Lesson 2: Does Every Complex Number Have a Square Root?

Classwork

Exercises 1–6

1. Use the geometric effect of complex multiplication to describe how to calculate a square root of $z = 119 + 120i$.

2. Calculate an estimate of a square root of $119 + 120i$.

3. Every real number has two square roots. Explain why.

4. Provide a convincing argument that every complex number must also have two square roots.

5. Explain how the polynomial identity $x^2 - b = (x - \sqrt{b})(x + \sqrt{b})$ relates to the argument that every number has two square roots.

6. What is the other square root of $119 + 120i$?

Example 1: Find the Square Roots of $119 + 120i$

Find the square roots of $119 + 120i$ algebraically.

Let $w = p + qi$ be the square root of $119 + 120i$. Then

$$w^2 = 119 + 120i$$

and

$$(p + qi)^2 = 119 + 120i.$$

a. Expand the left side of this equation.

b. Equate the real and imaginary parts, and solve for p and q.

c. What are the square roots of $119 + 120i$?

Exercises 7–9

7. Use the method in Example 1 to find the square roots of $1 + \sqrt{3}i$.

8. Find the square roots of each complex number.

   a. $5 + 12i$

   b. $5 - 12i$

9. Show that if $p + qi$ is a square root of $z = a + bi$, then $p - qi$ is a square root of the conjugate of z, $\bar{z} = a - bi$.

   a. Explain why $(p + qi)^2 = a + bi$.

   b. What do a and b equal in terms of p and q?

   c. Calculate $(p - qi)^2$. What is the real part, and what is the imaginary part?

   d. Explain why $(p - qi)^2 = a - bi$.

Lesson Summary

The square roots of a complex number $a + bi$ will be of the form $p + qi$ and $-p - qi$ and can be found by solving the equations $p^2 - q^2 = a$ and $2pq = b$.

Problem Set

Find the two square roots of each complex number by creating and solving polynomial equations.

1. $z = 15 - 8i$

2. $z = 8 - 6i$

3. $z = -3 + 4i$

4. $z = -5 - 12i$

5. $z = 21 - 20i$

6. $z = 16 - 30i$

7. $z = i$

A *Pythagorean triple* is a set of three positive integers a, b, and c such that $a^2 + b^2 = c^2$. Thus, these integers can be the lengths of the sides of a right triangle.

8. Show algebraically that for positive integers p and q, if

$$a = p^2 - q^2$$
$$b = 2pq$$
$$c = p^2 + q^2$$

then $a^2 + b^2 = c^2$,

9. Select two integers p and q, use the formulas in Problem 8 to find a, b, and c, and then show those numbers satisfy the equation $a^2 + b^2 = c^2$.

10. Use the formulas from Problem 8, and find values for p and q that give the following famous triples.
    a. $(3,4,5)$
    b. $(5,12,13)$
    c. $(7,24,25)$
    d. $(9,40,41)$

11. Is it possible to write the Pythagorean triple $(6,8,10)$ in the form $a = p^2 - q^2$, $b = 2pq$, $c = p^2 + q^2$ for some integers p and q? Verify your answer.

12. Choose your favorite Pythagorean triple (a, b, c) that has a and b sharing only 1 as a common factor, for example $(3,4,5)$, $(5,12,13)$, or $(7,24,25)$,… Find the square of the length of a square root of $a + bi$; that is, find $|p + qi|^2$, where $p + qi$ is a square root of $a + bi$. What do you observe?