Lesson 1: Solutions to Polynomial Equations

Student Outcomes

- Students determine all solutions of polynomial equations over the set of complex numbers and understand the implications of the fundamental theorem of algebra, especially for quadratic polynomials.

Lesson Notes

Students studied polynomial equations and the nature of the solutions of these equations extensively in Algebra II Module 1, extending factoring to the complex realm. The fundamental theorem of algebra indicates that any polynomial function of degree n will have n zeros (including repeated zeros). Establishing the fundamental theorem of algebra was one of the greatest achievements of nineteenth-century mathematics. It is worth spending time further exploring it now that students have a much broader understanding of complex numbers. This lesson reviews what they learned in previous grades and provides additional support for their understanding of what it means to solve polynomial equations over the set of complex numbers. Students work with equations with complex number solutions and apply identities such as $a^2 + b^2 = (a + bi)(a - bi)$ for real numbers a and b to solve equations and explore the implications of the fundamental theorem of algebra (**N-CN.C.8** and **N-CN.C.9**). Throughout the lesson, students will vary their reasoning by applying algebraic properties (MP.3) and examining the structure of expressions to support their solutions and make generalizations (MP.7 and MP.8). Relevant definitions introduced in Algebra II are provided in the student materials for this lesson.

A note on terminology: Equations have solutions, and functions have zeros. The distinction is subtle but important. For example, the equation $(x - 1)(x - 3) = 0$ has solutions 1 and 3, while the polynomial function $p(x) = (x - 1)(x - 3)$ has zeros at 1 and 3. Zeros of a function are the x-intercepts of the graph of the function; they are also known as *roots*.

Classwork

Opening Exercise (3 minutes)

Use this opening to activate prior knowledge about polynomial equations. Students may need to be reminded of the definition of a polynomial from previous grades. Have students work this exercise independently and then quickly share their answers with a partner. Lead a short discussion using the questions below.

Scaffolding:

- Remind students of the definition of a polynomial equation by using a Frayer model. For an example, see Module 1 Lesson 5.
- Be sure to include examples ($2x + 3 = 0$, $x^2 - 4 = 0$, $(x - 3)(x + 2) = 2x - 5$, $x^5 = 1$, etc.) and non-examples ($\sin(2x) - 1 = 0$, $\frac{1}{x} = 5$, $2^{3x} - 1 = 5$, etc.).

Opening Exercise

How many solutions are there to the equation $x^2 = 1$? Explain how you know.

There are two solutions to the equation: 1 and -1. I know these are the solutions because they make the equation true when each value is substituted for x.

- How do you know that there aren't any more real number solutions?
  - *If you sketch the graph of $f(x) = x^2$ and the graph of the line $y = 1$, they intersect in exactly two points. The x-coordinates of the intersection points are the solutions to the equation.*
- How can you show algebraically that this equation has just two solutions?
  - *Rewrite the equation $x^2 = 1$, and solve it by factoring. Then, apply the zero product property.*

$$x^2 - 1 = 0 \text{ or}$$
$$(x-1)(x+1) = 0$$
$$x - 1 = 0 \text{ or } x + 1 = 0$$
$$x = 1 \text{ or } x = -1$$

  - *So, the solutions are 1 and -1.*
- You just found and justified why this equation has only two real number solutions. How do we know that there aren't any complex number solutions to $x^2 = 1$?
  - *We would have to show that a second degree polynomial equation has exactly two solutions over the set of complex numbers. (Another acceptable answer would be the fundamental theorem of algebra, which states that a second degree polynomial equation has at most two solutions. Since we have found two solutions that are real, we know we have found all possible solutions.)*
- Why do the graphical approach and the algebraic approach not clearly provide an answer to the previous question?
  - *The graphical approach assumed you were working with real numbers. The algebraic approach does not clearly eliminate the possibility of complex number solutions. It only shows two real number solutions.*

Example 1 (5 minutes): Prove That A Quadratic Equation Has Only Two Solutions Over The Set Of Complex Numbers

This example illustrates an approach to showing that 1 and -1 are the only real or complex solutions to the quadratic equation $x^2 = 1$. Students may not have seen this approach before. However, they should be very familiar with operations with complex numbers after their work in Algebra II Module 1 and Module 1 of this course. We will be working with solutions to $x^n = 1$ in later lessons, and this approach will be most helpful.

MP.3

Example 1: Prove That A Quadratic Equation Has Only Two Solutions Over The Set Of Complex Numbers.

Prove that 1 and -1 are the only solutions to the equation $x^2 = 1$.

Let $x = a + bi$ be a complex number so that $x^2 = 1$.

a. Substitute $a + bi$ for x in the equation $x^2 = 1$.

$$(a + bi)^2 = 1$$

b. Rewrite both sides in standard form for a complex number.

$$a^2 + 2abi + b^2i^2 = 1$$
$$(a^2 - b^2) + 2abi = 1 + 0i$$

> *Scaffolding:*
> Ask advanced learners to develop the proof in Example 1 on their own without the leading questions.

MP.3

c. **Equate the real parts on each side of the equation and equate the imaginary parts on each side of the equation.**

$$a^2 - b^2 = 1 \text{ and } 2ab = 0$$

d. **Solve for a and b, and find the solutions for $x = a + bi$.**

For $2ab = 0$, either $a = 0$ or $b = 0$.

If $a = 0$, then $-b^2 = 1$, which gives us $b = i$ or $b = -i$.

Substituting into $x = a + bi$ gives us $0 + i \cdot i = -1$ or $0 - i \cdot i = 1$.

If $b = 0$, then $a^2 = 1$, which gives us $a = 1$ or $a = -1$.

Substituting into $x = a + bi$ gives us $1 + 0 \cdot i = 1$ or $-1 + 0 \cdot i = -1$.

Thus, the only complex number solutions to this equation are the complex numbers $1 + 0i$ or $-1 + 0i$.

The quadratic formula also "proves" that the only solutions to the equation are 1 and -1 even if we solve the equation over the set of complex numbers. If this did not come up earlier in this lesson as students share their thinking on the opening, discuss it now.

- What is the quadratic formula?
  - *The formula that provides the solutions to a quadratic equation when it is written in the form $ax^2 + bx + c = 0$ where $a \neq 0$.*
  - *If $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$, and the solutions to the equation are the complex numbers $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.*
- How does the quadratic formula guarantee that a quadratic equation has at most two solutions over the set of complex numbers?
  - *The quadratic formula is a general solution to the equation $ax^2 + bx + c = 0$, where $a \neq 0$. The formula shows two solutions: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. There will be only one solution if $b^2 - 4ac = 0$. There will be two distinct real number solutions when $b^2 - 4ac > 0$, and there will be two distinct complex number solutions when $b^2 - 4ac < 0$.*

Exercises 1–6 (5 minutes)

Allow students time to work these exercises individually, and then discuss as a class. Note that this is very similar to the Opening Exercise used in Lesson 40 of Module 1 in Algebra II with an added degree of difficulty since the coefficients of the polynomial are also complex. Exercises 1–6 are a review of patterns in the factors of the polynomials below for real numbers a and b.

$$a^2 - b^2 = (a + b)(a - b)$$
$$a^2 + b^2 = (a + bi)(a - bi)$$

Exercises 1 and 2 ask students to multiply binomials that have a product that is a sum or difference of squares. Exercises 3–6 ask students to factor polynomials that are sums and differences of squares.

Exercises

Find the product.

1. $(z-2)(z+2)$

   z^2-4

2. $(z+3i)(z-3i)$

   z^2+9

Write each of the following quadratic expressions as the product of two linear factors.

3. z^2-4

   $(z+2)(z-2)$

4. z^2+4

   $(z+2i)(z-2i)$

5. z^2-4i

   $(z+2\sqrt{i})(z-2\sqrt{i})$ *or* $(z+(1+i)\sqrt{2})(z-(1+i)\sqrt{2})$ [*using the fact that* $\sqrt{i}=\frac{1+i}{\sqrt{2}}$]

6. z^2+4i

   $(z+2i\sqrt{i})(z-2i\sqrt{i})$ *or* $(z+(i-1)\sqrt{2})(z-(i-1)\sqrt{2})$ [*using the fact that* $\sqrt{i}=\frac{1+i}{\sqrt{2}}$]

Students may be curious about the square roots of a complex number. They can recall from Precalculus Module 1 that we studied these when considering the polar form of a complex number. This question will also be addressed in Lesson 2 from an algebraic perspective.

- How did we know that each quadratic expression could be factored into two linear terms?
  - *The fundamental theorem of algebra guarantees that a polynomial of degree* 2 *can be factored into* 2 *linear factors. We proved this was true for quadratic expressions by using the solutions produced with the quadratic formula to write the expression as two linear factors.*
- Does the fundamental theorem of algebra apply even if the coefficients are non-real numbers?
  - *It still held true for Exercises 3 and 4, so it seems to, at least if the constant is a non-real number.*

Exercises 7–10 (10 minutes)

Students should work the following exercises in small groups. Have different groups come to the board and present their solutions to the class.

7. **Can a quadratic polynomial equation with real coefficients have one real solution and one complex solution? If so, give an example of such an equation. If not, explain why not.**

The quadratic formula shows that if the discriminant $b^2 - 4ac$ is negative, both solutions will be complex numbers that are complex conjugates. If it is positive, both solutions are real. If it is zero, there is one ("repeated") real solution. We cannot have a real solution coupled with a complex solution.

Scaffolding:

Provide several factored quadratic polynomial equations, and ask students to identify the solutions and write them in standard form.

$$(x-2)(x+2)=0$$
$$(2x-3)(2x+3)=0$$
$$(x-2i)(x+2i)=0$$
$$(2x-i)(2x+i)=0$$
$$(x-1+i)(x-1-i)=0$$
$$\big(x-(1+2i)\big)\big(x-(1-2i)\big)=0$$
$$\big(x-1+\sqrt{3}\big)\big(x-1-\sqrt{3}\big)=0$$

Recall from Algebra II that every quadratic expression can be written as a product of two linear factors, that is,

$$ax^2+bx+c=a(x-r_1)(x-r_2),$$

where r_1 and r_2 are solutions of the polynomial equation $ax^2+bx+c=0$.

8. **Solve each equation by factoring, and state the solutions.**

   a. $x^2+25=0$

   $(x+5i)(x-5i)=0$

   The solutions are $5i$ and $-5i$.

   b. $x^2+10x+25=0$

   $(x+5)(x+5)=0$

   The solution is -5.

9. **Give an example of a quadratic equation with $2+3i$ as one of its solutions.**

We know that if $2+3i$ is a solution of the equation, then its conjugate $2-3i$ must also be a solution.

$$\big(x-(2+3i)\big)\big(x-(2-3i)\big)=\big((x-2)-3i\big)\big((x-2)+3i\big)$$

Using the structure of this expression, we have $(a-bi)(a+bi)$ where $a=x-2$ and $b=3$. Since $(a+bi)(a-bi)=a^2+b^2$ for all real numbers a and b,

$$\begin{aligned}(x-2-3i)(x-2+3i)&=(x-2)^2+3^3\\&=x^2-4x+4+9\\&=x^2-4x+13.\end{aligned}$$

10. **A quadratic polynomial equation with real coefficients has a complex solution of the form $a+bi$ with $b\neq 0$. What must its other solution be, and why?**

The other solution is $a-bi$. If the polynomial equation must have real coefficients, then $\big(x-(a+bi)\big)\big(x-(a-bi)\big)$ when multiplied must yield an expression with real number coefficients.

$$\begin{aligned}\big(x-(a+bi)\big)\big(x-(a-bi)\big)&=(x-a-bi)(x-a+bi)\\&=(x-a)^2-b^2i^2\\&=(x-a)^2+b^2\\&=x^2-2ax+a^2+b^2\end{aligned}$$

Since a and b are real numbers, this expression will always have real number coefficients.

Debrief these exercises by having different groups share their approaches. Give students enough time to struggle with these exercises in their small groups. Use the results of these exercises to further plan for reteaching if students cannot recall what they learned in Algebra I and Algebra II.

Discussion (5 minutes)

Use this discussion to help students recall the fundamental theorem of algebra first introduced in Algebra II Module 1 Lesson 40.

MP.7 & MP.8

- What are the solutions to the polynomial equation $(x-1)(x+2i)(x-2i)=0$? What is the degree of this equation?
  - *The solutions are* 1, $2i$, *and* $-2i$. *This is a third degree equation.*
- What are the solutions to the polynomial equation $(x-1)(x+1)(x-2i)(x+2i)=0$? What is the degree of this equation?
  - *The solutions are* 1, -1, $2i$, *and* $-2i$. *This is a fourth degree equation.*
- Predict how many solutions the equation $x^5-3x^3+2x=0$ has. Justify your response.
  - *It should have at most* 5 *solutions. It is a fifth degree polynomial.*
- To find the solutions, we need to write x^5-3x^3+2x as a product of 5 linear factors. Explain how to factor this polynomial.
  - *Factor out* x, *giving us* $x(x^4-3x^2+2)$.
- How can you factor (x^4-3x^2+2)?
  - *We know* $x^4=(x^2)^2$, *so let* $u=x^2$. *This gives us a polynomial* u^2-3u+2 *in terms of* u, *which factors into* $(u-1)(u-2)$. *Now substitute* x^2 *for* u, *and we have* $(x^2-1)(x^2-2)$.
- What is the factored form of the equation?
  - $x(x^2-1)(x^2-2)=x(x+1)(x-1)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)=0$
- What are the solutions to the equation $x^5-3x^3+2x=0$?
  - *The solutions are* $0, 1, -1, \sqrt{2}$, *and* $-\sqrt{2}$.
- How many solutions will a degree n polynomial equation have? Explain your reasoning.
  - *If every polynomial equation can be written as the product of* n *linear factors, then there will be at most* n *solutions.*

Scaffolding:

Have advanced learners factor x^4-3x^2+2 without the hint.

This is an appropriate point to reintroduce the fundamental theorem of algebra. For more details or to provide additional background information, you can refer back to Algebra II Module 1.

Fundamental Theorem of Algebra

1. Every polynomial function of degree $n \geq 1$ with real or complex coefficients has at least one real or complex zero.
2. Every polynomial of degree $n \geq 1$ with real or complex coefficients can be factored into $\boldsymbol{n}$ linear terms with real or complex coefficients.

Continue the discussion, and have students write any examples and their summaries in the space below each question. Have students discuss both of these questions in their small groups before leading a whole group discussion.

- Could a polynomial function of degree n have more than n zeros? Explain your reasoning.
  - *If a polynomial function has n zeros, then it has n linear factors, which means the degree will be n.*
- Could a polynomial function of degree n have less than n zeros? Explain your reasoning.
  - *Yes. If a polynomial has repeated linear factors, then it will have less than n distinct zeros. For example, $p(x) = (x-1)^n$ has only one zero: the number 1.*

Exercises 11–15 (10 minutes)

Give students time to work on the exercises either individually or with a partner, and then share answers as a class. On Exercises 11 and 12, students will need to recall polynomial division from Algebra II. Students divided polynomials using both the reverse tabular method and long division. You may need to review one or both of these methods with the students. In Exercise 11, students may recall the sum and difference of cube formulas derive through polynomial long division in Algebra II. Students should have access to technology to aid with problems such as Exercise 12. However, in Exercise 12, you could ask the students to verify that $x = 2$ is a zero of p rather than having them use technology to locate the zero.

11. Write the left side of each equation as a product of linear factors, and state the solutions.

a. $x^3 - 1 = 0$

$$0 = x^3 - 1 = (x-1)(x^2 + x + 1)$$

Using the quadratic formula, $x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$, so the quadratic expression $x^2 + x + 1$ factors into

$$x^2 + x + 1 = \left(x - \left(\frac{-1 + i\sqrt{3}}{2}\right)\right)\left(x - \left(\frac{-1 - i\sqrt{3}}{2}\right)\right).$$

Then, the factored form of the equation is

$$(x-1)\left(x - \left(\frac{-1 + i\sqrt{3}}{2}\right)\right)\left(x - \left(\frac{-1 - i\sqrt{3}}{2}\right)\right) = 0.$$

The solutions are 1, $\frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$.

b. $x^3 + 8 = 0$

$$x^3 + 8 = (x+2)(x^2 - 2x + 4)$$

Using the quadratic formula, $x = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{-3}}{2} = 1 \pm i\sqrt{3}$, so the quadratic expression $x^2 - 2x + 4$ factors into

$$x^2 \pm 2 + 4 = \left(x - (1 + i\sqrt{3})\right)\left(x - (1 - i\sqrt{3})\right).$$

Then, the factored form of the equation is

$$(x+2)\left(x - (1 + i\sqrt{3})\right)\left(x - (1 - i\sqrt{3})\right) = 0.$$

The solutions are -2, $1 + i\sqrt{3}$, and $1 - i\sqrt{3}$.

Scaffolding:

Display the sum and difference of cube formulas on the board once students have completed Exercise 11 as a reminder for future work.

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

c. $x^4 + 7x^2 + 10 = 0$

$$(x^2+5)(x^2+2) = (x+i\sqrt{5})(x-i\sqrt{5})(x+i\sqrt{2})(x-i\sqrt{2}) = 0$$

The solutions are $\pm i\sqrt{5}$ and $\pm i\sqrt{2}$.

12. Consider the polynomial $p(x) = x^3 + 4x^2 + 6x - 36$.

MP.5

a. Graph $y = x^3 + 4x^2 + 6x - 36$, and find the real zero of polynomial p.

The graph of p has an x-intercept at $x = 2$. Therefore, $x = 2$ is a zero of p.

b. Write $p(x)$ as a product of linear factors.

$$p(x) = (x-2)(x+3-3i)(x+3+3i)$$

c. What are the solutions to the equation $p(x) = 0$?

The solutions are 2, $-3+3i$, and $-3-3i$.

13. Malaya was told that the volume of a box that is a cube is $4,096$ cubic inches. She knows the formula for the volume of a cube with side length x is $V(x) = x^3$, so she models the volume of the box with the equation $x^3 - 4096 = 0$.

a. Solve this equation for x.

$x^3 - 4096 = (x-16)(x^2+16x+256) = 0$

$x = 16, x = -8+8\sqrt{3}i, x = -8-8\sqrt{3}i$

b. Malaya shows her work to Tiffany and tells her that she has found three different values for the side length of the box. Tiffany looks over Malaya's work and sees that it is correct but explains to her that there is only one valid answer. Help Tiffany explain which answer is valid and why.

Since we are looking for the dimensions of a box, only real solutions are acceptable, so the answer is 16 inches.

14. Consider the polynomial $p(x) = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$.

a. Graph $y = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$, and state the number of real zeros of p.

There are no real zeros.

b. Verify that i is a zero of p.

$$\begin{aligned} p(i) &= i^6 - 2i^5 + 7i^4 - 10i^3 + 14i^2 - 8i + 8 \\ &= -1 - 2i + 7 + 10i - 14 - 8i + 8 \\ &= 0 \end{aligned}$$

c. Given that i is a zero of p, state another zero of p.

Another zero is $-i$.

MP.3

d. Given that $2i$ and $1+i$ are also zeros of p, explain why polynomial p cannot possibly have any real zeros.

Since $2i$ and $1+i$ are zeros, $-2i$ and $1-i$ must also be zeros of p. The fundamental theorem of algebra tells us that since p is a degree 6 polynomial it can be written as a product of 6 linear factors. We now know that p has 6 complex zeros and therefore cannot have any real zeros.

e. **What is the solution set to the equation $p(x) = 0$?**

The solution set is $\{i, -i, 2i, -2i, 1+i, 1-i\}$.

15. **Think of an example of a sixth degree polynomial equation that when written in standard form has integer coefficients, four real number solutions, and two imaginary number solutions. How can you be sure your equation will have integer coefficients?**

One correct response is $(x-1)(x+1)(x-2)(x+2)(x-2i)(x+2i) = 0$. *By selecting an imaginary number* bi*, where* b *is an integer, and its conjugate as solutions, we know by the identity* $(a+bi)(a-bi) = a^2 + b^2$ *that these factors will produce a quadratic expression with integer coefficients. If we choose the real solutions to also be integers, then when written in standard form the polynomial equation will have integer coefficients.*

Closing (4 minutes)

Review the information in the Lesson Summary box by asking students to choose one vocabulary term, theorem, or identity and paraphrase it with a partner. Select students to share their paraphrasing with the class.

Lesson Summary

Relevant Vocabulary

POLYNOMIAL FUNCTION: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \to \mathbb{R}$ such that for each real number x in the domain, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \to \mathbb{R}$ is a *polynomial function*, then there is some nonnegative integer n and collection of real numbers $a_0, a_1, a_2, \ldots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

for every real number x in the domain, which is called the *standard form of the polynomial function.* The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

DEGREE OF A POLYNOMIAL FUNCTION: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function. The degree is the highest degree of its terms.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x+1)^2 - (x-1)^2$ is 1 because when g is put into standard form, it is $g(x) = 4x$.

ZEROS OR ROOTS OF A FUNCTION: A *zero* (or *root*) *of a function* $f: \mathbb{R} \to \mathbb{R}$ is a number x of the domain such that $f(x) = 0$. A *zero of a function* is an element in the solution set of the equation $f(x) = 0$.

Given any two polynomial functions p and q, the solution set of the equation $p(x)q(x) = 0$ can be quickly found by solving the two equations $p(x) = 0$ and $q(x) = 0$ and combining the solutions into one set.

A number a is zero of a polynomial function p with multiplicity m if the factored form of p contains $(x-a)^m$.

Every polynomial function of degree n, for $n \geq 1$, has n zeros over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors.

Exit Ticket (3 minutes)

Name ____ Date ____________________

Lesson 1: Solutions to Polynomial Equations

Exit Ticket

1. Find the solutions of the equation $x^4 - x^2 - 12$. Show your work.

2. The number 1 is a zero of the polynomial $p(x) = x^3 - 3x^2 + 7x - 5$.
   a. Write $p(x)$ as a product of linear factors.

   b. What are the solutions to the equation $x^3 - 3x^2 + 7x - 5 = 0$?

Exit Ticket Sample Solutions

1. Find the solutions of the equation $x^4 - x^2 - 12$. Show your work.

   $x^4 - x^2 - 12 = (x^2+3)(x^2-4) = (x - i\sqrt{3})(x + i\sqrt{3})(x-2)(x+2) = 0$

   The solutions of the equation are $i\sqrt{3}, -i\sqrt{3}, 2, -2$.

2. The number 1 is a zero of the polynomial $p(x) = x^3 - 3x^2 + 7x - 5$.

   a. Write $p(x)$ as a product of linear factors.

   $(x-1)(x^2-2x+5)$

   $(x-1)(x-(1+2i))(x-(1-2i))$

   b. What are the solutions to the equation $x^3 - 3x^2 + 7x - 5 = 0$?

   The solutions are 1, $1+2i$, *and* $1-2i$.

Problem Set Sample Solutions

1. Find all solutions to the following quadratic equations, and write each equation in factored form.

   a. $x^2 + 25 = 0$

   $$x = \pm 5i,$$
   $$(x+5i)(x-5i) = 0$$

   b. $-x^2 - 16 = -7$

   $$x = \pm 3i,$$
   $$(x+3i)(x-3i) = 0$$

   c. $(x+2)^2 + 1 = 0$

   $$x^2 + 4x + 5 = 0, \qquad x = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i,$$
   $$(x+2+i)(x+2-i) = 0$$

   d. $(x+2)^2 = x$

   $$x^2 + 3x + 4 = 0, \qquad x = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{3}i}{2} = -\frac{3}{2} \pm \frac{\sqrt{3}i}{2}$$
   $$\left(x + \frac{3}{2} + \frac{\sqrt{3}i}{2}\right)\left(x + \frac{3}{2} - \frac{\sqrt{3}i}{2}\right) = 0$$

   e. $(x^2+1)^2 + 2(x^2+1) - 8 = 0$

   $$(x^2+1+4)(x^2+1-2) = 0, \qquad x^2 + 5 = 0, \qquad x = \pm\sqrt{5}i, \qquad x^2 - 1 = 0, \qquad x = \pm 1$$
   $$(x+\sqrt{5}i)(x-\sqrt{5}i)(x+1)(x-1) = 0$$

f. $(2x-1)^2 = (x+1)^2 - 3$

$$4x^2 - 4x + 1 - x^2 - 2x - 1 + 3 = 0, \quad 3x^2 - 6x + 3 = 0, \quad x^2 - 2x + 1 = 0,$$
$$(x-1)^2 = 0, \quad x = \pm 1$$
$$(x+1)(x-1) = 0$$

g. $x^3 + x^2 - 2x = 0$

$$x(x^2 + x - 2) = 0, \quad x(x+2)(x-1) = 0,$$
$$x = 0, \quad x = -2, x = 1$$
$$x(x+2)(x-1) = 0$$

h. $x^3 - 2x^2 + 4x - 8 = 0$

$$x^2(x-2) + 4(x-2) = 0, \quad (x-2)(x^2+4) = 0, \quad x = 2, \quad x = \pm 2i$$
$$(x-2)(x+2i)(x-2i) = 0$$

2. The following cubic equations all have at least one real solution. Find the remaining solutions.

a. $x^3 - 2x^2 - 5x + 6 = 0$

One real solution is -2; then $x^3 - 2x^2 - 5x + 6 = (x+2)(x^2 - 4x + 3) = (x+2)(x-1)(x-3)$.

The solutions are $-2, 1, 3$.

b. $x^3 - 4x^2 + 6x - 4 = 0$

One real solution is 2; then $x^3 - 4x^2 + 6x - 4) = (x-2)(x^2 - 2x + 2)$. Using the quadratic formula on $x^2 - 2x + 2 = 0$ gives $x = 1 \pm i$.

The solutions are $2, 1+i, 1-i$.

c. $x^3 + x^2 + 9x + 9 = 0$

One real solution is -1; then $x^3 + x^2 + 9x + 9 = (x+1)(x^2+9) = (x+1)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$.

The solutions are $-1, 3i, -3i$.

d. $x^3 + 4x = 0$

One real solution is 0; then $(x^3 + 4x) = x(x^2 + 4) = x(x+2i)(x-2i)$.

The solutions are $0, 2i, -2i$.

e. $x^3 + x^2 + 2x + 2 = 0$

One real solution is $x = -1$; then $x^3 + x^2 + 2x + 2 = (x+1)(x^2+2) = (x+1)\left(x - i\sqrt{2}\right)\left(x + i\sqrt{2}\right)$.

The solutions are $-1, i\sqrt{2}, -i\sqrt{2}$.

3. **Find the solutions of the following equations.**

a. $4x^4 - x^2 - 18 = 0$

Set $u = x^2$. Then

$$\begin{aligned} 4x^4 - x^2 - 18 &= 4u^2 - u - 18 \\ &= (4u - 9)(u + 2) \\ &= (4x^2 - 9)(x^2 + 2) \\ &= (2x - 3)(2x + 3)\left(x + i\sqrt{2}\right)\left(x - i\sqrt{2}\right). \end{aligned}$$

Solutions are $i\sqrt{2}, -i\sqrt{2}, -\frac{3}{2}, \frac{3}{2}$.

b. $x^3 - 8 = 0$

$$\begin{aligned} (x^3 - 8) &= (x - 2)(x^2 + 2x + 4) \\ &= (x - 2)\left(x - \left(-1 - i\sqrt{3}\right)\right)\left(x - \left(-1 + i\sqrt{3}\right)\right) \end{aligned}$$

Solutions are $2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$.

c. $8x^3 - 27 = 0$

$$\begin{aligned} (8x^3 - 27) &= (2x - 3)(4x^2 + 6x + 9) \\ &= (2x - 3)\left(2x - \left(\frac{-3 - 3i\sqrt{3}}{2}\right)\right)\left(2x - \left(\frac{-3 + 3i\sqrt{3}}{2}\right)\right) \end{aligned}$$

Solutions are $\frac{3}{2}, -\frac{3}{4} + \frac{3i\sqrt{3}}{4}, -\frac{3}{4} - \frac{3i\sqrt{3}}{4}$.

d. $x^4 - 1 = 0$

$$\begin{aligned} (x^4 - 1) &= (x^2 - 1)(x^2 + 1) \\ &= (x - 1)(x + 1)(x + i)(x - i) \end{aligned}$$

Solutions are $1, -1, i, -i$.

e. $81x^4 - 64 = 0$

$$\begin{aligned} (81x^4 - 64) &= (9x^2 - 8)(9x^2 + 8) \\ &= (3x - 2\sqrt{2})(3x + 2\sqrt{2})(3x + 2i\sqrt{2})(3x - 2i\sqrt{2}) \end{aligned}$$

Solutions are $\frac{2\sqrt{2}}{3}i, -\frac{2\sqrt{2}}{3}i, \frac{2\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3}$.

f. $20x^4 + 121x^2 - 25 = 0$

$$\begin{aligned} (20x^4 + 121x^2 - 25) &= (4x^2 + 25)(5x^2 - 1) \\ &= (2x + 5i)(2x - 5i)(\sqrt{5}x - 1)(\sqrt{5}x + 1) \end{aligned}$$

Solutions are $\frac{5}{2}i, -\frac{5}{2}i, \frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}$.

g. $64x^3 + 27 = 0$

$$(64x^3 + 27) = (4x + 3)(16x^2 - 12x + 9)$$
$$= (4x + 3)\left(8x - 3\left(1 - i\sqrt{3}\right)\right)\left(8x - 3\left(1 + i\sqrt{3}\right)\right)$$

Solutions are $\frac{3}{8} + \frac{3i\sqrt{3}}{8}, -\frac{3}{4}, \frac{3}{8} - \frac{3i\sqrt{3}}{8}$.

h. $x^3 + 125 = 0$

$$(x^3 + 125) = (x + 5)(x^2 - 5x + 25)$$
$$= (x + 5)\left(x - \frac{5\left(1 - i\sqrt{3}\right)}{2}\right)\left(x - \frac{5\left(1 + i\sqrt{3}\right)}{2}\right)$$

Solutions are $-5, \frac{5}{2} + \frac{5i\sqrt{3}}{2}, \frac{5}{2} - \frac{5i\sqrt{3}}{2}$.