Lesson 2: Does Every Complex Number Have a Square Root?

Student Outcomes

- Students apply their understanding of polynomial identities that have been extended to the complex numbers to find the square roots of complex numbers.

Lesson Notes

In Precalculus Module 1, students used the polar form of a complex number to find powers and roots of complex numbers. However, nearly all of the examples used in those lessons involved complex numbers with arguments that were multiples of special angles ($\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$). In this lesson, we return to the rectangular form of a complex number to show algebraically that we can find the square roots of any complex number without having to express it first in polar form. Students will use the properties of complex numbers and the fundamental theorem of algebra to find the square roots of any complex number by creating and solving polynomial equations (**N-NC.C.8** and **N-NC.C.9**). Students see that while solving these equations, we arrive at polynomial identities with two factors that guarantee two roots to the equation. Throughout the lesson, students will use algebraic properties to justify their reasoning (MP.3) and examine the structure of expressions to support their solutions and make generalizations (MP.7 and MP.8).

Classwork

Opening (5 minutes)

Organize your class into groups of 3–5 students. Start by displaying the question shown below. Give students time to consider an answer to this question on their own, and then allow them to discuss it in small groups. Have a representative from each group briefly summarize their small group discussions.

- Does every complex number have a square root? If yes, provide at least two examples. If no, explain why not.
  - *If we think about transformations, then squaring a complex number dilates a complex number by the modulus and rotates it by the argument. Thus, taking the square root of a complex number should divide the argument by 2 and have a modulus equal to the square root of the original modulus.*

    For example, the complex number i would have a square root with modulus equal to 1 and argument equal to $45°$, which is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Scaffolding:

- Create an anchor chart with the following formulas:

  For any complex number z,

  $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

  The nth roots of $z = re^{i\theta}$ are given by

  $\sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)\right)$

  for integers k and n such that $n > 0$ and $0 \le k < n$.
- Ask students to explain the formulas above for given values of n, r, and θ. For example, substitute $\frac{\pi}{2}$ for θ, 1 for r, and 2 for n into the formula for the nth roots of z, and ask students what the expressions represent. (The square root of i.)

Student discussions and responses will reveal rich information about what students do and do not remember from their work in Module 1 Lessons 18 and 19. In these lessons, students used the polar form of a complex number to find the n^{th} roots of a complex number in polar form. Based on student responses to this question, you may need to briefly review the polar form of a complex number and the formulas developed in Lessons 18 and 19 for finding powers of a complex number z and roots of a complex number z.

Exercises 1–6 (10 minutes)

Students should work these exercises with their group members. As students are working, be sure to circulate around the classroom monitoring progress of the groups. Pause as needed for whole group discussion and debriefing, especially if a majority of the groups are struggling to make progress.

Exercises 1–6

1. **Use the geometric effect of complex multiplication to describe how to calculate a square root of $z = 119 + 120i$.**

   The square root of this number would have a modulus equal to the square root of $|119 + 120i|$ and an argument equal to $\frac{\arg(119+120i)}{2}$.

2. **Calculate an estimate of a square root of $119 + 120i$.**

$$|z| = \sqrt{119^2 + 120^2} = 169$$
$$\arg(z) = \arctan\left(\frac{120}{119}\right)$$

   The square root's modulus is $\sqrt{169} = 13$, and the square root's argument Is $\frac{1}{2}\arctan\left(\frac{120}{119}\right)$. The square root is close to

$$13(\cos(22.6°) + i\sin(22.6°)) = 12 + 4.99i.$$

If students get stuck on the next exercises, lead a short discussion to help set the stage for establishing that every complex number will have two square roots that are opposites of one another.

- What are the square roots of 4?
  - *The square roots of 4 are 2 and -2 because $(2)^2 = 4$ and $(-2)^2 = 4$.*
- What are the square roots of 5?
  - *The square roots of 5 are $\sqrt{5}$ and $-\sqrt{5}$ because $\left(\sqrt{5}\right)^2 = 5$ and $\left(-\sqrt{5}\right)^5 = 5$.*

MP.3

3. **Every real number has two square roots. Explain why.**

   The fundamental theorem of algebra guarantees that a second degree polynomial equation has two solutions. To find the square roots of a real number b, we need to solve the equation $z^2 = b$, which is a second degree polynomial equation. Thus, the two solutions of $z^2 = b$ are the two square roots of b. If a is one of the square roots, then $-a$ is the other.

4. **Provide a convincing argument that every complex number must also have two square roots.**

   By the same reasoning, if w is a complex number, then the polynomial equation $z^2 = w$ has two solutions. The two solutions to this quadratic equation are the square roots of w. If $a + bi$ is one square root, then $a - bi$ is the other.

MP.3

5. Explain how the polynomial identity $x^2 - b = (x - \sqrt{b})(x + \sqrt{b})$ relates to the argument that every number has two square roots.

   To solve $x^2 = b$, we can solve $x^2 - b = 0$. Since this quadratic equation has two distinct solutions, we can find two square roots of b. The two square roots are opposites of each other.

6. What is the other square root of $119 + 120i$?

   It would be the opposite of $12 + 5i$, which is the complex number, $-12 - 5i$.

Example 1: Find the Square Roots of $119 + 120i$ (10 minutes)

The problem with using the polar form of a complex number to find its square roots is that the argument of these numbers is not an easily recognizable number unless we pick our values of a and b very carefully, such as $1 + \sqrt{3}i$.

Recall from the last lesson that we proved using complex numbers that the equation $x^2 = 1$ had exactly two solutions. Here is another approach to finding both square roots of a complex number that involves creating and solving a system of equations. The solutions to these equations will provide a way to define the square roots of a complex number. Students will have to solve a fourth degree polynomial equation that has both real and imaginary solutions by factoring using polynomial identities.

Scaffolding:

- Students may benefit from practice and review of factoring fourth degree polynomials. See Algebra II Module 1. Some sample problems are provided below.

  $$x^4 - 2x^2 - 8$$
  $$x^4 - 9x^2 - 112$$
  $$x^4 - x^2 - 12$$

- Post the following identities on the board:
  $a^2 - b^2 = (a + b)(a - b)$
  $a^2 + b^2 = (a + bi)(a - bi)$
- Students could use technology like Desmos to quickly find real number solutions to equations like those in part (b) by graphing each side and finding the x-coordinates of the intersection points.

Example 1: Find the Square Roots of $119 + 120i$

Let $w = p + qi$ be the square root of $119 + 120i$. Then

$$w^2 = 119 + 120i$$

and

$$(p + qi)^2 = 119 + 120i.$$

a. Expand the left side of this equation.

   $p^2 - q^2 + 2pqi = 119 + 120i$

b. Equate the real and imaginary parts, and solve for p and q.

   $p^2 - q^2 = 119$ and $2pq = 120$. Solving for q and substituting gives

   $$p^2 - \left(\frac{60}{p}\right)^2 = 119.$$

   Multiplying by p^2 gives the equation

   $$p^4 - 119p^2 - 3600 = 0.$$

   And this equation can be solved by factoring.

   $$(p^2 + 25)(p^2 - 144) = 0$$

   We now have two polynomial expressions that we know how to factor: the sum and difference of squares.

   $$(p + 5i)(p - 5i)(p - 12)(p + 12) = 0$$

   The solutions are $5i$, $-5i$, 12, and -12. Since p must be a real number by the definition of complex number, we can choose 12 or -12 for p. Using the equation $2pq = 120$, when $p = 12$, $q = 5$, and when $p = -12$, $q = -5$.

c. **What are the square roots of $119 + 120i$?**

Thus, the square roots of $119 + 120i$ are $12 + 5i$ and $-12 - 5i$.

Debrief this example by discussing how this process could be generalized for any complex number $z = a + bi$. Be sure to specifically mention that the polynomial identities (sum and difference of squares) each have two factors, so we get two solutions when setting those factors equal to zero.

- If we solved this problem again using different values of a and b instead of 119 and 120, would we still get exactly two square roots of the form $w = p + qi$ and $w = -p - qi$ that satisfy $w^2 = a + bi$? Explain your reasoning.
  - *Yes. When calculating w^2, we would get a fourth degree equation that would factor into two second degree polynomial factors: one that is a difference of squares and one that is a sum of squares. The difference of squares will have two linear factors with real coefficients giving us two values of p that can be used to find the two square roots $p + qi$.*

Exercises 7–9 (12 minutes)

Students can work these exercises in groups. Be sure to have at least one group present their solution to Exercise 8. Before Exercise 7, use the discussion questions that follow if needed to activate students' prior knowledge about complex conjugates.

Exercises 7–9

7. **Use the method in Example 1 to find the square roots of $1 + \sqrt{3}i$.**

   $p^2 - q^2 = 1$ *and* $2pq = \sqrt{3}$

   Substituting and solving for p,

$$\begin{aligned} p^2 - \left(\frac{\sqrt{3}}{2p}\right)^2 &= 1 \\ 4p^4 - 3 &= 4p^2 \\ 4p^4 - 4p^2 - 3 &= 0 \\ (2p^2 - 3)(2p^2 + 1) &= 0 \end{aligned}$$

   gives the real solutions $p = \sqrt{\frac{3}{2}}$ or $p = -\sqrt{\frac{3}{2}}$. The values of q would then be

   $q = \dfrac{\sqrt{3}}{2\sqrt{\frac{3}{2}}} = \dfrac{\sqrt{2}}{2}$ *and* $q = -\dfrac{\sqrt{2}}{2}$.

   The square roots of $1 + \sqrt{3}i$ are $\dfrac{\sqrt{6}}{2} + \dfrac{\sqrt{2}}{2}i$ and $-\dfrac{\sqrt{6}}{2} - \dfrac{\sqrt{2}}{2}i$.

8. Find the square roots of each complex number.

a. $5 + 12i$

The square roots of $5 + 12i$ *will satisfy the equation* $(p + qi)^2 = 5 + 12i$.

Expanding $(p + qi)^2$ *and equating the real and imaginary parts gives*

$$p^2 - q^2 = 5$$
$$2pq = 12.$$

Substituting $q = \frac{6}{p}$ *into* $p^2 - q^2 = 5$ *gives*

$$p^2 - \left(\frac{6}{p}\right)^2 = 5$$
$$p^4 - 36 = 5p^2$$
$$p^4 - 5p^2 - 36 = 0$$
$$(p^2 - 9)(p^2 + 4) = 0.$$

The one positive real solution to this equation is 3. *Let* $p = 3$. *Then* $q = 2$. *A square root of* $5 + 12i$ *is* $3 + 2i$.

The other square root is when $p = -3$ *and* $q = -2$. *Therefore,* $-3 - 2i$ *is the other square root of* $5 + 12i$.

b. $5 - 12i$

The square roots of $5 - 12i$ *will satisfy the equation* $(p + qi)^2 = 5 - 12i$.

Expanding $(p + qi)^2$ *and equating the real and imaginary parts gives*

$$p^2 - q^2 = 5$$
$$2pq = -12.$$

Substituting $q = -\frac{6}{p}$ *into* $p^2 - q^2 = 5$ *gives*

$$p^2 - \left(-\frac{6}{p}\right)^2 = 5$$
$$p^4 - 36 = 5p^2$$
$$p^4 - 5p^2 - 36 = 0$$
$$(p^2 - 9)(p^2 + 4) = 0.$$

The one positive real solution to this equation is 3. *Let* $p = 3$. *Then* $q = -2$. *A square root of* $5 - 12i$ *is* $3 - 2i$.

The other square root is when $p = -3$ *and* $q = 2$. *Therefore,* $-3 + 2i$ *is the other square root of* $5 - 12i$.

- What do we call complex numbers of the form $a + bi$ and $a - bi$?
  - *They are called complex conjugates.*
- Based on Exercise 6, how are square roots of conjugates related? Why do you think this relationship exists?
  - *It appears that the square roots are conjugates as well. When we solved the equation, the only difference was that* $2pq = -12$, *which ended up not mattering when we squared* $-\frac{6}{p}$.
- What is the conjugate of $119 + 120i$?
  - *The conjugate is* $119 - 120i$.

- What do you think the square roots of $119 - 120i$ would be? Explain your reasoning.
  - *Since we only reflected this number across the real axis, the square roots should also simply be a reflection across the real axis; thus, the square roots should be $12 - 5i$ and $-12 + 5i$.*

9. Show that if $p + qi$ is a square root of $z = a + bi$, then $p - qi$ is a square root of the conjugate of z, $\bar{z} = a - bi$.

a. Explain why $(p + qi)^2 = a + bi$.

If $p + qi$ is a square root of $a + bi$, then it must satisfy the definition of a square root. The square root of a number raised to the second power should equal the number.

b. What do a and b equal in terms of p and q?

Expanding $(p + qi)^2 = p^2 - q^2 + 2pqi$. Thus, $a = p^2 - q^2$ and $b = 2pq$.

c. Calculate $(p - qi)^2$. What is the real part, and what is the imaginary part?

$(p - qi)^2 = p^2 - q^2 - 2pqi$ The real part is $p^2 - q^2$, and the imaginary part is $-2pq$.

d. Explain why $(p - qi)^2 = a - bi$.

From part (c), $p^2 - q^2 = a$ and $2pq = b$. Substituting,

$(p - qi)^2 = p^2 - q^2 - 2pqi = a - bi$.

Closing (3 minutes)

Students can respond to the following questions either in writing or by discussing them with a partner.

- How are square roots of complex numbers found by solving a polynomial equation?
  - *If $p + qi$ is a square root of a complex number $a + bi$, then $(p + qi)^2 = a + bi$. Expanding the expression on the left and equating the real and imaginary parts leads to equations $p^2 - q^2 = a$ and $2pq = b$. Then, solve this resulting system of equations for p and q.*
- Explain why the fundamental theorem of algebra guarantees that every complex number has two square roots.
  - *According to the fundamental theorem of algebra, every second degree polynomial equation factors in to linear terms. The square roots of a complex number w are solutions to the equation $z^2 = w$, which can be rewritten as $z^2 - w = 0$. This equation will have two solutions, each of which is a square root of w.*

Lesson Summary

The square roots of a complex number $a + bi$ will be of the form $p + qi$ and $-p - qi$ and can be found by solving the equations $p^2 - q^2 = a$ and $2pq = b$.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 2: Does Every Complex Number Have a Square Root?

Exit Ticket

1. Find the two square roots of $5 - 12i$.

2. Find the two square roots of $3 - 4i$.

Exit Ticket Sample Solutions

1. **Find the two square roots of $5 - 12i$.**

   $(p + qi)^2 = 5 - 12i, p^2 - q^2 + 2pqi = 5 - 12i,$

   $p^2 - q^2 = 5,\ pq = -6, q = -\frac{6}{p}$

   $p^2 - \frac{36}{p^2} - 5 = 0, p^4 - 5p^2 - 36 = 0, (p^2 + 4)(p^2 - 9) = 0, p = \pm 3,$

   $p = 3, q = -2; p = -3,\ q = 2$

   Therefore, the square roots are $3 - 2i$ and $-3 + 2i$.

2. **Find the two square roots of $3 - 4i$.**

   Let the square roots have the form $p + qi$. Then $(p + qi)^2 = 3 - 4i$. This gives $p^2 - q^2 = 3$ and $2pq = -4$. Then $p = -\frac{2}{q}$, so $p^2 - q^2 = \frac{4}{q^2} - q^2 = 3$, and $q^4 + 3q^2 - 4 = 0$. This expression factors into $(q + 1)(q - 1)(q + 2i)(q - 2i) = 0$, and we see that the only real solutions are $q = 1$ and $q = -1$. If $q = 1$, then $p = -2$, and if $q = -1$, then $p = 2$. Thus, the two square roots of $3 - 4i$ are $-2 + i$ and $2 - i$.

Problem Set Sample Solutions

Find the two square roots of each complex number by creating and solving polynomial equations.

1. $z = 15 - 8i$

   $(p + qi)^2 = 15 - 8i, p^2 - q^2 + 2pqi = 15 - 8i,$

   $p^2 - q^2 = 15, pq = -4, q = -\frac{4}{p}$

   $p^2 - \frac{16}{p^2} - 15 = 0,\ p^4 - 15p^2 - 16 = 0, (p^2 + 1)(p^2 - 16) = 0, p = \pm 4,$

   $p = 4,\ q = -1; p = -4, q = 1$

   Therefore, the square roots of $15 - 8i$ are $4 - i$ and $-4 + i$.

2. $z = 8 - 6i$

   $(p + qi)^2 = 8 - 6i, p^2 - q^2 + 2pqi = 8 - 6i,$

   $p^2 - q^2 = 8, pq = -3, q = -\frac{3}{p}$

   $p^2 - \frac{9}{p^2} - 8 = 0, p^4 - 8p^2 - 9 = 0, (p^2 + 1)(p^2 - 9) = 0, p = \pm 3,$

   $p = 3, q = -1; p = -3, q = 1$

   Therefore, the square roots of $8 - 6i$ are $3 - i$ and $-3 + i$.

3. $z = -3 + 4i$

$(p + qi)^2 = -3 + 4i, p^2 - q^2 + 2pqi = -3 + 4i,$

$p^2 - q^2 = -3, pq = 2, q = \frac{2}{p}$

$p^2 - \frac{4}{p^2} + 3 = 0, p^4 + 3p^2 - 4 = 0, (p^2 + 4)(p^2 - 1) = 0, p = \pm 1,$

$p = 1, q = 2; p = -1, q = -2$

Therefore, the square roots of $-3 + 4i$ *are* $1 + 2i$ *and* $-1 - 2i$.

4. $z = -5 - 12i$

$(p + qi)^2 = -5 - 12i, p^2 - q^2 + 2pqi = -5 - 12i,$

$p^2 - q^2 = -5, pq = -6, q = \frac{-6}{p}$

$p^2 - \frac{6}{p^2} + 5 = 0, p^4 + 5p^2 - 36 = 0, (p^2 + 9)(p^2 - 4) = 0, p = \pm 2,$

$p = 2, q = -3; p = -2, q = 3$

Therefore, the square roots of $-5 - 12i$ *are* $2 - 3i$ *and* $-2 + 3i$.

5. $z = 21 - 20i$

$(p + qi)^2 = 21 - 20i, p^2 - q^2 + 2pqi = 21 - 20i,$

$p^2 - q^2 = 21, pq = -10, q = -\frac{10}{p}$

$p^2 - \frac{100}{p^2} - 21 = 0, p^4 - 21p^2 - 100 = 0, (p^2 + 4)(p^2 - 25) = 0, p = \pm 5,$

$p = 5, q = -2; p = -5, q = 2$

Therefore, the square roots of $21 - 20i$ *are* $5 - 2i$ *and* $-5 + 2i$.

6. $z = 16 - 30i$

$(p + qi)^2 = 16 - 30i, p^2 - q^2 + 2pqi = 16 - 30i,$

$p^2 - q^2 = 16, pq = -15, q = -\frac{15}{p}$

$p^2 - \frac{225}{p^2} - 16 = 0, p^4 - 16p^2 - 225 = 0, (p^2 + 9)(p^2 - 25) = 0, p = \pm 5,$

$p = 5, q = -3; p = -5, q = 3$

Therefore, the square roots of $16 - 30i$ *are* $5 - 3i$ *and* $-5 + 3i$.

7. $z = i$

$(p+qi)^2 = 0 + i, p^2 - q^2 + 2pqi = 0 + i,$

$p^2 - q^2 = 0, pq = \frac{1}{2}, q = \frac{1}{2p}$

$p^2 - \frac{1}{4p^2} = 0,\ 4p^4 - 1 = 0, (2p^2+1)(2p^2-1) = 0, p = \pm\frac{\sqrt{2}}{2}$

$p = \frac{\sqrt{2}}{2}, q = \frac{\sqrt{2}}{2}; p = -\frac{\sqrt{2}}{2}, q = -\frac{\sqrt{2}}{2}$

Therefore, the square roots of i are $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ *and* $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

A *Pythagorean triple* is a set of three positive integers a, b, and c such that $a^2 + b^2 = c^2$. Thus, these integers can be the lengths of the sides of a right triangle.

8. Show algebraically that for positive integers p and q, if

$$a = p^2 - q^2$$
$$b = 2pq$$
$$c = p^2 + q^2$$

then $a^2 + b^2 = c^2$,

$$\begin{aligned} a^2 + b^2 &= (p^2 - q^2)^2 + (2pq)^2 \\ &= p^4 - 2p^2q^2 + q^4 + 4p^2q^2 \\ &= p^4 + 2p^2q^2 + q^4 \\ &= p^4 + 2p^2q^2 + q^4 \\ &= c^2. \end{aligned}$$

9. Select two integers p and q, use the formulas in Problem 8 to find a, b, and c, and then show those numbers satisfy the equation $a^2 + b^2 = c^2$.

Let $p = 3$ and $q = 2$. Calculate the values of a, b, and c.

$$a = 3^2 - 2^2 = 5$$
$$b = 2(3)(2) = 12$$
$$c = 3^2 + 2^2 = 13$$
$$a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169 = 13^2 = c^2$$

10. Use the formulas from Problem 8, and find values for p and q that give the following famous triples.

   a. $(3, 4, 5)$

   $a = p^2 - q^2 = 3, b = 2pq = 4, c = p^2 + q^2 = 5$

   $2p^2 = 8, p = 2, q = 1$

   $(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

   $(2^2 - 1^2)^2 + (2(2)(1))^2 = (2^2 + 1^2)^2$

   $(4-1)^2 + 4^2 = (4+1)^2, (3)^2 + (4)^2 = (5)^2$

b. $(5, 12, 13)$

$a = p^2 - q^2 = 5, b = 2pq = 12, c = p^2 + q^2 = 13$

$2p^2 = 18,\ p = 3, q = 2$

$(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

$(3^2 - 2^2)^2 + (2(3)(2))^2 = (3^2 + 2^2)^2$

$(9 - 4)^2 + 12^2 = (9 + 4)^2, (5)^2 + (12)^2 = (13)^2$

c. $(7, 24, 25)$

$a = p^2 - q^2 = 7, b = 2pq = 24, c = p^2 + q^2 = 25,$

$2p^2 = 32, p = 4, q = 3$

$(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

$(4^2 - 3^2)^2 + (2(4)(3))^2 = (4^2 + 3^2)^2$

$(16 - 9)^2 + 24^2 = (16 + 9)^2, (7)^2 + (24)^2 = (25)^2$

d. $(9, 40, 41)$

$a = p^2 - q^2 = 9, b = 2pq = 40, c = p^2 + q^2 = 41,$

$2p^2 = 50, p = 5, q = 4$

$(p^2 - q^2)^2 + (2pq)^2 = (p^2 - q^2)^2$

$(5^2 - 4^2)^2 + (2(5)(4))^2 = (5^2 + 4^2)^2$

$(25 - 16)^2 + 40^2 = (25 + 16)^2, (9)^2 + (40)^2 = (41)^2$

11. Is it possible to write the Pythagorean triple $(6, 8, 10)$ in the form $a = p^2 - q^2, b = 2pq, c = p^2 + q^2$ for some integers p and q? Verify your answer.

$a = p^2 - q^2 = 6, b = 2pq = 8, c = p^2 + q^2 = 10,$

$2p^2 = 16, p^2 = 8, p = \pm 2\sqrt{2}, q = \pm\sqrt{2}$

The Pythagorean triple $(6, 8, 10)$ cannot be written in the form $a = p^2 - q^2$, $b = 2pq$, $c = p^2 + q^2$, for any integers p and q.

12. Choose your favorite Pythagorean triple (a, b, c) that has a and b sharing only 1 as a common factor, for example $(3, 4, 5)$, $(5, 12, 13)$, or $(7, 24, 25)$,… Find the square of the length of a square root of $a + bi$; that is, find $|p + qi|^2$, where $p + qi$ is a square root of $a + bi$. What do you observe?

For $(3, 4, 5)$, $a = 3, b = 4, a + bi = (p + qi)^2 = (-p - qi)^2$,

$a + bi = 3 + 4i = (p + qi)^2 = p^2 - q^2 + 2pqi$; *therefore,* $p^2 - q^2 = 3, 2pq = 4, q = \frac{2}{p}$

$p^2 - \frac{4}{p^2} = 3, p^4 - 3p^2 - 4 = 0, (p^2 + 1)(p^2 - 4) = 0, p = \pm 2$

$p = 2, q = 1$; *therefore,* $p + qi = 2 + i$.

$p = -2, q = -1$; *therefore,* $p + qi = -2 - i$.

The two square roots of $a + bi$ are $2 + i$ and $-2 - i$. Both of these have length $\sqrt{5}$, so the squared length is 5. This is the third value c in the Pythagorean triple (a, b, c).