each alternating sum of a row in Pascal's triangle is 0.**

The binomial theorem states that $(u+v)^n = u^n + A_1u^{n-1}v + A_2u^{n-2}v^2 + A_{n-1}uv^{n-1} + n$.

So, $0 = (1+(-1))^n = 1^n + A_1(1^{n-1})(-1) + A_2(1^{n-2})(-1)^2 \ldots + A_{n-1}(1)(-1)^{n-1} + (-1)^n = 1 - A_1 + A_2 \ldots - A_{n-1} + 1$ *for all even values of* n *and*

$= 1 - A_1 + A_2 \ldots + A_{n-1} - 1$ *for all odd values of* n.

Exercises 1–2 (15 minutes)

The students should be placed into small groups. The groups will be assigned to complete either Exercise 1 or Exercise 2. After about 5 minutes, the groups assigned to the same exercise could work together to discuss their findings and to organize their thoughts in order to prepare to present the exercise to the rest of the class. During the final 5 minutes, the students will present their solutions to the class.

Scaffolding:

- Students in groups that are struggling with how to approach Exercise 1 could be prompted to write 2^n as $(1+1)^n$.
- Students in groups that are struggling with how to approach Exercise 2 could be prompted to write 11^n as$(10+1)^n$.
- Help struggling students convert from the Pascal's triangle format to base-ten format of 11^n by writing $1|5|10|10|5|1$ in expanded form, i.e., $1\times(100{,}000) + 5\times(10{,}000) + 10\times(1{,}000) + 10\times(100) + 5\times(10) + 1 = 161{,}051$.

Advanced students could explore a different pattern they covered related to Pascal's triangle.

Exercises 1–2

1. **Consider the Rows 0–6 of Pascal's triangle.**

a. **Find the sum of each row.**

$$
\begin{array}{cccccccccccccc}
 & & & & & & & & & & & & & \textit{SUM} \\
 & & & & & & 1 & & & & & & & 1 \\
 & & & & & 1 & & 1 & & & & & & 2 \\
 & & & & 1 & & 2 & & 1 & & & & & 4 \\
 & & & 1 & & 3 & & 3 & & 1 & & & & 8 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & 16 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & 32 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & 64
\end{array}
$$

b. **What pattern do you notice in the sums computed?**

The sum of the coefficients in the n^{th} row appears to be 2^n.

c. **Use the binomial theorem to explain this pattern.**

$2^n = (1+1)^n$

The binomial theorem states that $(u+v)^n = u^n + A_1u^{n-1}v + A_2u^{n-2}v^2 \ldots + A_{n-1}uv^{n-1} + v^n$.

So $(1+1)^n = 1^n + A_1(1^{n-1})(1) + A_2(1^{n-2})(1)^2 \ldots + A_{n-1}(1)(1)^{n-1} + 1^n$.
$= 1 + A_1 + A_2 \ldots + A_{n-1} + 1$, *which is the sum of the coefficients of the* n*th row of Pascal's triangle.*

2. **Consider the expression** 11^n.

a. **Calculate** 11^n, **where** $n = 0, 1, 2, 3, 4$.

$11^0 = 1$
$11^1 = 11$
$11^2 = 121$
$11^3 = 1331$
$11^4 = 14641$

b. **What pattern do you notice in the successive powers?**

The digits of 11^n *correspond to the coefficients in the* n^{th} *row of Pascal's triangle.*

MP.2 & MP.7

c. **Use the binomial theorem to demonstrate why this pattern arises.**

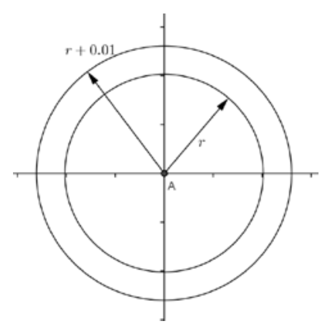
$11^n = (10+1)^n = 10^n + A_1(10)^{n-1}(1) + A_2(10)^{n-2}(1)^2 \ldots + A_{n-1}(10)(1)^{n-1} + 1^n$
$= 10^n + A_1(10)^{n-1} + A_2(10)^{n-2} \ldots + A_{n-1}(10) + 1$

d. **Use a calculator to find the value of** 11^5. **Explain whether this value represents what would be expected based on the pattern seen in lower powers of** 11.

If we continued the pattern seen in 11^n, *where* $n = 0, 1, 2, 3, 4$, *we would expect* 11^5 *to comprise the digits of the fifth row in Pascal's triangle. In other words, we could conjecture that* $11^5 = 1|5|10|10|5|1$. *Because we cannot represent a* 10 *as a single digit, the number on the calculator would be* $161{,}051$.

Scaffolding:

- Students that are strong visual learners may benefit from being shown a model of the situation, e.g., a solid sphere with radius $r + 0.01$ units with a sphere with radius r removed. Think of it as the shell of a tennis ball with this cross-section.



Example 2 (7 minutes)

This example should be completed as a teacher-led discussion. It will provide an opportunity for the students to apply the binomial theorem to a geometric context.

- We have explored patterns using the binomial theorem. Let's see how it can also be applied to solving problems with geometric solids. In Example 2, how can we calculate the increase in volume from a sphere with radius r to radius $r + 0.01$ units?
  - *Subtract the volume of the larger sphere from that of the smaller sphere.*

How can we represent this mathematically?

  - $V(r+0.01) - V(r) = \frac{4}{3}\pi(r+0.01)^3 - \frac{4}{3}\pi r^3$
- How can we use the binomial theorem to simplify this expression?
  - *We can use it to expand* $\frac{4}{3}\pi(r+0.01)^3$, *where* $u = r, v = 0.01$, *and* $n = 3$.

- Once we expand the expression and combine like terms, we are left with $0.04\pi(r)^2 + 0.0004\pi r + 0.000001\pi$. How can we use this expression to find the average rate of change of the volume?
  - *Divide the expression by the change in the radius, which is* 0.01.
- This gives us $4\pi r^2 + 0.04\pi r + 0.0001\pi$. How can we say that this approximates the surface area $S(r)$ when there are three terms in our expression?
  - *For most values of* r*, the values* 0.04π *and* 0.0001π *are negligible in comparison to the value of* $4\pi r^2$*, so we can reasonably approximate the expression as* $4\pi r^2$*, which is the surface area* $S(r)$.
- Why does the expression $V(r + 0.01) - V(r)$ represent a shell with thickness 0.01units covering the outer surface of the sphere with radius r?
  - *It is the surface that results from starting with a solid sphere with radius* $r + 0.01$ *and removing from it the solid sphere with radius* r.
- Why can we approximate the volume of the shell as $0.01 \cdot S(r)$?
  - *Answers will vary. An example of an appropriate answer would be that if we were to "unroll" the shell so the surface area of its inside lay flat, the expression* $0.01 \cdot S(r)$ *would be equal to the area of the flat surface multiplied by its height.*
- And using this expression for volume, how could we calculate the average rate of change?
  - *Divide by* 0.01*, which results in* $S(r)$.

Example 2

We know that the volume $V(r)$ and surface area $S(r)$ of a sphere of radius r are given by these formulas:

$$V(r) = \frac{4}{3}\pi r^3$$

$$S(r) = 4\pi r^2$$

Suppose we increase the radius of a sphere by 0.01 units from r to $r + 0.01$.

a. Use the binomial theorem to write an expression for the increase in volume.

$$V(r+0.01) - V(r) = \frac{4}{3}\pi(r+0.01)^3 - \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi((r)^3 + 3(r)^2(0.01) + 3r(0.01)^2 + (0.01)^3) - \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi r^3 + 0.04\pi(r)^2 + 0.0004\pi r + 0.000001\pi - \frac{4}{3}\pi r^3$$
$$= 0.04\pi(r)^2 + 0.0004\pi r + 0.000001\pi$$

b. Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.01$.

$$\text{Average rate of change} = \frac{V(r+0.01) - V(r)}{(r+0.01) - r}$$

c. Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.01$.

$$\text{Average rate of change} = \frac{0.04\pi(r)^2 + 0.0004\pi r + 0.000001\pi}{0.01} = 4\pi r^2 + 0.04\pi r + 0.0001\pi$$

d. **What does the expression from part (c) resemble?**

Surface area of the sphere with radius r.

e. **Why does it make sense that the average rate of change should approximate the surface area? Think about the geometric figure formed by $V(r+0.01)-V(r)$. What does this represent?**

It is a shell of volume, a layer 0.01 units thick, covering the surface area of the inner sphere of radius r.

f. **How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?**

The volume is approximately $S(r)\cdot(0.01)$; rate of change of volume is approximately $\frac{S(r)\times 0.01}{0.01}=S(r)$, which is the surface area of the sphere, $S(r)$.

Closing (5 minutes)

Have the students respond in writing to the prompt. After a few minutes, select several students to share their responses.

- Why is it beneficial to understand and be able to apply the binomial theorem? After you respond, share your thoughts with a partner.
  - *The binomial theorem can be used to help determine whether complex numbers are solutions to polynomial functions.*
  - *The binomial theorem can be used to explain mathematical patterns of numbers and patterns seen in Pascal's triangle.*
  - *The binomial theorem can be used to solve problems involving geometric solids.*
  - *The binomial theorem expedites the process of expanding binomials raised to whole number powers greater than 1.*

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 5: The Binomial Theorem

Exit Ticket

The area and circumference of a circle of radius r are given by:

$$A(r) = \pi r^2$$

$$C(r) = 2\pi r$$

a. Show mathematically that the average rate of change of the area of the circle as the radius increases from r to $r + 0.01$ units is very close to the perimeter of the circle.

b. Explain why this makes sense geometrically.

Exit Ticket Sample Solutions

The area and circumference of a circle of radius r are given by:

$$A(r) = \pi r^2$$

$$C(r) = 2\pi r$$

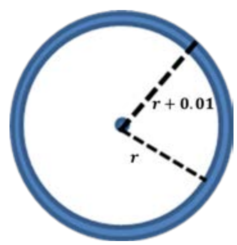
a. Show mathematically that the average rate of change of the area of the circle as the radius increases from r to $r + 0.01$ units is very close to the perimeter of the circle.

Average rate of change: $\frac{A(r+0.01)-A(r)}{(r+0.01)-r}$

$$A(r+0.01) - A(r) = \pi(r+0.01)^2 - \pi r^2 = \pi r^2 + 2\pi(r)(0.01) + \pi(0.01)^2 - \pi r^2$$
$$= 0.02\pi r + 0.0001\pi$$

Average rate of change: $\frac{0.02\pi r+0.0001\pi}{0.01} = 2\pi r + 0.01\pi$, *which is approximately equal to* $C(r)$.

b. Explain why this makes sense geometrically.



The difference $A(r+0.01) - A(r)$ *represents the area of a ring of width* 0.01 *units, where the inner circle forming the ring is a circle with radius* r. *The area of the ring could be approximated by the expression* $0.01 \cdot 2\pi r$. *The average rate of change of the area is* $\frac{0.01 \cdot 2\pi r}{0.01} = 2\pi r$, *which is the circumference of the circle with radius* r.

Problem Set Sample Solutions

1. Consider the binomial $(2u - 3v)^6$.

   a. Find the term that contains v^4.

   $15(2u)^2(3v)^4 = 4860u^2v^4$

   b. Find the term that contains u^3.

   $20(2u)^3(3v)^3 = 4320u^3v^3$

   c. Find the third term.

   $15(2u)^4(3v)^2 = 2160u^4v^2$

2. Consider the binomial $(u^2 - v^3)^6$.

a. Find the term that contains v^6.

$v^6 = (v^3)^2, 15(u^2)^4(v^3)^2 = 15u^8v^6$

b. Find the term that contains u^6.

$u^6 = (u^2)^3, 20(u^2)^3(v^3)^3 = 20u^6v^9$

c. Find the fifth term.

$15(u^2)^2(v^3)^4 = 15u^4v^{12})$

3. Find the sum of all coefficients in the following binomial expansion.

a. $(2u + v)^{10}$

3^{10}

b. $(2u - v)^{10}$

1

c. $(2u - 3v)^{11}$

-1

d. $(u - 3v)^{11}$

-2^{11}

e. $(1 + i)^{10}$

$((1 + i)^2)^5 = (1 + 2i - 1)^5 = (2i)^5 = 32i$

f. $(1 - i)^{10}$

$((1 - i)^2)^5 = (1 - 2i - 1)^5 = (-2i)^5 = -32i$

g. $(1 + i)^{200}$

$((1 + i)^2)^{100} = (1 + 2i - 1)^{100} = (2i)^{100} = 2^{100}$

h. $(1 + v)^{201}$

$(1 + i)(1 + i)^{200} = (1 + i)((1 + i)^2)^{100} = (1 + i)2^{100}$

4. Expand the binomial $\left(1 + \sqrt{2}i\right)^6$.

$1 + 6(1)^5\left(\sqrt{2}i\right) + 15(1)^4\left(\sqrt{2}i\right)^2 + 20(1)^3\left(\sqrt{2}i\right)^3 + 15(1)^2\left(\sqrt{2}i\right)^4 + 6(1)\left(\sqrt{2}i\right)^5 + \left(\sqrt{2}i\right)^6$

$1 + 6\sqrt{2}i - 30 - 40\sqrt{2}i + 60 + 24\sqrt{2}i - 8 = 23 - 10\sqrt{2}i$

5. Show that $\left(2+\sqrt{2}i\right)^{20}+\left(2-\sqrt{2}i\right)^{20}$ is an integer.

Let's get the first few terms for both to see whether it has any patterns that we can simply.

$$\left(2+\sqrt{2}i\right)^{2}=2^{20}+A_1(2)^{19}\left(\sqrt{2}i\right)^{1}+A_2(2)^{18}\left(\sqrt{2}i\right)^{2}+A_3(2)^{17}\left(\sqrt{2}i\right)^{3}+A_4(2)^{18}\left(\sqrt{2}i\right)^{4}+A_5(2)^{18}\left(\sqrt{2}i\right)^{5}+\cdots$$
$$-A_{17}(2)^{3}\left(\sqrt{2}i\right)^{17}+A_{18}(2)^{2}\left(\sqrt{2}i\right)^{18}+A_{19}(2)^{1}\left(\sqrt{2}i\right)^{19}+\left(\sqrt{2}i\right)^{20}$$

$$\left(2-\sqrt{2}i\right)^{2}=2^{20}-A_1(2)^{19}\left(\sqrt{2}i\right)^{1}+A_2(2)^{18}\left(\sqrt{2}i\right)^{2}-A_3(2)^{17}\left(\sqrt{2}i\right)^{3}+A_4(2)^{18}\left(\sqrt{2}i\right)^{4}+A_5(2)^{18}\left(\sqrt{2}i\right)^{5}+\cdots$$
$$-A_{17}(2)^{3}\left(\sqrt{2}i\right)^{17}+A_{18}(2)^{2}\left(\sqrt{2}i\right)^{18}-A_{19}(2)^{1}\left(\sqrt{2}i\right)^{19}+\left(\sqrt{2}i\right)^{20}$$

All the even terms will be canceled, and the remaining terms would involve multiplying powers of 2 with even powers of $\sqrt{2}$. So, the result would be the sum of some positive and negative integers, which is $116\,967\,424$.

6. We know $(u+v)^2=u^2+2uv+v^2=u^2+v^2+2uv$. Use this pattern to predict what the expanded form of each expression would be. Then, expand the expression, and compare your results.

a. $(u+v+w)^2$

$u^2+v^2+w^2+2uv+2uw+2vw$

b. $(a+b+c+d)^2$

$a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$

7. Look at the powers of 101 up to the fourth power on a calculator. Explain what you see. Predict the value of 101^5, and then find the answer on a calculator. Are they the same?

1
1 0 1
1 0 2 0 1
1 0 3 0 3 0 1
1 0 4 0 6 0 4 0 1
1 0 5 0 10 0 10 0 5 0 1

It is similar to Pascal's triangle, but zeros are inserted between the numbers in each row. The answers are not the same; the expanded form of $1|0|5|0|10|0|10|0|5|0|1$ is 10510100501; however, if we could represent a 10 as a single digit in our base-ten arithmetic, then both answers would be the same.

8. Can Pascal's triangle be applied to $\left(\frac{1}{u}+\frac{1}{v}\right)^n$ given $u,v\neq 0$?

Yes. The answer will be $\left(\frac{1}{u}+\frac{1}{v}\right)^n=\frac{1}{u^n}+\frac{A_1}{u^{(n-1)}v}+\frac{A_2}{u^{(n-2)}v^2}+\cdots+\frac{A_{n-2}}{u^2v^{(n-2)}}+\frac{A_{n-1}}{uv^{(n-1)}}+\frac{1}{v^n}$.

9. The volume and surface area of a sphere are given by $\mathrm{V}=\frac{4}{3}\pi\mathrm{r}^3$ and $\mathrm{S}=4\pi\mathrm{r}^2$. Suppose we increase the radius of a sphere by 0.001 units from r to $r+0.001$.

a. Use the binomial theorem to write an expression for the increase in volume $V(r+0.001)-V(r)$ as the sum of three terms.

$$\begin{aligned}V(r+0.001)-V(r)&=\frac{4}{3}\pi(r+0.001)^3-\frac{4}{3}\pi r^3\\&=\frac{4}{3}\pi[(r)^3+3(r)^2(0.001)+3r(0.001)^2+(0.001)^3]-\frac{4}{3}\pi r^3\\&=\frac{4}{3}\pi r^3+0.004\pi(r)^2+0.000004\pi r+\frac{0.000000004\pi}{3}-\frac{4}{3}\pi r^3\\&=0.004\pi(r)^2+0.000004\pi r+\frac{0.000000004\pi}{3}\end{aligned}$$

b. Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.001$.

Average rate of change: $\frac{V(r+0.001)-V(r)}{(r+0.001)-r}$

c. Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.001$.

Average rate of change: $\frac{0.004\pi(r)^2+0.000004\pi r+\frac{0.000000004\pi}{3}}{0.001} = 4\pi r^2 + 0.004\pi r + 0.000004\pi$

d. What does the expression from part (c) resemble?

Surface area of the sphere with radius r.

e. Why does it make sense that the average rate of changes should approximate the surface area? Think about the geometric figure formed by $V(r + 0.001) - V(r)$. What does this represent?

It is a shell of the volume, a layer 0.001 units thick, covering the surface area of the inner sphere of radius r.

f. How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?

The volume is approximately $S(r) \cdot 0.001$; the rate of change of volume is approximately $\frac{S(r)\times 0.001}{0.001}$, which is the surface area of the sphere, $S(r)$.

g. Find the difference between the average rate of change of the volume and $S(r)$ when $r = 1$.

$\text{Average rate of change} = 4\pi(1)^2 + 0.004\pi(1) + 0.000004\pi = 12.579$

$S(1) = 4\pi(1)^2 = 12.566$

$12.579 - 12.566 = 0.013$

10. The area and circumference of a circle of radius r are given by $A(r) = \pi r^2$ and $C(r) = 2\pi r$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.

a. Use the binomial theorem to write an expression for the increase in area volume $A(r + 0.001) - A(r)$ as a sum of three terms.

$A(r + 0.001) - V(r) = \pi(r + 0.001)^2 - \pi r^2 = \pi r^2 + 0.002\pi r + 0.000001\pi - \pi r^2$

$= 0.002\pi r + 0.000001\pi$

b. Write an expression for the average rate of change of the area as the radius increases from r to $r + 0.001$.

Average rate of change: $\frac{A(r+0.001)-A(r)}{(r+0.001)-r}$

c. Simplify the expression in part (b) to compute the average rate of change of the area of a circle as the radius increases from r to $r + 0.001$.

Average rate of change: $\frac{0.002\pi r+0.000001\pi}{0.001} = 2\pi r + 0.001\pi$

d. **What does the expression from part (c) resemble?**

Surface area of the circle with radius r.

e. **Why does it make sense that the average rate of change should approximate the area of a circle? Think about the geometric figure formed by $A(r + 0.001) - A(r)$. What does this represent?**

It is a shell of volume, a layer 0.01 units thick, covering the surface area of the inner circle of radius r.

f. **How could we approximate the area of the shell using circumference? And the average rate of change for the area?**

The volume is approximately $A(s) \cdot 0.001$; rate of change of volume is approximately $\frac{A(s)\times 0.001}{0.001}$, which is the surface area of the circle, $A(s)$.

g. **Find the difference between the average rate of change of the area and $C(r)$ when $r = 1$.**

$\text{Average rate of change} = 2\pi(1) + 0.001(1) = 6.284$

$C(1) = 2\pi(1) = 6.283$

$6.284 - 6.283 = 0.001$