Lesson 12: Tangent Segments

Student Outcomes

- Students use *tangent segments* and *radii of circles* to conjecture and prove geometric statements, especially those that rely on the congruency of tangent segments to a circle from a given point.
- Students recognize and use the fact if a circle is tangent to both rays of an angle, then its center lies on the *angle bisector*.

Lesson Notes

The common theme of all the lesson activities is tangent segments and radii of circles can be used to conjecture and prove geometric statements.

Students first conjecture and prove that if a circle is tangent to both rays of an angle, then its center lies on the angle bisector. After extrapolating that every point on an angle bisector can be the center of a circle tangent to both rays of the angle, students show that there exists a circle simultaneously tangent to two angles with a common side. Finally, students conjecture and prove that the three angle bisectors of a triangle intersect at a single point and prove that this single point is the center of a circle inscribed in the triangle.

Classwork

Opening Exercise (5 minutes)

Students apply the theorem from Lesson 11, two segments tangent to a circle from a point outside the circle are congruent. This theorem will be used in proofs of this lesson's main results. Once you state the theorem below and ask the question, review Exercise 1 from Lesson 11, or have students discuss the proof to make sure students understand the theorem.

Scaffolding:

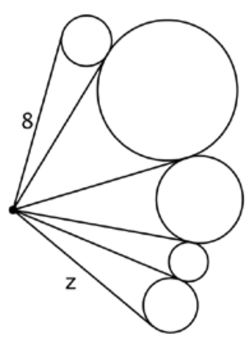
- Give students the theorem from Lesson 11 if they need help getting started.
- **THEOREM:** Two segments tangent to a circle from an exterior point are congruent.

MP.7

Opening Exercise

In the diagram to the right, what do you think the length of z could be? How do you know?

$z = 8$ because each successive segment is tangent to the same circle so the segments are congruent.



- Can someone say, in your own words, the theorem used to determine z?
  - *Students explain the theorem in their own words.*

- How did you use this theorem to find z?
  - *Each successive segment was tangent to the same circle, so they were congruent.*
- How many times did you use this theorem?
  - 5 *times*
- Summary: By applying this theorem and transitivity over and over again, you found that each of the tangent segments has equal length, so $z = 8$.

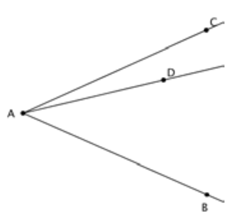
Example 1 (7 minutes)

The point of this example is to understand why the following statement holds: *If a circle is tangent to both rays of an angle, then its center lies on the angle bisector.* Students explore this statement by investigating the contrapositive: attempting to draw such circles when the center is not on the angle bisector and reasoning why this cannot be done.

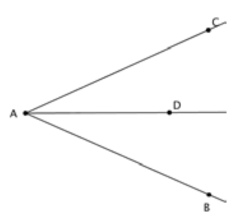
Example 1

In each diagram, try to draw a circle with center D that is tangent to both rays of the angle $\angle BAC$.

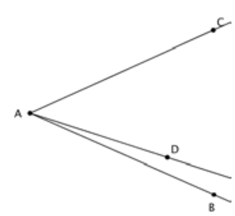
a.



b.



c.



Which diagrams did it seem impossible to draw such a circle? Why did it seem impossible?

(a) Impossible; (b) possible; (c) impossible. It's not possible to draw a circle tangent to both rays of the angle in (a) and (c) because D is a different distance from the two sides of the angle. The distance is the radius of a circle tangent to that side.

MP.3

What do you conjecture about circles tangent to both rays of an angle? Why do you think that?

If a circle is tangent to both rays of an angle, then the center lies on the angle bisector. We saw this in all examples.

- How many people were able to draw a circle for (a)? (b)? (c)?
  - *It was not possible to draw such a circle for (a) and (c), but it did seem possible for (b).*
- What is special about (b) that wasn't true for (a) and (c)?
  - *The point D was more in the middle of the angle in (b) than in (a) or (c).*
  - *The point D is on the angle bisector.*
- Why did this make a difference?
  - *You couldn't make a circle that was tangent to both sides at the same time because the center was too far or too close to one side of the angle.*

- (If students do not see that D must be on the angle bisector) Suppose you were given a different angle. Create different angles or have students draw some samples. Which of these points would you pick as the center of a circle tangent to both rays? How about these? How about these? What do you notice about the possible centers? (Draw two points, one that could be on the angle bisector and one that is obviously not; continue giving different examples until it is apparent that the set of viable centers is likely the angle bisector.)
- State what we have just discovered to your neighbor.
  - **CONJECTURE:** *If a circle is tangent to both rays of an angle, then the center of the circle lies on the angle bisector.*

Exercises 1–5 (25 minutes)

Allow students to work in pairs or groups to complete the exercises. You may assign certain groups particular problems and call the class together to share results. Some groups may need more guidance on these exercises.

Students first prove the conjecture made in Example 1, and it becomes a theorem. This theorem allows us to resolve the mystery opened in the last lesson: Does every triangle have an inscribed circle? Exercises 2–5 trace the mathematical steps from the proof of the Example 1 conjecture to the construction of the circle inscribed in a given triangle.

Throughout these exercises, emphasize that the definition of *angle bisector* is the set of points equidistant from the rays of an angle. Make sure students understand this means *both* that given any point on the angle bisector, the perpendiculars dropped from this point to the rays of the angle must be the same length *and* that if the dropped perpendiculars are the same length, then the point from which the perpendiculars are dropped must be on the angle bisector. These observations are critical for all the exercises and especially for Exercises 3–5.

Exercise 1 notes. The point of Exercise 1 is the following theorem.

THEOREM: *If a circle is tangent to both rays of an angle, then its center lies on the angle bisector.*

To get at the proof of this theorem, you might first poll students as to whether they used congruent triangles to show the conjecture; ask which triangles. If students ask how this proof is different from the Opening Exercise, you can point out that both proofs use the same pair of congruent triangles to draw their conclusions. However, in the Opening Exercise, the application of CPCTC (corresponding parts of congruent triangles are congruent) is for showing two legs are congruent, whereas here is it for two angles.

Exercise 2 notes. The point of Exercise 2 is the following construction, whose mathematical validity is a consequence of the theorem proven in Exercise 1.

CONSTRUCTION: The following constructs a circle tangent to both rays of a given angle: (1) Construct the angle bisector of the given angle; (2) Select a point of the bisector to be the center; (3) Drop a perpendicular from the selected point to the angle to find the radius; (4) Construct a circle with that center and radius.

A discussion could proceed: How did you construct the center of the circle? The radius? (Select a center from the points on the angle bisector; drop a perpendicular to find the radius.) We sometimes say that the angle bisector is the set of points equidistant from the two rays of the angle. How do you know that the distance from each of your centers to the two rays of the angle is the same? (If P is the center of a circle with tangents to the circle at C and B, then PC and PB are radii of the circle, and all radii have the same length) Why do we know that the radius is the distance from the center to the tangent line? (The radii are perpendicular to the rays.)

Exercise 3 notes. The point of Exercise 3 is applying Exercise 2 to the condition that the desired circle must have a center lying on angle bisectors of two angles. The key points of the argument are (1) finding a potential center of the circle, (2) finding a potential radius of the circle, and (3) establishing that a circle with this center and radius is tangent to both rays of both angles.

The reasoning for the key points could be the following: (1) The angle bisectors of two angles intersect in only one point, and if there is such a circle, the center of that circle must be the intersection point. (2) A potential radius is the length of the perpendicular segment from this center to the common side shared by the angles; a line intersects a circle in one point if and only if it is perpendicular to the radius at that point. (3) The circle with this radius and center is tangent to both rays of both angles, and it is the only circle. This is because the angle bisector is the set of points equidistant from the two sides, so the perpendicular segments from the potential center to the rays of the angle are all congruent (since the distance is defined as the length of the perpendicular segment). There is only one such circle because there is only one intersection point, and the distance from this point to the rays of the angle is well defined.

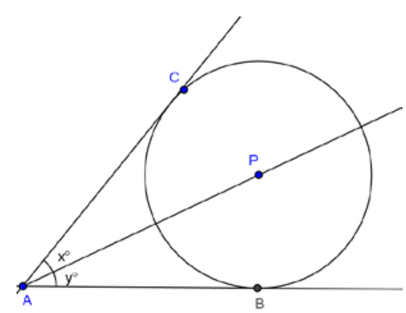
Exercise 4 notes. The point of Exercise 4 is extending the reasoning from Exercise 3 to conclude that all three-angle bisectors of a triangle meet at a single point, so finding the intersection of any two points suffices. Central to Exercise 4 is the definition of angle bisector. The key idea of the argument is that the intersection point of the angle bisectors of any two consecutive angles is the same distance from both rays of both angles.

Exercise 5 notes. Exercise 5 applies Exercise 4.

Exercises 1–5

1. **You conjectured that *if a circle is tangent to both rays of a circle, then the center lies on the angle bisector.***

   a. **Rewrite this conjecture in terms of the notation suggested by the diagram.**



   Given:

   Circle with center P and tangent to AB and AC; $x = m\angle CAP$; $y = \angle BAP$

   Need to show:

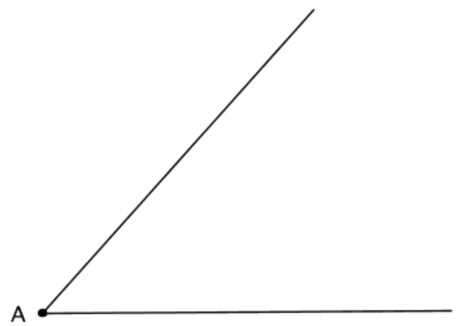
   $x = y$

   b. **Prove your conjecture using a two column proof.**

$m\angle CAP = \text{x}, m\angle BAP = \text{y}$	Given
$\overline{PC} \perp \overline{AC}, \overline{PB} \perp \overline{AB}$	*Tangent is perpendicular to the radius at the point of tangency.*
ΔABP and ΔACP are right triangles	*Definition of right triangle*
$AP = AP$	*Reflexive property*
$PB = PC$	*Radii of the same circle are equal in length.*
$\Delta ABP \cong \Delta ACP$	*HL*
$m\angle BAP = m\angle CAP$	*CPCTC*
$x = y$	*Substitution*
$\overline{AP}$ *bisects* $\angle CAB$	*Definition of angle bisector*

2. An angle is shown below.

   a. Draw at least 3 different circles that are tangent to both rays of the given angle.

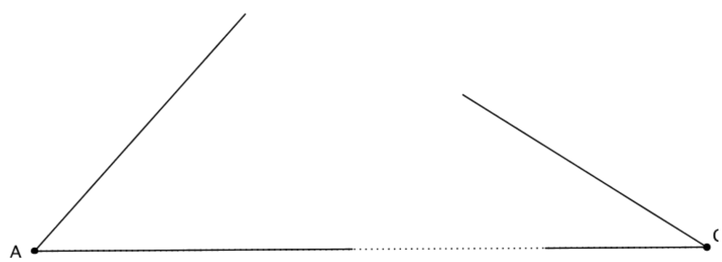


Many circles are possible. The main idea is that the center must be a point on the angle bisector, and the radius is the perpendicular from the center to the rays of the angle.

   b. Label the center of one of your circles with P. How does the distance between P and the rays of the angle compare to the radius of the circle? How do you know?

The distance between P and the rays of the angle is the same length as the radius of the circle because it is perpendicular to tangent segments as proved last lesson.

3. Construct as many circles as you can that are tangent to both the given angles at the same time. You can extend the rays as needed. These two angles share a side.



Explain how many circles you can draw to meet the above conditions and how you know.

There is only one circle. The center of the circle has to be on the angle bisector of each angle; the angle bisectors only intersect in one point. There is only one circle with that point as a center that is tangent to the rays. It is the one with the radius you get by dropping the perpendicular from the point to the side A.

4. In a triangle, let P be the location where two angle bisectors meet. Must P be on the third angle bisector as well? Explain your reasoning.

The angle bisectors of any two consecutive angles are the same distance from both rays of both angles. Let the vertices of the triangle be A, B, and C where $m\angle A = 2x$, $m\angle B = 2y$, and $\angle C = 2z$. If point P is on the angle bisector of $\angle A$ and $\angle B$, then it the same distance from AB and AC, and it is the same distance from BA and BC. Thus, it is the same distance from CA and CB; therefore, it is on the angle bisector of $\angle C$.

5. **Using a straightedge, draw a large triangle ABC.**

a. **Construct a circle inscribed in the given triangle.**

Construct angle bisectors between any two angles. Their intersection point will be the center of the circle. Drop a perpendicular from the intersection point to any side. This will be a radius. Draw the circle with that center and radius.

b. **Explain why your construction works.**

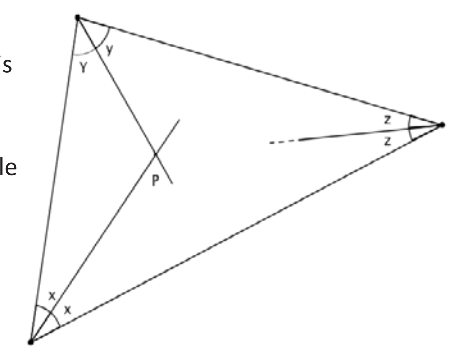
The point P from Exercise 4 is the same distance from all three sides of ΔABC. Thus, the perpendicular segments from P to each side are the same length. A line is tangent to a circle if and only if the line is perpendicular to a radius where the radius meets the circle. Therefore, a circle with center P and radius the length of the perpendicular segments from P to the sides is inscribed in the triangle.

c. **Do you know another name for the intersection of the angle bisectors in relation to the triangle?**

The intersection of the angle bisectors is the incenter of the triangle.

Closing (3 minutes)

Have a whole-class discussion of the topics studied today. Also, go through any questions that came up during the exercises.



Today we saw why any point on an angle bisector is the center of a circle that is tangent to both rays of an angle but that any point that's not on the angle bisector cannot possibly be the center of such a circle. This mean that we can construct a circle tangent to both rays of an angle by first constructing the angle bisector, selecting a point on it, and then dropping a perpendicular to find the radius. We put this all together to solve a mystery we raised yesterday:

- Do all triangles have inscribed circles?
  - *We found the answer is yes!*

The theme of all our constructions today, and for the homework, is that tangent segments and radii of circles are incredibly useful for conjecturing and proving geometric statements.

- What did we discover from our constructions today?
  - *The two tangent segments to a circle from an exterior point are congruent.*
  - *If a circle is tangent to both rays of an angle, then its center lies on the angle bisector.*
  - *Every triangle contains an inscribed circle whose center is the intersection of the triangle's angle bisectors.*

Lesson Summary

THEOREMS:

- The two tangent segments to a circle from an exterior point are congruent.
- If a circle is tangent to both rays of an angle, then its center lies on the angle bisector.
- Every triangle contains an inscribed circle whose center is the intersection of the triangle's angle bisectors.

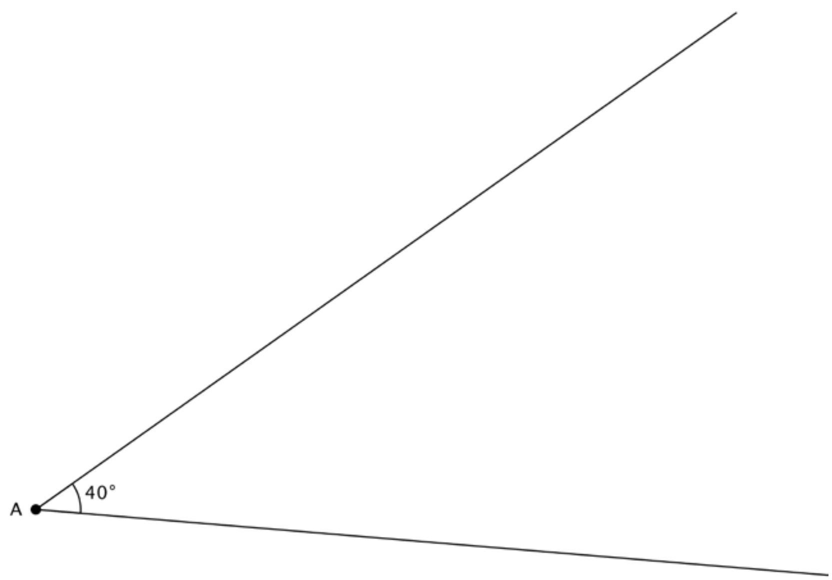
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 12: Tangent Segments

Exit Ticket

1. Draw a circle tangent to both rays of this angle.



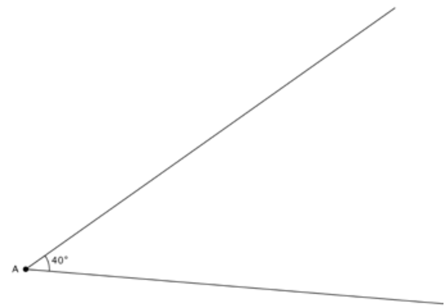
2. Let B and C be the points of tangency of your circle. Find the measures of $\angle ABC$ and $\angle ACB$. Explain how you determined your answer.

3. Let P be the center of your circle. Find the measures of the angles in ΔAPB.

Exit Ticket Sample Solutions

1. **Construct a circle tangent to both rays of this angle.**

   Lots of circles are possible. Check that the center of the circle is on the angle bisector and the radius is from a perpendicular from the center to a ray of the angle.



2. **Let B and C be the points of tangency of your circle. Find the measures of $\angle ABC$ and $\angle ACB$. Explain how you determined your answer.**

   Triangle ΔABC is isosceles because the tangent segments AB and AC are congruent, so

   $$m\angle ABC = m\angle ACB = \frac{180^\circ - 40^\circ}{2} = 70^\circ$$

3. **Let P be the center of your circle. Find the measures of the angles in ΔAPB.**

   In triangle ΔAPB, the angle at A measures 20°, at P measures 70°, and at B is 90°.

Problem Set Sample Solutions

It is recommended to assign Problem 8. This problem is used to open Lesson 12.

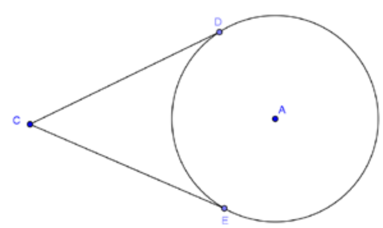
Problems 1 – 7 rely heavily on the fact that *two tangents from a given exterior point are congruent* and, hence, that *if a circle is tangent to both rays of an angle, then the center of the circle lies on the angle bisector*; these problems may also do arithmetic on lengths of tangent segments.

Problem 8 examines angles between tangent segments and chords.

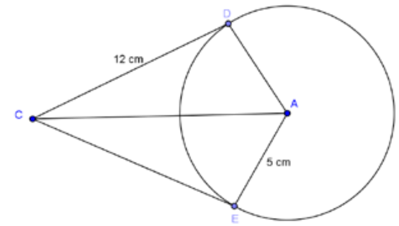
Problems involving proofs may take a while, so they can be assigned as student choice.

1. **On a piece of paper, draw a circle with center A and a point, C, outside of the circle.**
   a. **How many tangents can you draw from C to the circle?**

      2 tangents.



b. Draw two tangents from C to the circle, and label the tangency points D and E. Fold your paper along the line AC. What do you notice about the lengths of CD and CE? About the measures of the angles $\angle DCA$ and $\angle ECA$?



The lengths are the same; the angles are congruent.

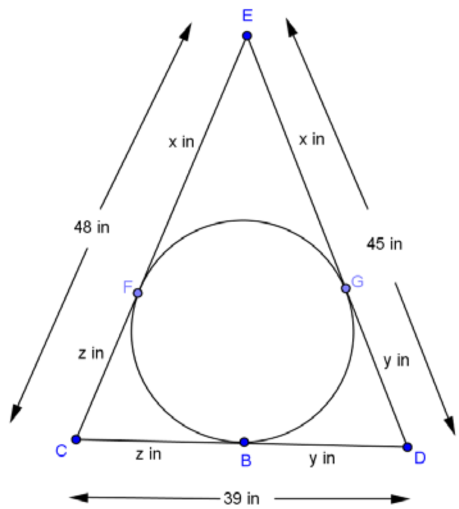
c. AC is the ____________________ of $\angle DCE$.

Angle bisector

d. $\overline{CD}$ and $\overline{CE}$ are tangent to circle A. Find AC.

$AC = 13$ by the Pythagorean theorem.

2. In the figure at right, the three segments are tangent to the circle at points B, F, and G. If $y = \frac{2}{3}x$, find x, y, and z.



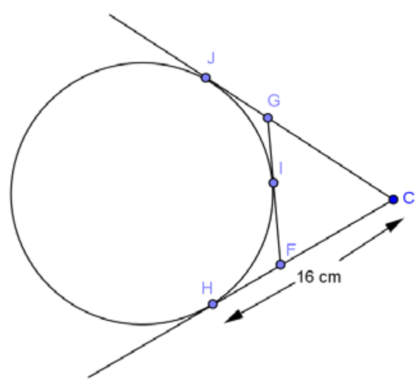
Tangents to a circle from a given point are congruent. So $EF = EG = x$, $GD = BD = y$, and $BC = CF = z$. This allows us to set up a system of simultaneous linear equations that can be solved for $x, y,$ and z.

$x + z = 48, x + y = 45, y + z = 39$

$x = 27, y = 18, z = 21$

3. In the figure given, the three segments are tangent to the circle at points J, I, and H.

a. Prove $GF = GJ + HF$



$GJ = GI$ and $FI = FH$ because tangents to a circle from a given point are congruent.

$GF = GI + IF$ sum of segments

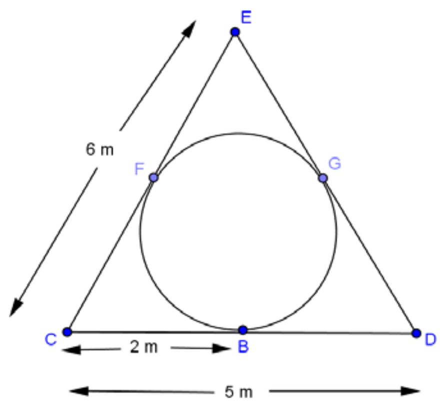
$GF = GJ + HF$ by substitution

b. Find the perimeter of ΔGCF.

$CH = 16$
$GJ = GI, IF = FH, CJ = CH$ tangents to a circle from a given point are congruent.
$CJ = GC + GJ, CH = CF + FH$ sum of segments
Perimeter $= GC + CF + GF$
$GC + CF + (GI + IF)$ by substitution
$GC + GI + CF + IF$
$GC + GJ + CF + FH$ by substitution
$CJ + CH$ by substitution
$2CH = 2(16\ cm) = 32\ cm$

4. In the given figure, the three segments are tangent to the circle at point F, B, and G. Find DE.

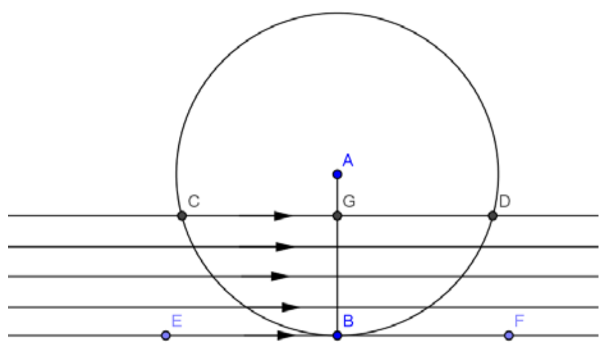
7 m



5. $\overleftrightarrow{EF}$ is tangent to circle A. If points C and D are the intersection points of circle A and any line parallel to $\overleftrightarrow{EF}$, answer the following.

a. Does $CG = GD$ for any line parallel to $\overleftrightarrow{EF}$? Explain.

Yes. No matter what, ΔACD is an isosceles triangle, and AG is the altitude to the base of the triangle and, therefore, the angle bisector of the angle opposite the base.



b. Suppose that $\overleftrightarrow{CD}$ coincides with $\overleftrightarrow{EF}$. Would C, G, and D all coincide with B?

Yes. If they approached B at different times, then at some point G would not be on the same line as C and D, but G is defined to be contained in the chord $\overline{CD}$.

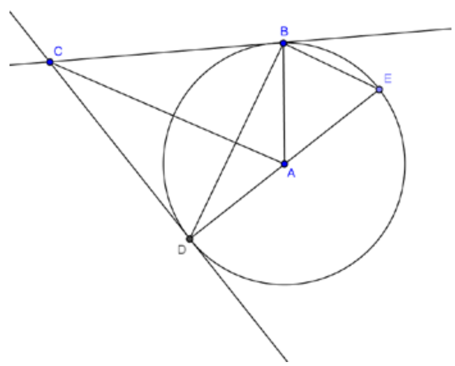
c. Suppose C, G, and D have now reached B, so $\overleftrightarrow{CD}$ is tangent to the circle. What is the angle between the line $\overleftrightarrow{CD}$ and $\overline{AB}$?

$90°$. A line tangent to a circle is perpendicular to the radius at the point of tangency.

d. Draw another line tangent to the circle from some point, P, in the exterior of the circle. If the point of tangency is point T, what is the measure of $\angle PTA$?

The measure of angle PTA is $90°$ because a line tangent to a circle is perpendicular to a radius through the tangency point.

6. The segments are tangent to circle A at points B and D. $\overline{ED}$ is a diameter of the circle.



a. Prove $\overline{BE} \| \overline{CA}$.

$m\angle E = \frac{1}{2} m\angle BAD$	*Inscribed angle is half the central angle that intercepts the same arc.*
$\overline{AC}$ *bisects* $\angle BAD$	*If a circle is tangent to both rays of an angle, then its center lies on the angle bisector.*
$m\angle E = m\angle BAC$	*Substitution*
$\overline{BE} \| \overline{CA}$	*If two lines are cut by a transversal such that the alternate interior angles are congruent, then the lines are parallel.*

b. Prove quadrilateral $ABCD$ is a kite.

$CB = CD$	*tangents to a circle from a given point are congruent.*
$AB = AD$	*radii are congruent.*

Quadrilateral $ABCD$ is a kite because two pairs of adjacent sides are congruent.

7. In the diagram shown, $\overleftrightarrow{BH}$ is tangent to the circle at point B. What is the relationship between $\angle DBH$, the angle between the tangent and a chord, and the arc subtended by that chord and its inscribed angle $\angle DCB$?

$m\angle DBH = m\angle DCB$