Lesson 19: Equations for Tangent Lines to Circles

Student Outcomes

- Given a circle, students find the equations of two lines tangent to the circle with specified slopes.
- Given a circle and a point outside the circle, students find the equation of the line tangent to the circle from that point.

Lesson Notes

This lesson builds on the understanding of equations of circles in Lessons 17 and 18 and on the understanding of tangent lines developed in Lesson 11. Further, the work in this lesson relies on knowledge from Module 4 related to **G-GPE.B.4** and **G-GPE.B.5**. Specifically, students must be able to show that a particular point lies on a circle, compute the slope of a line, and derive the equation of a line that is perpendicular to a radius. The goal is for students to understand how to find equations for both lines with the same slope tangent to a given circle and to find the equation for a line tangent to a given circle from a point outside the circle.

Classwork

Opening Exercise (5 minutes)

Students are guided to determine of the equation of a line perpendicular to a chord of a given circle.

Opening Exercise

A circle of radius 5 passes through points $A(-3,3)$ and $B(3,1)$.

a. **What is the special name for segment AB?**

Segment AB is called a chord.

b. **How many circles can be drawn that meet the given criteria? Explain how you know.**

Two circles of radius 5 pass through points A and B. Two distinct circles, at most, can have two points in common.

c. **What is the slope of $\overline{AB}$?**

The slope of $\overline{AB}$ is $-\frac{1}{3}$.

d. **Find the midpoint of $\overline{AB}$.**

The midpoint of $\overline{AB}$ is $(0,2)$.

Scaffolding:

- Provide struggling students with a picture of the two circles containing the indicated points.
- Alternatively, struggling students may benefit from simply graphing the two points.

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e. **Find the equations of the line containing a diameter of the given circle perpendicular to $\overline{AB}$.**

$\mathbf{y - 2 = 3(x - 0)}$

f. **Is there more than one answer possible for question 5?**

Although two circles may be drawn that meet the given criteria, the diameters of both lie on the line perpendicular to $\overline{AB}$. That line is the perpendicular bisector of $\overline{AB}$.

Example 1 (10 minutes)

Example 1

Consider the circle with equation $(x-3)^2 + (y-5)^2 = 20$. Find the equations of two tangent lines to the circle that each have slope $-\frac{1}{2}$.

MP.1

Provide time for students to think about this in pairs or small groups. If necessary, guide their thinking by reminding students of their work in Lesson 11 on tangent lines and their work in Lessons 17–18 on equations of circles. Allow individual students or groups of students to share their reasoning as to how to determine the needed equations. Once students have shared their thinking, continue with the reasoning below.

- What is the center of the circle? The radius of the circle?
  - *The center is $(3,5)$, and the radius is $\sqrt{20}$.*
- If the tangent lines are to have a slope of $-\frac{1}{2}$, what must be the slope of the radii to those tangent lines? Why?
  - *The slope of each radius must be 2. A tangent is perpendicular to the radius at the point of tangency. Since the tangent lines must have slopes of $-\frac{1}{2}$, the radii must have slopes of 2, the negative reciprocal of $-\frac{1}{2}$.*
- Label the center O. We need to find a point $A(x, y)$ on the circle with a slope of 2. We have $\frac{y-5}{x-3} = 2$, and $(x-3)^2 + (y-5)^2 = 20$.
  Since $y - 5 = 2(x-3)$,
  then $(y-5)^2 = 4(x-3)^2$.
  Substituting into the equation for the circle results in $(x-3)^2 + 4(x-3)^2 = 20$.
- At this point, ask students to solve the equation for x and call on volunteers to share their results.
  Ask if students noticed that using the distributive property made solving the equation easier.
  $(x-3)^2(1+4) = 20$
  $(x-3)^2 = 4$ $\quad$ $x - 3 = 2$ or $x - 3 = -2$ $\quad$ $x = 5,1$
- As expected, there are two possible values for x, 1 *or* 5. Why are two values expected?
  - *There should be two lines tangent to a circle for any given slope.*
- What are the coordinates of the points of tangency? How can you determine the y-coordinates?
  - *$(1,1)$ and $(5,9)$; it is easiest to find the y-coordinate by plugging each x-coordinate into the slope formula above ($\frac{y-5}{x-3} = 2$).*

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- How can we verify that these two points lie on the circle?
  - *Substituting the coordinates into the equation given for the circle proves that they, in fact, do lie on the circle.*
- How can we find the equations of these tangent lines? What are the equations?
  - *We know both the slope and a point on each of the two tangent lines, so we can use the point-slope formula. The two equations are $y - 1 = -\frac{1}{2}(x - 1)$ and $y - 9 = -\frac{1}{2}(x - 5)$.*

Provide students time to discuss the process for finding the equations of the tangent lines; then, ask students how the solution would have changed if we were looking for two tangent lines whose slopes were 4 instead of $-\frac{1}{2}$. Students should respond that the slope of the radii to the tangent lines would change to $-\frac{1}{4}$, which would have impacted all other calculations related to slope and finding the equations.

Exercise 1 (5 minutes, optional)

The exercise below can be used to check for understanding of the process used to find the equations of tangent lines to circles. This exercise should be assigned to groups of students who struggled to respond to the last question from the previous discussion. Consider asking the question again—how would the solution have changed if the slopes of the tangent lines were $\frac{1}{3}$ instead of 2—after students finish work on Exercise 1. If this exercise is not used, Exercises 3–4 can be assigned at the end of the lesson.

Exercise 1

Consider the circle with equation $(x-4)^2 + (y-5)^2 = 20$. Find the equations of two tangent lines to the circle that each have slope 2.

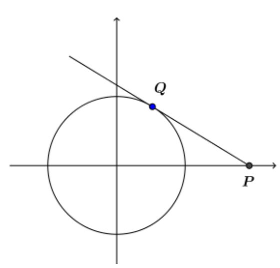
$y - 7 = 2(x - 0)$ *and* $y - 3 = 2(x - 8)$

Example 2 (10 minutes)

Example 2

Refer to the diagram below.

Let $p > 1$. What is the equation of the tangent line to the circle $x^2 + y^2 = 1$ through the point $(p, 0)$ on the x-axis with a point of tangency in the upper half-plane?



Scaffolding:

Consider labeling point Q as $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ for struggling students. Ask them to show that the point does lie on the circle. Then, ask them to find the slope of the tangent line at that point.

- Use $Q(x, y)$ as the point of tangency, as shown in the diagram provided. Label the center as $O(0,0)$. What do we know about segments OP and OQ?
  - *They are perpendicular.*

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- Write an equation that considers this.
  - $\frac{y}{x} = \frac{x-p}{y-o}$, *giving* $x(x-p) + y^2 = 0$, *or* $x^2 - xp + y^2 = 0$
- Combine the two equations to find an expression for x.
  - *Since* $x^2 + y^2 = 1$, *we get* $1 - xp = 0$, *or* $x = \frac{1}{p}$.
- Use the expression for x to find an expression for y.
  - $y = \sqrt{1 - \frac{1}{p^2}}$, *which simplifies to* $y = \frac{1}{p}\sqrt{p^2 - 1}$.
- What are the coordinates of the point Q (the point of tangency)?
  - $(\frac{1}{p}, \frac{1}{p}\sqrt{p^2 - 1})$
- What is the slope of $\overline{OQ}$ in terms of p?
  - $\sqrt{p^2 - 1}$
- What is the slope of $\overline{QP}$ in terms of p?
  - $-\frac{1}{\sqrt{p^2-1}} = \frac{\sqrt{p^2-1}}{1-p^2}$
- What is the equation of line QP?
  - $y = \frac{\sqrt{p^2-1}}{1-p^2}(x - p)$

Exercise 2 (3 minutes)

The following exercise continues the thinking that began in Example 2. Allow students to work on the exercise in pairs or small groups if necessary.

Exercises

2. Use the same diagram from Example 2 above, but label the point of tangency in the lower half-plane as Q'.

a. What are the coordinates of Q'?

$\left(\frac{1}{p}, -\frac{1}{p}\sqrt{p^2 - 1}\right)$

b. What is the slope of $\overline{OQ'}$?

$-\sqrt{p^2 - 1}$

c. What is the slope of $\overline{Q'P}$?

$\frac{\sqrt{p^2 - 1}}{p^2 - 1}$

d. Find the equation of the second tangent line to the circle through $(p, 0)$.

$y = \frac{\sqrt{p^2 - 1}}{p^2 - 1}(x - p)$

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Discussion (4 minutes)

- As the point $(p, 0)$ on the x-axis slides to the right, that is, as we choose larger and larger values of p, to what coordinate pair does the point of tangency (Q) of the first tangent line (in Example 2) converge? Hint: It might be helpful to rewrite the coordinates of Q as $\left(\frac{1}{p}, \sqrt{1-\frac{1}{p^2}}\right)$.
  - $(0,1)$
- What is the equation of the tangent line in this limit case?
  - $y = 1$
- Suppose instead that we let the value of p be a value very close to $p = 1$. What can you say about the point of tangency and the tangent line to the circle in this case?
  - *The point of tangency would approach $(1,0)$, and the tangent line would have the equation $x = 1$.*
- For the case of $p = 2$, what angle does the tangent line make with the x-axis?
  - $30°$*; a right triangle is formed with base 1 and hypotenuse 2.*
- What value of p gives a tangent line intersecting the x-axis at a $45°$ angle?
  - *There is no solution that gives a $45°$ angle. If $p = 1$ (which results in the required number of degrees), then the tangent line is $x = 1$, which is perpendicular to the x-axis and, therefore, not at a $45°$ angle.*
- What is the length of $\overline{QP}$?
  - $\sqrt{p^2 - 1}$ *(Pythagorean theorem)*

Exercises 3–4 (5 minutes)

The following two exercises can be completed in class, if time, or assigned as part of the problem set. Consider posing this follow up question to Exercise 4: How can we change the given equations so that they would represent lines tangent to the circle. Students should respond that the slope of the first equation should be $-\frac{4}{3}$, and the slope of the second equation should be $-\frac{3}{4}$.

MP.3

3. Show that a circle with equation $(x-2)^2 + (y+3)^2 = 160$ has two tangent lines with equations $y + 15 = \frac{1}{3}(x-6)$ and $y - 9 = \frac{1}{3}(x+2)$.

Assume that the circle has the tangent lines given by the equations above. Then, the tangent lines have slope $\frac{1}{3}$, and the slope of the radius to those lines must be -3. If we can show that the points $(6,-15)$ and $(-2,9)$ are on the circle, and that the slope of the radius to the tangent lines is -3, then we will have shown that the given circle has the two tangent lines given.

$$\begin{aligned}&(6-2)^2 + (-15+3)^2\\&= 4^2 + (-12)^2\\&= 160\end{aligned}$$

$$\begin{aligned}&(-2-2)^2 + (9+3)^2\\&= (-4)^2 + 12^2\\&= 160\end{aligned}$$

Since both points satisfy the equation, then the points $(6,-15)$ and $(-2,9)$ are on the circle.

$$m = \frac{-15-(-3)}{6-2} = -\frac{12}{4} = -3$$

$$m = \frac{9-(-3)}{-2-2} = -\frac{12}{4} = -3$$

The slope of the radius is -3.

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4. **Could a circle given by the equation $(x-5)^2+(y-1)^2=25$ have tangent lines given by the equations $y-4=\frac{4}{3}(x-1)$ and $y-5=\frac{3}{4}(x-8)$? Explain how you know.**

Though the points $(1,4)$ and $(8,5)$ are on the circle, the given equations cannot represent tangent lines. For the equation $-4=\frac{4}{3}(x-1)$, the slope of the tangent line is $\frac{4}{3}$. To be tangent, the slope of the radius must be $-\frac{3}{4}$, but the slope of the radius is $\frac{3}{4}$; therefore, the equation does not represent a tangent line. Similarly, for the second equation, the slope is $\frac{3}{4}$; to be tangent to the circle, the radius must have slope $-\frac{4}{3}$, but the slope of the radius is $\frac{4}{3}$. Neither of the given equations represents lines that are tangent to the circle.

Closing (3 minutes)

Have students summarize the main points of the lesson in writing, by talking to a partner, or as a whole class discussion. Use the questions below, if necessary.

- Describe how to find the equations of lines that are tangent to a given circle.
- Describe how to find the equation of a tangent line given a circle and a point outside of the circle.

Lesson Summary

THEOREMS:

A tangent line to a circle is perpendicular to the radius of the circle drawn to the point of tangency.

Relevant Vocabulary

TANGENT TO A CIRCLE: A *tangent line to a circle* is a line in the same plane that intersects the circle in one and only one point. This point is called the *point of tangency*.

Exit Ticket (5 minutes)

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Name ____ Date ____________________

Lesson 19: Equations for Tangent Lines to Circles

Exit Ticket

Consider the circle $(x+2)^2 + (y-3)^2 = 9$. There are two lines tangent to this circle having a slope of -1.

1. Find the coordinates of the two points of tangency.

2. Find the equations of the two tangent lines.

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Exit Ticket Sample Solutions

Consider the circle $(x+2)^2+(y-3)^2=9$. There are two lines tangent to this circle having a slope of -1.

1. Find the coordinates of the two points of tangency.

   $\left(\frac{3\sqrt{2}-4}{2}, \frac{3\sqrt{2}+6}{2}\right)$ *and* $\left(\frac{-3\sqrt{2}-4}{2}, \frac{-3\sqrt{2}+6}{2}\right)$ *or* ~$(0.12, 5.12)$ *and* $(-4.12, 0.88)$

2. Find the equations of the two tangent lines.

   $y-\frac{3\sqrt{2}+6}{2}=-\left(x-\frac{3\sqrt{2}-4}{2}\right)$ *and* $y-\frac{-3\sqrt{2}+6}{2}=-\left(x-\frac{-3\sqrt{2}-4}{2}\right)$ *or* $y-5.12=-(x-0.12)$ *and* $y-0.88=-(x+4.12)$

Problem Set Sample Solutions

1. Consider the circle $(x-1)^2+(y-2)^2=16$. There are two lines tangent to this circle having a slope of 0.

   a. Find the coordinates of the points of tangency.

      $(1,6)$ *and* $(1,-2)$

   b. Find the equations of the two tangent lines.

      $y=6$ *and* $y=-2$

2. Consider the circle $x^2-4x+y^2+10y+13=0$. There are two lines tangent to this circle having a slope of $\frac{2}{3}$.

   a. Find the coordinates of the two points of tangency.

      $\left(\frac{-8\sqrt{13}+26}{13}, \frac{12\sqrt{13}-65}{13}\right)$ *and* $\left(\frac{8\sqrt{13}+26}{13}, \frac{-12\sqrt{13}-65}{13}\right)$ *or* ~$(-0.2, -1.7)$ *and* $(4.2, -8.3)$

   b. Find the equations of the two tangent lines.

      $y-\frac{12\sqrt{13}-65}{13}=\frac{2}{3}\left(x+\frac{8\sqrt{13}+26}{13}\right)$ *and* $y+\frac{12\sqrt{13}-65}{13}=\frac{2}{3}\left(x-\frac{8\sqrt{13}+26}{13}\right)$

      or $y+1.7=\frac{2}{3}(x+0.2)$ *and* $y+8.3=\frac{2}{3}(x-4.2)$

3. What are the coordinates of the points of tangency of the two tangent lines through the point $(1,1)$ each tangent to the circle $x^2+y^2=1$?

   $(0,1)$ *and* $(1,0)$

4. What are the coordinates of the points of tangency of the two tangent lines through the point $(-1,-1)$ each tangent to the circle $x^2+y^2=1$?

   $(0,-1)$ *and* $(-1,0)$

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5. What is the equation of the tangent line to the circle $x^2 + y^2 = 1$ through the point $(6, 0)$?

$$y = -\frac{\sqrt{35}}{35}(x - 6)$$

6. D'Andre said that a circle with equation $(x - 2)^2 + (y - 7)^2 = 13$ has a tangent line represented by the equation $y - 5 = -\frac{3}{2}(x + 1)$. Is he correct? Explain.

Yes, D'Andre is correct. The point $(-1, 5)$ is on the circle, and the slopes are negative reciprocals.

MP.3

7. Kamal gives the following proof that $y - 1 = \frac{8}{9}(x + 10)$ is the equation of a line that is tangent to a circle given by $(x + 1)^2 + (y - 9)^2 = 145$.

   The circle has center $(-1, 9)$ and radius 12. The point $(-10, 1)$ is on the circle because

   $$(-10 + 1)^2 + (1 - 9)^2 = (-9)^2 + (-8)^2 = 145.$$

   The slope of the radius is $\frac{9-1}{-1-10} = \frac{8}{9}$; therefore, the equation of the tangent line is $y - 1 = \frac{8}{9}(x + 10)$.

   a. Kerry said that Kamal has made an error. What was Kamal's error? Explain what he did wrong.

   Kamal used the slope of the radius as the slope of the tangent line. To be tangent, the slopes must be negative reciprocals of one another, not the same.

   b. What should the equation for the tangent line be?

   $y - 1 = -\frac{9}{8}(x + 10)$

8. Describe a similarity transformation that maps a circle given by $x^2 + 6x + y^2 - 2y = 71$ to a circle of radius 3 that is tangent to both axes in the first quadrant.

The given circle has center $(-3, 1)$ and radius 9. A circle that is tangent to both axes in the first quadrant with radius 3 must have a center at $(3, 3)$. Then, a translation of $x^2 + 6x + y^2 - 2y = 71$ along a vector from $(-3, 1)$ to $(3, 3)$, and a dilation by a scale factor of $\frac{1}{3}$ from the center $(3, 3)$ will map circle $x^2 + 6x + y^2 - 2y = 71$ onto the circle in the first quadrant with radius 3.

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