Lesson 21: Ptolemy’s Theorem

Classwork

Opening Exercise

*Ptolemy’s theorem* says that for a cyclic quadrilateral , .

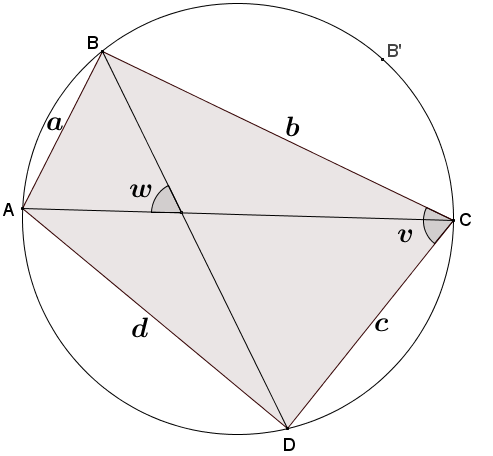
With ruler and a compass, draw an example of a cyclic quadrilateral. Label its vertices , , , and .

Draw the two diagonals and .

With a ruler, test whether or not the claim that seems to hold true.

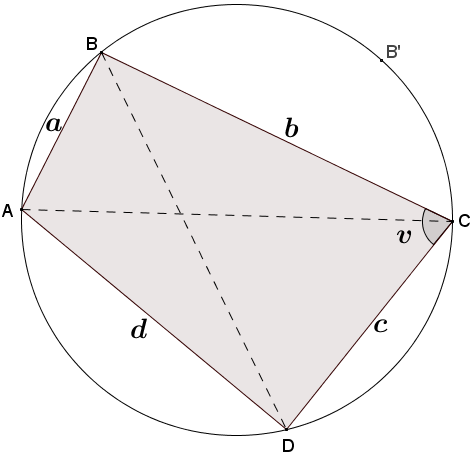
Repeat for a second example of a cyclic quadrilateral.

**Challenge:**  Draw a cyclic quadrilateral with one side of length zero. What shape is the this cyclic quadrilateral? Does Ptolemy’s claim hold true for it?

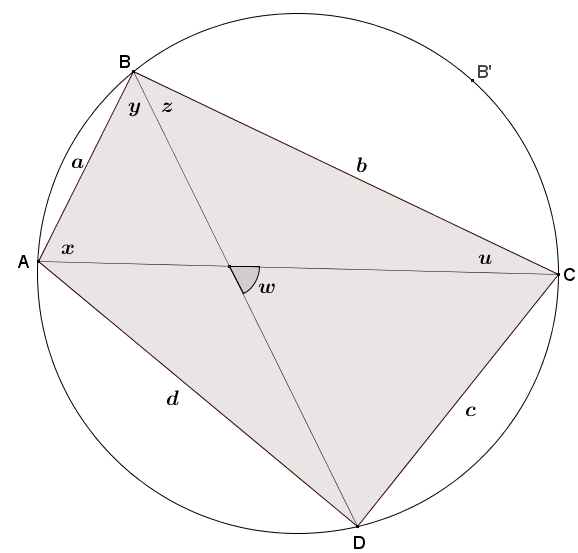
**Exploratory Challenge: A Journey to Ptolemy’s Theorem**

The diagram shows cyclic quadrilateral with diagonals and intersecting to form an acute angle with degree measure . , , , and .

* 1. From last lesson, what is the area of quadrilateral in terms of the lengths of its diagonals and the angle ? Remember this formula for later on!
  2. Explain why one of the angles, or , has a measure less than or equal to .
  3. Let’s assume that in our diagram is the angle with a measure less than or equal to . Call its measure degrees. What is the area of triangle in terms of , , and ? What is the area of triangle in terms of , , and ? What is the area of quadrilateral in terms of , , , , and ?

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* 1. We now have two different expressions representing the area of the same cyclic quadrilateral . Does it seem to you that we are close to a proof of Ptolemy’s claim?
  2. Trace the circle and points , , , and onto a sheet of patty paper. Reflect triangle about the perpendicular bisector of diagonal . Let , , and be the images of the points , , and , respectively.
     1. What does the reflection do with points and ?
     2. Is it correct to draw as on the circle? Explain why or why not.
     3. Explain why quadrilateral has the same area as quadrilateral .
  3. The diagram shows angles having degree measures , , ,, and . Find and label any other angles having degree measures ,,, , or , and justify your answers.



* 1. Explain why in your diagram from part (f).
  2. Identify angles of measures , , ,, and in your diagram of the cyclic quadrilateral from part (e).
  3. Write a formula for the area of triangle in terms of , , and . Write a formula for the area of triangle in terms of ,, and .
  4. Based on the results of part (i), write a formula for the area of cyclic quadrilateral In terms of , ,, ,and .
  5. Going back to part (a), now establish Ptolemy’s theorem.

Lesson Summary

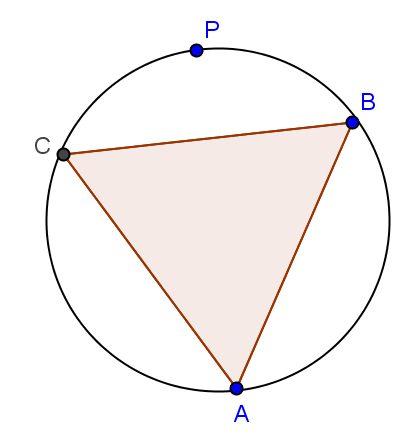
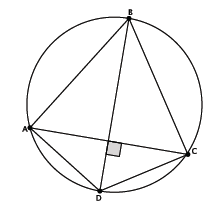
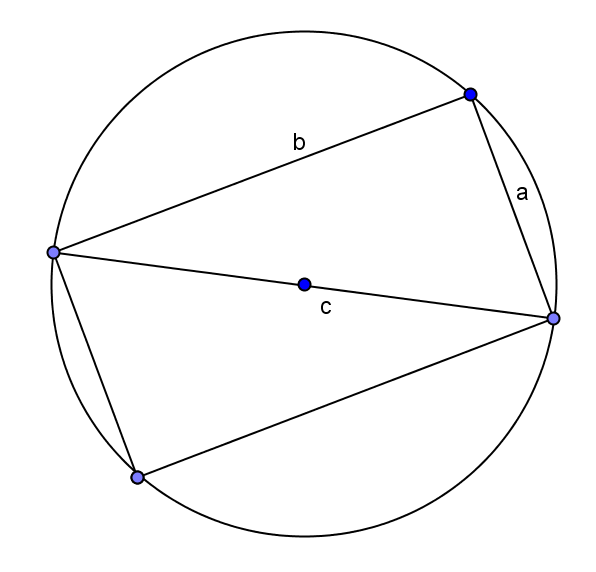
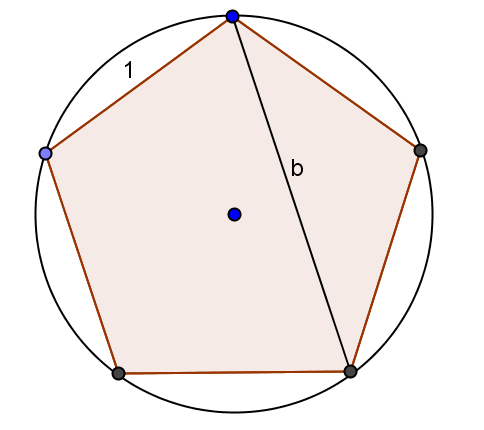
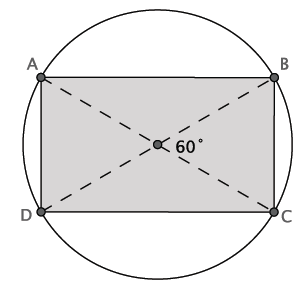
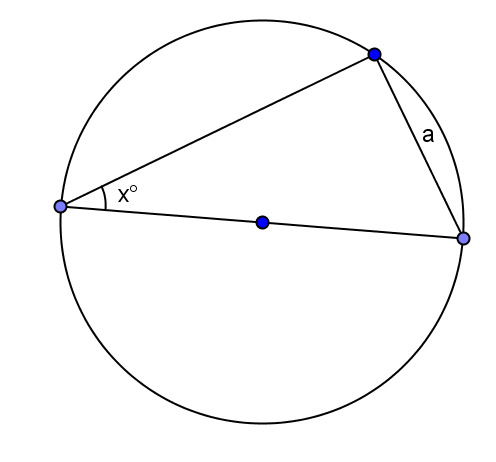
**Theorems**

Ptolemy’s theorem: For a cyclic quadrilateral, .

**Relevant Vocabulary**

Cyclic Quadrilateral: A quadrilateral with all vertices lying on a circle is known as a cyclic quadrilateral.

Problem Set

1. An equilateral triangle is inscribed in a circle. If is a point on the circle, what does Ptolemy’s theorem have to say about the distances from this point to the three vertices of the triangle?
2. Kite is inscribed in a circle. The kite has an area of , and the ratio of the lengths of the non-congruent adjacent sides is . What is the perimeter of the kite?
3. ****Draw a right triangle with leg lengths and , and hypotenuse length . Draw a rotated copy of the triangle such that the figures form a rectangle. What does Ptolemy have to say about this rectangle?
4. Draw a regular pentagon of side length in a circle. Let be the length of its diagonals. What does Ptolemy’s theorem say about the quadrilateral formed by four of the vertices of the pentagon?
5. The area of the inscribed quadrilateral is . Determine the circumference of the circle.
6. Extension: Suppose and are two acute angles, and the circle has a diameter of unit. Find , , , and in terms of and . Apply Ptolemy’s theorem, and determine the exact value of .
   1. Explain why equals the diameter of the circle.
   2. If the circle has a diameter of , what is ?
   3. Use Thales’ theorem to write the side lengths in the original diagram in terms of and .
   4. If one diagonal of the cyclic quadrilateral is , what is the other?
   5. **What does Ptolemy’s theorem give?
   6. Using the result from part (e), determine the exact value of .