



Lesson 4: The Binomial Theorem

Student Outcomes

- Students discover patterns in the expansion of binomials, leading to the understanding of the binomial theorem.
- Students use Pascal's triangle to find the coefficients of binomial expansions.
- Students use binomial coefficients $C(n, k)$ to find the coefficients of binomial expansions.

Lesson Notes

The students will begin the lesson by working through an exercise verifying that a given complex number is a solution to a given polynomial. By carrying out the tedious process of repeatedly multiplying binomial factors together, they should come to appreciate the usefulness of finding a quicker way to expand binomials raised to whole number powers.

Students generate Pascal's triangle recursively, and then the binomial coefficients $C(n, k) = \frac{n!}{k!(n-k)!}$ are introduced, and students connect the binomial coefficient $C(n, k)$ to the k^{th} element of row n of Pascal's triangle (counting the top row of the triangle as row 0). Students then connect the entries in row n of Pascal's triangle to the coefficients of the expansion of the binomial $(u + v)^n$. These connections are made explicit in the binomial theorem, which students apply to expand binomial expressions and to find specific terms in expansions (**A-APR.C.5**).

You might choose to split this lesson over two days, introducing Pascal's triangle and the binomial coefficients $C(n, k)$ on the first day and connecting these to the binomial expansion $(u + v)^n$ and presenting the binomial theorem on the second day.

Classwork

Exercises 1–2 (4 minutes)

Assign half of the students to complete Exercise 1 and half of them to complete Exercise 2. Students should complete the exercise individually. After a few minutes, they could verify their responses with a partner assigned to the same exercise. Early finishers could write their solving process on chart paper to be displayed and explained when their classmates have completed the exercise. Review the exercises as an entire class. The purpose of this exercise is to show how tedious it can be to expand a binomial to higher powers so that students see the value in the formula presented in the binomial theorem.

Scaffolding:

- Show that $z = 1 + i$ is a solution to a simpler polynomial such as $z^2 - 2z + 2$.
- Challenge advanced learners to explain how properties of complex conjugates could be used to verify $1 - i$ is a solution if they know that $1 + i$ is a solution.

Exercises

1. Show that $z = 1 + i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.

If $(1 + i)^4 - (1 + i)^3 + 3(1 + i)^2 - 4(1 + i) + 6 = 0$, then $z = 1 + i$ is a solution.

$$(1 + i)^2 = 1^2 + 2(i)(i) + i^2 = 2i$$

$$(1 + i)^3 = (1 + i)(1 + i)^2 = (1 + i)(2i) = -2 + 2i$$

$$(1 + i)^4 = ((1 + i)^2)^2 = (2i)^2 = 4i^2 = -4$$

$$\begin{aligned}(1 + i)^4 - (1 + i)^3 + 3(1 + i)^2 - 4(1 + i) + 6 &= -4 - (-2 + 2i) + 3(2i) - 4(1 + i) + 6 \\ &= -4 + 2 - 2i + 6i - 4 - 4i + 6 \\ &= 0\end{aligned}$$

2. Show that $z = 1 - i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.

If $(1 - i)^4 - (1 - i)^3 + 3(1 - i)^2 - 4(1 - i) + 6 = 0$, then $z = 1 - i$ is a solution.

$$(1 - i)^2 = 1^2 + 2(1)(-i) + (-i)^2 = -2i$$

$$(1 - i)^3 = (1 - i)(1 - i)^2 = (1 - i)(-2i) = -2 - 2i$$

$$(1 - i)^4 = ((1 - i)^2)^2 = (-2i)^2 = 4i^2 = -4$$

$$\begin{aligned}(1 - i)^4 - (1 - i)^3 + 3(1 - i)^2 - 4(1 - i) + 6 &= -4 - (-2 - 2i) + 3(-2i) - 4(1 - i) + 6 \\ &= -4 + 2 + 2i - 6i - 4 + 4i + 6 = 0\end{aligned}$$

Discussion (8 minutes)

- What did you find most challenging or frustrating about verifying the solution in the exercises just completed?
 - *Answers may vary but will probably address that it is tedious to carry out all the arithmetic needed to verify the solution.*
- How do you think these issues would be affected by the degree of the polynomial for which you are verifying a solution?
 - *Answers may vary but might include that the process will become even more tedious and time consuming as the number of terms and degree of the polynomial increase.*
- Though it is tedious to substitute and simplify expressions to verify the solutions to polynomials, it is an important component to solving polynomials with complex solutions. What strategies did you use to try to expedite the process of simplifying in the exercise?
 - *Answers will vary but might address simplifying binomials of lesser degree and then applying those expressions to simplify binomials of higher degree.*
- Instead of looking at specific complex numbers such as $1 + i$ or $1 - i$, let's look at any binomial expression $u + v$ where u and v can be numbers or expressions such as x , $2xy$, ab , etc. Let's suppose that we could write any expression $(u + v)^n$ in expanded form without having to multiply binomials repeatedly. We know how to do this for a few values of n . How can we write an expression equivalent to $(u + v)^0$ in expanded form? What about $(u + v)^1$?
 - $(u + v)^0 = 1$; $(u + v)^1 = u + v$

- You might also know an expanded expression to represent quadratic polynomials. What is the expanded form of the expression $(u + v)^2$?
 - $u^2 + 2uv + v^2$
- Our goal for this lesson is to find a quick way to expand polynomials of higher degree. We will return to this task in a bit.
- Pascal's triangle is a triangular configuration of numbers that is constructed recursively.

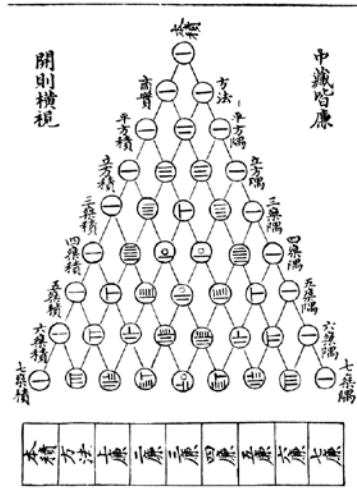
Row 0:				1			
Row 1:			1		1		
Row 2:			1	2		1	
Row 3:		1	3		3	1	
Row 4:	1	4	6		4	1	
Row 5:	1	5	10	10	5	1	
	:	:	:	:	:	:	:

- Rows in the triangle are generated recursively. The row at the top containing a single 1 is counted as row 0. Row 1 contains two 1's. To build a row from the row above it, you start with a 1, written to the left of the position of the 1 from the previous row. In the next space, which is positioned horizontally between two elements in the row above, we add the elements in the upper row that are to the left and the right of the current position. That is, to generate Row 5 of the table, we start with a 1, then add $1 + 4 = 5$, $4 + 6 = 10$, $6 + 4 = 10$, $4 + 1 = 5$ across the row and end with another 1.
- Now we want to generate Row 6 of Pascal's triangle. Allow students to suggest the entries as you record them in the triangle. Talk through the process of calculating each entry in Row 6: 1, then $1 + 5 = 6$ for the second entry, then $5 + 10 = 15$ for the third entry, etc.

Row 0:					1		
Row 1:				1		1	
Row 2:			1	2		1	
Row 3:		1	3		3	1	
Row 4:		1	4	6		4	1
Row 5:	1	5	10	10	5	1	
Row 6:	1	6	15	20	15	6	1

- Though its name comes from the French mathematician Blaise Pascal (1623–1662), who published the triangle in the *Treatise on the Arithmetic Triangle* in France in 1654, the use of the triangle predates Pascal. The figure below was used as early as the thirteenth century and was known in China as Yang Hui's triangle. The markings in the circles below are Chinese rod numbers, and they indicate the same numbers that we have in Pascal's triangle above.

圖方藥七法古



“Yanghui Triangle” by Yáng Huī is licensed under CC BY 2.0

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- The Persian mathematician Omar Khayyam (1044–1123 AD) also mentioned the triangle in his works.
- What patterns do you notice in the coefficients in the triangle? Think about this, and then share your ideas with a partner.
 - *Answers will vary but need to address that each row starts and ends with 1, and each of the rest of the coefficients are found by summing the two terms above it.*

Exercise 3 (2 minutes)

The students should complete the exercise individually and verify their responses with a partner when they are done. Have a student provide the coefficients for each row at the appropriate time.

3. Based on the patterns seen in Pascal's triangle, what would be the coefficients of Rows 7 and 8 in the triangle? Write the coefficients of the triangle beneath the part of the triangle shown.

Row 0:							1												
Row 1:							1		1										
Row 2:						1		2		1									
Row 3:					1		3		3		1								
Row 4:				1		4		6		4		1							
Row 5:			1		5		10		10		5		1						
Row 6:		1		6		15		20		15		6		1					
Row 7:	1		7		21		35		35		21		7		1				
Row 8:	1		8		28		56		70		56		28		8		1		

Discussion (2 minutes)

- There is a way to calculate an entry of Pascal's triangle without writing out the whole triangle, but we first need the idea of a *factorial*, which we denote by $n!$ for integers $n \geq 0$. First, we define $0! = 1$. Then, if $n > 0$, we define $n!$ to be the product of all positive integers less than or equal to n . For example, $2! = 2 \cdot 1$ and $3! = 3 \cdot 2 \cdot 1 = 6$. What are $4!$ and $5!$?
 - $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
- Then, for integers $n \geq 0$ and $k \geq n$, we define the quantity $C(n, k) = \frac{n!}{k!(n-k)!}$. For example,

$$C(6, 4) = \frac{6!}{4! \cdot 2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{6 \cdot 5}{2 \cdot 1} = 15.$$

Exercises 4–7 (6 minutes)

In these exercises, students practice calculating simple factorials and dividing factorials. Encourage students to write out the products and simplify the expression before calculating the factorials. Have students complete these exercises in pairs. Quickly debrief the answers before continuing, making sure that students draw the conclusion that we can generate Row n of Pascal's triangle by calculating the quantities $C(n, k)$ for $0 \leq k \leq n$.

4. Calculate the following factorials.

a. $6!$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

b. $10!$

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3\,628\,800$$

5. Calculate the value of the following factorial expressions.

a. $\frac{7!}{6!}$

$$\frac{7!}{6!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7$$

b. $\frac{10!}{6!}$

$$\frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

c. $\frac{8!}{5!}$

$$\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 = 336$$

d. $\frac{12!}{10!}$

$$\frac{12!}{10!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 12 \cdot 11 = 132$$

6. Calculate the following quantities.

a. $C(1, 0)$ and $C(1, 1)$

$$C(1, 0) = \frac{1!}{0!1!} = 1 \text{ and } C(1, 1) = \frac{1!}{0!1!} = 1$$

b. $C(2, 0)$, $C(2, 1)$, and $C(2, 2)$

$$C(2, 0) = \frac{2!}{0!2!} = 1, C(2, 1) = \frac{2!}{1!1!} = 2, \text{ and } C(2, 2) = \frac{2!}{2!0!} = 1$$

c. $C(3, 0)$, $C(3, 1)$, $C(3, 2)$, and $C(3, 3)$

$$C(3, 0) = \frac{3!}{0!3!} = 1, C(3, 1) = \frac{3!}{1!2!} = 3, C(3, 2) = \frac{3!}{2!1!} = 3, \text{ and } C(3, 3) = \frac{3!}{3!0!} = 1$$

d. $C(4, 0)$, $C(4, 1)$, $C(4, 2)$, $C(4, 3)$, and $C(4, 4)$

$$C(4, 0) = \frac{4!}{0!4!} = 1, C(4, 1) = \frac{4!}{1!3!} = 4, C(4, 2) = \frac{4!}{2!2!} = 6, C(4, 3) = \frac{4!}{3!1!} = 4, \text{ and } C(4, 4) = \frac{4!}{4!0!} = 1$$

7. What patterns do you see in Exercise 5?

The numbers $C(n, k)$ for $1 \leq n \leq 4$ give the same numbers as in Pascal's triangle.

Also, it appears that $C(n, 0) = 1$ and $C(n, n) = 1$ for each n .

MP.7

Exercises 8–11 (7 minutes)

We now return to looking for a shortcut to expanding a binomial expression. In these exercises, students connect the binomial coefficients $C(n, k)$ to the coefficients of the binomial expansion $(u + v)^n$.

Have the students complete these exercises in pairs. At an appropriate time, have students display their solutions.

8. Expand the expression $(u + v)^3$.

$$\begin{aligned} (u + v)^3 &= (u + v)(u^2 + 2uv + v^2) \\ &= u^3 + 2u^2v + uv^2 + vu^2 + 2uv^2 + v^3 \\ &= u^3 + 3u^2v + 3uv^2 + v^3 \end{aligned}$$

9. Expand the expression $(u + v)^4$.

$$\begin{aligned} (u + v)^4 &= (u + v)(u^3 + 3u^2v + 3uv^2 + v^3) \\ &= u^4 + 3u^3v + 3u^2v^2 + uv^3 + vu^3 + 3u^2v^2 + 3uv^3 + v^4 \\ &= u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 \end{aligned}$$

Scaffolding:

Encourage students to write each term in the expansion with the power of u preceding the power of v if they are struggling with recognizing like terms.

10.

- a. Multiply the expression you wrote in Exercise 4 by u .

$$u(u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4) = u^5 + 4u^4v + 6u^3v^2 + 4u^2v^3 + uv^4$$

- b. Multiply the expression you wrote in Exercise 4 by v .

$$v(u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4) = u^4v + 4u^3v^2 + 6u^2v^3 + 4uv^4 + v^5$$

- c. How can you use the results from parts (a) and (b) to find the expanded form of the expression $(u + v)^5$?

Because $(u + v)^5 = (u + v)(u + v)^4 = u(u + v)^4 + v(u + v)^4$, we have

$$\begin{aligned}(u + v)^5 &= (u^5 + 4u^4v + 6u^3v^2 + 4u^2v^3 + uv^4) + (u^4v + 4u^3v^2 + 6u^2v^3 + 4uv^4 + v^5) \\ &= u^5 + 5u^4v + 10u^3v^2 + 10u^2v^3 + 5uv^4 + v^5.\end{aligned}$$

11. What do you notice about your expansions for $(u + v)^4$ and $(u + v)^5$? Does your observation hold for other powers of $(u + v)$?

The coefficients of $(u + v)^4$ are the numbers in Row 4 of Pascal's triangle. The coefficients of $(u + v)^5$ are the numbers in Row 5 of Pascal's triangle. The same pattern holds for $(u + v)$, $(u + v)^2$, and $(u + v)^3$.

Discussion (5 minutes)

This teacher-led discussion will reiterate the connection that the students made in the previous exercises between the coefficients of a binomial expansion and Pascal's triangle.

- So now we have computed the expansions for $(u + v)^n$ starting with $n = 0$ to $n = 5$. Let's arrange them vertically, from least power to the greatest and with all the coefficients written explicitly.

$$(u + v)^0 = 1$$

$$(u + v)^1 = 1u + 1v$$

$$(u + v)^2 = 1u^2 + 2uv + 1v^2$$

$$(u + v)^3 = 1u^3 + 3u^2v + 3uv^2 + 1v^3$$

$$(u + v)^4 = 1u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + 1v^4$$

$$(u + v)^5 = 1u^5 + 5u^4v + 10u^3v^2 + 10u^2v^3 + 5uv^4 + 1v^5$$

- What patterns do you notice in the expansion? Think about this for a minute, and then share your ideas with a partner.
 - Answers will vary but should address that within each row, the power of u decreases from left to right, and the power of v increases from left to right; the sum of the powers in each term is equal to the power of the binomial; each row begins and ends with terms that have a coefficient of 1; the number of terms is one greater than the power of the binomial.
 - The coefficients of the binomial expansion are the numbers in the corresponding row of Pascal's triangle.
 - The coefficients of the binomial expansion $(u + v)^n$ are $C(n, k)$ as k increases from 0 to n .

- The correspondence between the numbers in Row n of Pascal's triangle and the coefficients of the expanded expression $(u + v)^n$ is known as *the binomial theorem*. The numbers $C(n, k)$ are called *binomial coefficients*.
- THE BINOMIAL THEOREM:** For any expressions u and v ,

$$(u + v)^n = u^n + C(n, 1)u^{n-1}v + C(n, 2)u^{n-2}v^2 + \cdots + C(n, k)u^{n-k}v^k + \cdots + C(n, n-1)u v^{n-1} + v^n.$$
That is, the coefficients of the expanded binomial $(u + v)^n$ are exactly the numbers in Row n of Pascal's triangle.

Exercise 12 (4 minutes)

Have students work on these exercises in pairs or small groups. If time permits, have students share their results with the class either on the document camera, using individual white boards, or by writing on the board.

12. Use the binomial theorem to expand the following binomial expressions.

a. $(x + y)^6$

$$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

b. $(x + 2y)^3$

$$x^3 + 6x^2y + 12xy^2 + 8y^3$$

c. $(ab + bc)^4$

$$a^4b^4 + 4a^3b^4c + 6a^2b^4c^2 + 4ab^4c^3 + b^4c^4$$

d. $(3xy - 2z)^3$

$$27x^3y^3 - 54x^2y^2z + 36xyz^2 - 8z^3$$

e. $(4p^2qr - qr^2)^5$

$$1024p^{10}q^5r^5 - 1280p^8q^5r^6 + 640p^6q^5r^7 - 160p^4q^5r^8 + 20p^2q^5r^9 - q^5r^{10}$$

Closing (3 minutes)

Have the students reflect on the questions. After a minute, have them share their responses with a partner. If time permits, a few students could share their reflections with the rest of the class.

- When is it helpful to apply the binomial theorem?
 - Answers will vary. An example of an appropriate response would be that it can be used to expand binomials in the form $(u + v)^n$ without having to multiply several factors together.*
- How is Pascal's triangle helpful when applying the binomial theorem?
 - The entries in Row n represent the coefficients of the expansion of $(u + v)^n$.*

- How are the binomial coefficients $C(n, k)$ helpful when applying the binomial theorem?
 - *The binomial coefficients allow us to calculate Row n of Pascal's triangle without writing out all of the previous rows.*

Lesson Summary

Pascal's triangle is an arrangement of numbers generated recursively:

Row 0:				1			
Row 1:				1		1	
Row 2:				1		2	
Row 3:				1		3	
Row 4:				1		4	
Row 5:				1		5	
				1		6	
				1		10	
				1		10	
				1		5	
				1		1	

For an integer $n \geq 1$, the number $n!$ is the product of all positive integers less than or equal to n . We define $0! = 1$.

The binomial coefficients $C(n, k)$ are given by $C(n, k) = \frac{n!}{k!(n-k)!}$ for integers $n \geq 0$ and $0 \leq k \leq n$.

THE BINOMIAL THEOREM: For any expressions u and v ,

$$(u + v)^n = u^n + C(n, 1)u^{n-1}v + C(n, 2)u^{n-2}v^2 + \cdots + C(n, k)u^{n-k}v^k + \cdots + C(n, n-1)u v^{n-1} + v^n.$$

That is, the coefficients of the expanded binomial $(u + v)^n$ are exactly the numbers in Row n of Pascal's triangle.

Exit Ticket (4 minutes)

Name _____

Date _____

Lesson 4: The Binomial Theorem

Exit Ticket

1. Evaluate the following expressions.

a. $5!$

b. $\frac{8!}{6!}$

c. $C(7,3)$

2. Find the coefficients of the terms below in the expansion of $(u + v)^8$. Explain your reasoning.

a. u^2v^6

b. u^3v^5

c. u^4v^4

Exit Ticket Sample Solutions

1. Evaluate the following expressions.

a. $5!$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

b. $\frac{8!}{6!}$

$$\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$$

c. $C(7, 3)$

$$C(7, 3) = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

Alternatively, students could use the corresponding entry of Row 7 of Pascal's triangle, 7 21 35 35 21 7 1, which is 35.

2. Find the coefficients of the terms below in the expansion of $(u + v)^8$. Explain your reasoning.

a. u^2v^6

The binomial theorem says that the u^2v^6 term of the expansion is $C(8, 2)u^2v^6$, so the coefficient is $C(8, 2) = \frac{8!}{2!6!} = 28$. Alternatively, Row 8 of Pascal's triangle is 1 8 28 56 70 56 28 8 1, and the entry corresponding to u^2v^6 is 28.

b. u^3v^5

The binomial theorem says that the u^3v^5 term of the expansion is $C(8, 3)u^3v^5$, so the coefficient is $C(8, 3) = \frac{8!}{3!5!} = 56$. Alternatively, Row 8 of Pascal's triangle is 1 8 28 56 70 56 28 8 1, and the entry corresponding to u^3v^5 is 56.

c. u^4v^4

The binomial theorem says that the u^4v^4 term of the expansion is $C(8, 4)u^4v^4$, so the coefficient is $C(8, 4) = \frac{8!}{4!4!} = 70$. Alternatively, Row 8 of Pascal's triangle is 1 8 28 56 70 56 28 8 1, and the entry corresponding to u^4v^4 is 70.

Problem Set Sample Solutions

1. Evaluate the following expressions.

a. $\frac{9!}{8!}$

$$\frac{9!}{8!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 9$$

b. $\frac{7!}{5!}$

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 = 42$$

c. $\frac{21!}{19!}$

$$\frac{21!}{19!} = \frac{21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{19 \cdot 18 \cdot 17 \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 21 \cdot 20 = 420$$

d. $\frac{8!}{4!}$

$$\frac{8!}{4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

2. Use the binomial theorem to expand the following binomial expressions.

a. $(x + y)^4$

$$\begin{aligned}(x + y)^4 &= x^4 + C(4, 1)x^3y + C(4, 2)x^2y^2 + C(4, 3)xy^3 + y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

b. $(x + 2y)^4$

$$\begin{aligned}(x + 2y)^4 &= x^4 + C(4, 1)x^3(2y) + C(4, 2)x^2(2y)^2 + C(4, 3)x(2y)^3 + (2y)^4 \\ &= x^4 + 4x^3(2y) + 6x^2(4y^2) + 4x(8y^3) + 16y^4 \\ &= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4\end{aligned}$$

c. $(x + 2xy)^4$

$$\begin{aligned}(x + 2xy)^4 &= x^4 + C(4, 1)x^3(2xy) + C(4, 2)x^2(2xy)^2 + C(4, 3)x(2xy)^3 + (2xy)^4 \\ &= x^4 + 4x^3(2xy) + 6x^2(4x^2y^2) + 4x(8x^3y^3) + 16x^4y^4 \\ &= x^4 + 8x^4y + 24x^4y^2 + 32x^4y^3 + 16x^4y^4\end{aligned}$$

d. $(x - y)^4$

$$\begin{aligned}(x - y)^4 &= (x + (-y))^4 \\ &= x^4 + C(4, 1)x^3(-y) + C(4, 2)x^2(-y)^2 + C(4, 3)x(-y)^3 + (-y)^4 \\ &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4\end{aligned}$$

e. $(x - 2xy)^4$

$$\begin{aligned}(x - 2xy)^4 &= x^4 + C(4,1)x^3(-2xy) + C(4,2)x^2(-2xy)^2 + C(4,3)x(-2xy)^3 + (-2xy)^4 \\ &= x^4 + 4x^3(-2xy) + 6x^2(4x^2y^2) + 4x(-8x^3y^3) + 16x^4y^4 \\ &= x^4 - 8x^4y + 24x^4y^2 - 32x^4y^3 + 16x^4y^4\end{aligned}$$

3. Use the binomial theorem to expand the following binomial expressions.

a. $(1 + \sqrt{2})^5$

$$\begin{aligned}(1 + \sqrt{2})^5 &= 1^5 + C(5,1)1^4\sqrt{2} + C(5,2)1^3(\sqrt{2})^2 + C(5,3)1^2(\sqrt{2})^3 + C(5,4)1 \cdot (\sqrt{2})^4 + (\sqrt{2})^5 \\ &= 1 + 5\sqrt{2} + 10 \cdot 2 + 10 \cdot 2\sqrt{2} + 5 \cdot 4 + 4\sqrt{2} \\ &= 41 + 29\sqrt{2}\end{aligned}$$

b. $(1 + i)^9$

$$\begin{aligned}(1 + i)^9 &= 1 + 9i + 36i^2 + 84i^3 + 126i^4 + 126i^5 + 84i^6 + 36i^7 + 9i^8 + i^9 \\ &= 1 + 9i - 36 - 84i + 126 + 126i - 84 - 36i + 9 + i \\ &= 16 + 16i\end{aligned}$$

c. $(1 - \pi)^5$ (Hint: $1 - \pi = 1 + (-\pi)$.)

$$\begin{aligned}(1 - \pi)^5 &= 1 + C(5,1)(-\pi) + C(5,2)(-\pi)^2 + C(5,3)(-\pi)^3 + C(5,4)(-\pi)^4 + (-\pi)^5 \\ &= 1 - 5\pi + 10\pi^2 - 10\pi^3 + 5\pi^4 - \pi^5\end{aligned}$$

d. $(\sqrt{2} + i)^6$

$$\begin{aligned}(\sqrt{2} + i)^6 &= (\sqrt{2})^6 + C(6,1)(\sqrt{2})^5i + C(6,2)(\sqrt{2})^4i^2 + C(6,3)(\sqrt{2})^3i^3 + C(6,4)(\sqrt{2})^2i^4 + C(6,5)\sqrt{2}i^5 \\ &\quad + i^6 \\ &= 8 + 6 \cdot 4\sqrt{2}i + 15 \cdot 4(-1) + 20 \cdot 2\sqrt{2}(-i) + 15 \cdot 2 \cdot 1 + 6\sqrt{2}(i) + (-1) \\ &= -23 - 10\sqrt{2}i\end{aligned}$$

e. $(2 - i)^6$

$$\begin{aligned}(2 - i)^6 &= (2 + (-i))^6 \\ &= 2^6 + C(6,1)2^5(-i) + C(6,2)2^4(-i)^2 + C(6,3)2^3(-i)^3 + C(6,4)2^2(-i)^4 + C(6,5)2(-i)^5 \\ &\quad + (-i)^6 \\ &= 64 - 6 \cdot 32i + 15 \cdot 16(-1) + 20 \cdot 8(-i) + 15 \cdot 4 \cdot 1 + 6 \cdot 2i + 1 \\ &= -115 - 340i\end{aligned}$$

4. Consider the expansion of $(a + b)^{12}$. Determine the coefficients for the terms with the powers of a and b shown.

a. a^2b^{10}

$$(a + b)^{12} = a^{12} + C(12,1)a^{11}b + \cdots + C(12,10)a^2b^{10} + C(12,11)ab^{11} + b^{12}$$

$$\text{So, the coefficient of } a^2b^{10} \text{ is } C(12,10) = \frac{12!}{2!10!} = \frac{12 \cdot 11}{2 \cdot 1} = 66.$$

b. a^5b^7

The coefficient of a^5b^7 is $C(12, 7) = \frac{12!}{5!7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792$.

c. a^8b^4

The coefficient of a^8b^4 is $C(12, 4) = \frac{12!}{8!4!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495$.

5. Consider the expansion of $(x + 2y)^{10}$. Determine the coefficients for the terms with the powers of x and y shown.

a. x^2y^8

The x^2y^8 term is $C(10, 8)x^2(2y)^8 = 45x^2 \cdot 256y^8 = 11520 x^2y^8$, so the coefficient of x^2y^8 is 11,520.

b. x^4y^6

The x^4y^6 term is $C(10, 6)x^4(2y)^6 = 210x^4 \cdot 64y^6 = 13440 x^4y^6$, so the coefficient of x^4y^6 is 13,440.

c. x^5y^5

The x^5y^5 term is $C(10, 5)x^5(2y)^5 = 252x^5 \cdot 32y^5 = 8064 x^5y^5$, so the coefficient of x^5y^5 is 8,064.

6. Consider the expansion of $(5p + 2q)^6$. Determine the coefficients for the terms with the powers of p and q shown.

a. p^2q^4

Since $C(6, 2) = 15$ and $15(5p)^2(2q)^4 = 15(25p^2)(16q^4) = 6000p^2q^4$, the coefficient is 6,000.

b. p^5q

Since $C(6, 5) = 6$ and $6(5p)^5(2q) = 6(3125p^5)(2q) = 37500p^5q$, the coefficient is 37,500.

c. p^3q^3

Since $C(6, 3) = 20$ and $20(5p)^3(2q)^3 = 20(125p^3)(8q^3) = 20000p^3q^3$, the coefficient is 20,000.

7. Explain why the coefficient of the term that contains u^n is 1 in the expansion of $(u + v)^n$.

The corresponding binomial coefficient is $C(n, 0) = \frac{n!}{0!n!} = \frac{1}{0!} = 1$.

8. Explain why the coefficient of the term that contains $u^{n-1}v$ is n in the expansion of $(u + v)^n$.

The corresponding binomial coefficient is $C(n, 1) = \frac{n!}{1!(n-1)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{(1)((n-1) \cdot (n-2) \cdots 2 \cdot 1)} = n$.

9. Explain why the rows of Pascal's triangle are symmetric. That is, explain why $C(n, k) = C(n, (n - k))$.

Using the formula for the binomial coefficients, $C(n, k) = \frac{n!}{k!(n-k)!}$ and $C(n, n - k) = \frac{n!}{(n-k)!(n-(n-k))!}$
so $C(n, n - k) = \frac{n!}{(n-k)!n!} = C(n, k)$.