



Topic A:

Polynomial Functions and the Fundamental Theorem of Algebra

N-CN.C.8, N-CN.C.9, A-APR.C.5, G-GMD.A.2, G-GPE.A.3

Focus Standards:	N-CN.C.8	(+) Extend polynomial identities to the complex numbers. <i>For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.</i>
	N-CN.C.9	(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.
	A-APR.C.5	(+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.
	G-GMD.A.2	(+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
	G-GPE.A.3	(+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
Instructional Days:	9	
Lesson 1:	Solutions to Polynomial Equations (P) ¹	
Lesson 2:	Does Every Complex Number Have a Square Root? (P)	
Lesson 3:	Roots of Unity (E)	
Lessons 4–5:	The Binomial Theorem (P, P)	
Lesson 6:	Curves in the Complex Plane (P)	
Lessons 7–8:	Curves from Geometry (S, S)	
Lesson 9:	Volume and Cavalieri's Principle (S)	

In Algebra II Module 1, students encountered the Fundamental Theorem of Algebra, which states that every polynomial function has at least one zero in the realm of the complex numbers (**N-CN.C.9**). Topic A of this module brings students back to the study of both real and complex roots of polynomial functions. Students first complete a brief review of quadratic and cubic functions by extending familiar polynomial identities to the complex numbers (**N-CN.C.8**) and then review general polynomial functions.

¹ Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

In Lesson 1, students work with equations with complex number solutions and apply identities such as $a^2 + b^2 = (a + bi)(a - bi)$ for real numbers a and b to solve equations and explore the implications of the Fundamental Theorem of Algebra (**N-CN.C.8** and **N-CN.C.9**). In Precalculus Module 1, students used the polar form of a complex number to find powers and roots of complex numbers. However, nearly all of the examples used in that module involved complex numbers with arguments that were multiples of special angles ($\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$). In Lesson 2, we return to the rectangular form of a complex number to show algebraically that we can find the square roots of any complex number without having to express it first in polar form. Students use the properties of complex numbers and the Fundamental Theorem of Algebra to find the square roots of any complex number by creating and solving polynomial equations (**N-CN.C.8** and **N-CN.C.9**). Lesson 3 connects the work within the algebra strand of solving polynomial equations to the geometry and arithmetic of complex numbers in the complex plane. Students determine solutions to polynomial equations using various methods and interpret the results in the real and complex planes. Students extend factoring to the complex numbers (**N-CN.C.8**) and more fully understand the conclusion of the Fundamental Theorem of Algebra (**N-CN.C.9**) by seeing that a polynomial of degree n has n complex roots when they consider the roots of unity graphed in the complex plane. These lessons can be supported with graphing software that is available free of charge such as GeoGebra.

In Lesson 4, students carry out the tedious process of repeatedly multiplying binomial factors together, leading them to see the usefulness of finding a quicker way to expand binomials raised to whole number powers. By exploring binomial expansions, students recognize patterns in the coefficients of terms illustrated in Pascal's Triangle. They verify patterns in the powers of terms in binomial expansions and are introduced to the formal Binomial theorem, which they prove by combinatorics. Students then apply the Binomial theorem to expand binomials and to find specific terms in expansions (**A-APR.C.5**). Lesson 5 provides students with opportunities to explore additional patterns formed by the coefficients of binomial expansions. They apply the Binomial theorem to find a mathematical basis for the patterns observed. They also apply the theorem to explore the average rate of change for the volume of a sphere with a changing radius (**A-APR.C.5**). In Lesson 6, students review the characteristics of the graphs of the numbers $z = r(\cos\theta + i\sin\theta)$, recognizing that they represent circles centered at the origin with radius equal to the modulus. They explore sets of complex numbers written in the form $z = a\cos\theta + b\sin\theta$, identifying the graphs as ellipses. Students convert between the complex and real forms of equations for ellipses, including those whose centers are not the origin (**G-GPE.A.3**). In Lesson 7, students study ellipses as the set of all points in a plane such that the sum of the distances from two points (foci) to any point on the line containing the foci is constant. Formally, an ellipse is a set of points that satisfy a constraint $PF + PG = k$, where points F and G are the foci of the ellipse, P is any point distinct from points F and G , and k is a constant. The goal of this lesson is to connect these two definitions. That is, if a point $P(x, y)$ satisfies the constraint $PF + PG = k$, then it also satisfies an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for suitable values of a and b (**G-GPE.A.3**). Lesson 8 introduces students to the hyperbola as they graph hyperbolas and derive the standard equation of a hyperbola knowing its foci (**G-GPE.A.3**). Students learn that the hyperbola is a set of points in a plane whose distances to two fixed points (foci) have a constant difference. Students see the connection between ellipses, parabolas, and hyperbolas in the context of the orbital path of a satellite. Topic A ends with Lesson 9 as students see ellipses and hyperbolas as cross sections of a cone. Cavalieri's volume principle is based on cross-sectional area and is used to explore the volume of pyramids, cones, and spheres (**G-GMD.A.2**). Students were first introduced to

this principle in Geometry (Module 3, Lesson 10) when studying the volume of prisms and cylinders. In this lesson, students use the principle to find the volumes of pyramids, cones, and spheres.

Throughout Topic A, students make conjectures about the roots of polynomials and solve problems by applying algebraic properties. They construct arguments and critique the reasoning of others in these algebraic contexts (MP.3). They examine the structure of expressions to support their solutions and make generalizations about roots of polynomials and to find patterns of the coefficients of expanded polynomials (MP.7 and MP.8).