Lesson 5: The Binomial Theorem

Classwork

Opening Exercise

Write the first six rows of Pascal's triangle. Then, use the triangle to find the coefficients of the terms with the powers of u and v shown, assuming that all expansions are in the form $(u+v)^n$. Explain how Pascal's triangle allows you to determine the coefficient.

a. u^2v^4

b. u^3v^2

c. u^2v^2

d. v^{10}

Example 1

Look at the alternating sums of the rows of Pascal's triangle. An alternating sum alternately subtracts and then adds values. For example, the alternating sum of Row 2 would be $1 - 2 + 1$, and the alternating sum of Row 3 would be $1 - 3 + 3 - 1$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

a. Compute the alternating sum for each row of the triangle shown.

b. Use the binomial theorem to explain why each alternating sum of a row in Pascal's triangle is 0.

Exercises 1–2

1. Consider the Rows 0–6 of Pascal's triangle.

   a. Find the sum of each row.

SUM

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

   b. What pattern do you notice in the sums computed?

   c. Use the binomial theorem to explain this pattern.

2. Consider the expression 11^n.

   a. Calculate 11^n, where $n = 0, 1, 2, 3, 4$.

   b. What pattern do you notice in the successive powers?

c. Use the binomial theorem to demonstrate why this pattern arises.

d. Use a calculator to find the value of 11^5. Explain whether this value represents what would be expected based on the pattern seen in lower powers of 11.

Example 2

We know that the volume $V(r)$ and surface area $S(r)$ of a sphere of radius r are given by these formulas:

$$V(r) = \frac{4}{3}\pi r^3$$

$$S(r) = 4\pi r^2$$

Suppose we increase the radius of a sphere by 0.01 units from r to $r + 0.01$.

a. Use the binomial theorem to write an expression for the increase in volume.

b. Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.01$.

c. Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.01$.

d. What does the expression from part (c) resemble?

e. Why does it make sense that the average rate of change should approximate the surface area? Think about the geometric figure formed by $V(r + 0.01) - V(r)$. What does this represent?

f. How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?

Problem Set

1. Consider the binomial $(2u - 3v)^6$.
   a. Find the term that contains v^4.
   b. Find the term that contains u^3.
   c. Find the third term.

2. Consider the binomial $(u^2 - v^3)^6$.
   a. Find the term that contains v^6.
   b. Find the term that contains u^6.
   c. Find the fifth term.

3. Find the sum of all coefficients in the following binomial expansion.
   a. $(2u + v)^{10}$
   b. $(2u - v)^{10}$
   c. $(2u - 3v)^{11}$
   d. $(u - 3v)^{11}$
   e. $(1 + i)^{10}$
   f. $(1 - i)^{10}$
   g. $(1 + i)^{200}$
   h. $(1 + v)^{201}$

4. Expand the binomial $\left(1 + \sqrt{2}i\right)^6$.

5. Show that $\left(2 + \sqrt{2}i\right)^{20} + \left(2 - \sqrt{2}i\right)^{20}$ is an integer.

6. We know $(u + v)^2 = u^2 + 2uv + v^2 = u^2 + v^2 + 2uv$. Use this pattern to predict what the expanded form of each expression would be. Then, expand the expression, and compare your results.
   a. $(u + v + w)^2$
   b. $(a + b + c + d)^2$

7. Look at the powers of 101 up to the fourth power on a calculator. Explain what you see. Predict the value of 101^5, and then find the answer on a calculator. Are they the same?

8. Can Pascal's triangle be applied to $\left(\frac{1}{u} + \frac{1}{v}\right)^n$ given $u, v \neq 0$?

9. The volume and surface area of a sphere are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.
   a. Use the binomial theorem to write an expression for the increase in volume $V(r + 0.001) - V(r)$ as the sum of three terms.
   b. Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.001$.
   c. Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.001$.
   d. What does the expression from part (c) resemble?
   e. Why does it make sense that the average rate of changes should approximate the surface area? Think about the geometric figure formed by $V(r + 0.001) - V(r)$. What does this represent?
   f. How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?
   g. Find the difference between the average rate of change of the volume and $S(r)$ when $r = 1$.

10. The area and circumference of a circle of radius r are given by $A(r) = \pi r^2$ and $C(r) = 2\pi r$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.
   a. Use the binomial theorem to write an expression for the increase in area volume $A(r + 0.001) - A(r)$ as a sum of three terms.
   b. Write an expression for the average rate of change of the area as the radius increases from r to $r + 0.001$.
   c. Simplify the expression in part (b) to compute the average rate of change of the area of a circle as the radius increases from r to $r + 0.001$.
   d. What does the expression from part (c) resemble?
   e. Why does it make sense that the average rate of change should approximate the area of a circle? Think about the geometric figure formed by $A(r + 0.001) - A(r)$. What does this represent?
   f. How could we approximate the area of the shell using circumference? And the average rate of change for the area?
   g. Find the difference between the average rate of change of the area and $C(r)$ when $r = 1$.