Lesson 2: Circles, Chords, Diameters, and Their Relationships

Student Outcomes

- Identify the relationships between the diameters of a circle and other chords of the circle.

Lesson Notes

Students are asked to construct the perpendicular bisector of a line segment and draw conclusions about points on that bisector and the endpoints of the segment. They relate the construction to the theorem stating that any perpendicular bisector of a chord must pass through the center of the circle.

Classwork

Opening Exercise (4 minutes)

Opening Exercise

Construct the perpendicular bisector of line segment $\overline{AB}$ below (as you did in Module 1).

Draw another line that bisects $\overline{AB}$ but is not perpendicular to it.

List one similarity and one difference between the two bisectors.

Answers will vary. All points on the perpendicular bisector are equidistant from points A and B.. Points on the other bisector are not equidistant from points A and B. The perpendicular bisector meets $\overline{AB}$ at right angles. The other bisector meets at angles that are not congruent.

Scaffolding:

Post a diagram and display the steps to create a perpendicular bisector used in Lesson 4 of Module 1.

- Label the endpoints of the segment A and B.
- Draw circle A with center A and radius $\overline{AB}$.
- Draw circle B with center B and radius $\overline{BA}$.
- Label the points of intersection as C and D.
- Draw $\overleftrightarrow{CD}$.

You may wish to recall for students the definition of *equidistant*:

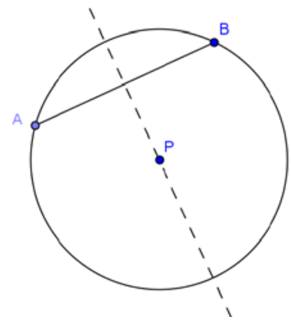
EQUIDISTANT:. A point A is said to be *equidistant* from two different points B and C if $AB = AC$.

Points B and C can be replaced in the definition above with other figures (lines, etc.) as long as the distance to those figures is given meaning first. In this lesson, we will define the distance from the center of a circle to a chord. This definition will allow us to talk about the center of a circle as being equidistant from two chords.

Discussion (12 minutes)

Ask students independently or in groups to each draw chords and describe what they notice. Answers will vary depending on what each student drew.

Lead students to relate the perpendicular bisector of a line segment to the points on a circle, guiding them toward seeing the relationship between the perpendicular bisector of a chord and the center of a circle.

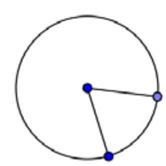


- Construct a circle of any radius, and identify the center as point P.
- Draw a chord, and label it $\overline{AB}$.
- Construct the perpendicular bisector of $\overline{AB}$.
- What do you notice about the perpendicular bisector of $\overline{AB}$?
  - *It passes through point P, the center of the circle.*
- Draw another chord and label it $\overline{CD}$.
- Construct the perpendicular bisector of $\overline{CD}$.
- What do you notice about the perpendicular bisector of $\overline{CD}$?
  - *It passes through point P, the center of the circle.*
- What can you say about the points on a circle in relation to the center of the circle?
  - *The center of the circle is equidistant from any two points on the circle.*

MP.3 & MP.7

- Look at the circles, chords, and perpendicular bisectors created by your neighbors. What statement can you make about the perpendicular bisector of any chord of a circle? Why?
  - *It must contain the center of the circle. The center of the circle is equidistant from the two endpoints of the chord because they lie on the circle. Therefore, the center lies on the perpendicular bisector of the chord. That is, the perpendicular bisector contains the center.*
- How does this relate to the definition of the perpendicular bisector of a line segment?
  - *The set of all points equidistant from two given points (endpoints of a line segment) is precisely the set of all points on the perpendicular bisector of the line segment.*

Scaffolding:
- Review the definition of *central angle* by posting a visual guide.
- A central angle of a circle is an angle whose vertex is the center of a circle.



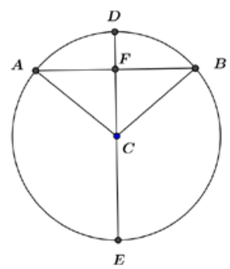
Exercises 1–6 (20 minutes)

Assign one proof to each group, and then jigsaw, share, and gallery walk as students present their work.

Exercises 1–6

1. **Prove the theorem:** ***If a diameter of a circle bisects a chord, then it must be perpendicular to the chord.***

Draw a diagram similar to that shown below.



Given: Circle C with diameter $\overline{DE}$, chord $\overline{AB}$, and $AF = BF$.

Prove: $\overline{DE} \perp \overline{AB}$

$AF = BF$	*Given*
$FC = FC$	*Reflexive property*
$AC = BC$	*Radii of the same circle are equal in measure*
$\triangle AFC \cong \triangle BFC$	*SSS*
$m\angle AFC = m\angle BFC$	*Corresponding angles of congruent triangles are equal in measure*
$\angle AFC$ *and* $\angle BFC$ *are right angles*	*Equal angles that form a linear pair each measure* $90°$
$\overline{DE} \perp \overline{AB}$	*Definition of perpendicular lines*

OR

$AF = BF$	*Given*
$AC = BC$	*Radii of the same circle are equal in measure*
$m\angle FAC = m\angle FBC$	*Base angles of an isosceles are equal in measure*
$\triangle AFC \cong \triangle BFC$	*SAS*
$m\angle AFC = m\angle BFC$	*Corresponding angles of congruent triangles are equal in measure*
$\angle AFC$ *and* $\angle BFC$ *are right angles*	*Equal angles that form a linear pair each measure* $90°$
$\overline{DE} \perp \overline{AB}$	*Definition of perpendicular lines*

2. Prove the theorem: ***If a diameter of a circle is perpendicular to a chord, then it bisects the chord.***

Use a diagram similar to that in Exercise 1 above.

Given: Circle C with diameter $\overline{DE}$, chord $\overline{AB}$, and $\overline{DE} \perp \overline{AB}$

Prove: $\overline{DE}$ bisects $\overline{AB}$

$\overline{DE} \perp \overline{AB}$	*Given*
$\angle AFC$ and $\angle BFC$ are right angles	*Definition of perpendicular lines*
$\triangle AFC$ and $\triangle BFC$ are right triangles	*Definition of right triangle*
$\angle AFC \cong \angle BFC$	*All right angles are congruent*
$FC = FC$	*Reflexive property*
$AC = BC$	*Radii of the same circle are equal in measure*
$\triangle AFC \cong \triangle BFC$	*HL*
$AF = BF$	*Corresponding sides of congruent triangles are equal in length*
$\overline{DE}$ bisects $\overline{AB}$	*Definition of segment bisector*

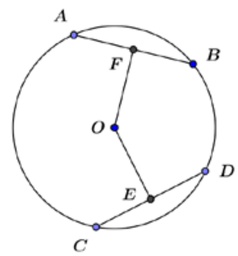
OR

$\overline{DE} \perp \overline{AB}$	*Given*
$\angle AFC$ and $\angle BFC$ are right angles	*Definition of perpendicular lines*
$\angle AFC \cong \angle BFC$	*All right angles are congruent*
$AC = BC$	*Radii of the same circle are equal in measure*
$m\angle FAC = m\angle FBC$	*Base angles of an isosceles triangle are congruent*
$m\angle ACF = m\angle BCF$	*Two angles of triangle are equal in measure, so third angles are equal*
$\triangle AFC \cong \triangle BFC$	*ASA*
$AF = BF$	*Corresponding sides of congruent triangles are equal in length*
$\overline{DE}$ bisects $\overline{AB}$	*Definition of segment bisector*

3. **The distance from the center of a circle to a chord is defined as the length of the perpendicular segment from the center to the chord. Note that, since this perpendicular segment may be extended to create a diameter of the circle, therefore, the segment also bisects the chord, as proved in Exercise 2 above.**

Prove the theorem: *In a circle, if two chords are congruent, then the center is equidistant from the two chords.*

Use the diagram below.



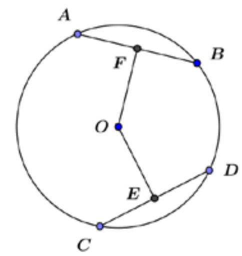
Given: Circle O with chords $\overline{AB}$ and $\overline{CD}$; $AB = CD$; F is the midpoint of $\overline{AB}$ and E is the midpoint of CD.

Prove: $OF = OE$

$AB = CD$	*Given*
$\overline{OF} \perp \overline{AB}$; $\overline{OE} \perp \overline{CD}$	*If a diameter of a circle bisects a chord, then the diameter must be perpendicular to the chord.*
$\angle AFO$ *and* $\angle DEO$ *are right angles*	*Definition of perpendicular lines*
$\triangle AFO$ *and* $\triangle DEO$ *are right triangles*	*Definition of right triangle*
E and F are midpoints of $\overline{CD}$ *and* $\overline{AB}$	*Given*
$AF = DE$	$AB = CD$ *and* F *and* E *are midpoints of* $\overline{AB}$ *and* $\overline{CD}$
$AO = DO$	*All radii of a circle are equal in measure*
$\triangle AFO \cong \triangle DEO$	*HL*
$OE = OF$	*Corresponding sides of congruent triangles are equal in length*

4. Prove the theorem: ***In a circle, if the center is equidistant from two chords, then the two chords are congruent.***

Use the diagram below.



Given: Circle O with chords $\overline{AB}$ and $\overline{CD}$; $OF = OE$; F is the midpoint of $\overline{AB}$ and E is the midpoint of CD

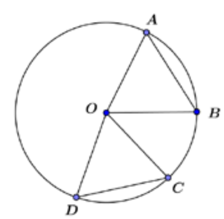
Prove: $AB = CD$

$OF = OE$	*Given*
$\overline{OF} \perp \overline{AB}$; $\overline{OE} \perp \overline{CD}$	*If a diameter of a circle bisects a chord, then it must be perpendicular to the chord.*
$\angle AFO$ *and* $\angle DEO$ *are right angles*	*Definition of perpendicular lines.*
$\triangle AFO$ *and* $\triangle DEO$ *are right triangles*	*Definition of right triangle.*
$AO = DO$	*All radii of a circle are equal in measure.*
$\triangle AFO \cong \triangle DEO$	*HL*
$AF = DE$	*Corresponding sides of congruent triangles are equal in length.*
F is the midpoint of $\overline{AB}$, and E is the midpoint of CD	*Given*
$AB = CD$	*$AF = DE$ and F and E are midpoints of $\overline{AB}$ and $\overline{CD}$.*

5. **A central angle defined by a chord is an angle whose vertex is the center of the circle and whose rays intersect the circle. The points at which the angle's rays intersect the circle form the endpoints of the chord defined by the central angle.**

Prove the theorem: ***In a circle, congruent chords define central angles equal in measure.***

Use the diagram below.



We are given that the two chords ($\overline{AB}$ and $\overline{CD}$) are congruent. Since all radii of a circle are congruent, $\overline{AO} \cong \overline{BO} \cong \overline{CO} \cong \overline{DO}$. Therefore, $\triangle ABO \cong \triangle CDO$ by SSS. $\angle AOB \cong \angle COD$ since corresponding angles of congruent triangles are equal in measure.

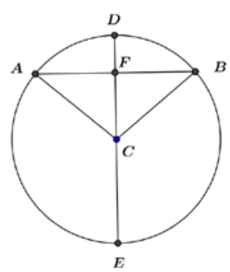
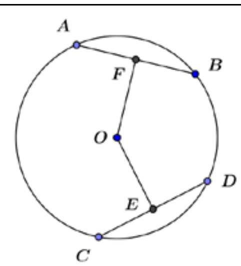
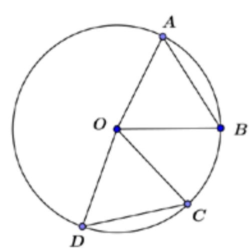
6. **Prove the theorem: In a circle, if two chords define central angles equal in measure, then they are congruent.**

Using the diagram from Exercise 5 above, we now are given that $m\angle AOB = m\angle COD$. Since all radii of a circle are congruent, $\overline{AO} \cong \overline{BO} \cong \overline{CO} \cong \overline{DO}$. Therefore, $\triangle ABO \cong \triangle CDO$ by SAS. $\overline{AB} \cong \overline{CD}$ because corresponding sides of congruent triangles are congruent

Closing (4 minutes)

Have students write all they know to be true about the diagrams below. Bring the class together, go through the Lesson Summary, having students complete the list that they started, and discuss each point.

A reproducible version of the graphic organizer shown is included at the end of the lesson.

Diagram	Explanation of Diagram	Theorem or Relationship
	*Diameter of a circle bisecting a chord*	*If a diameter of a circle bisects a chord, then it must be perpendicular to the chord.* *If a diameter of a circle is perpendicular to a chord, then it bisects the chord.*
	*Two congruent chords equidistance from center*	*If two chords are congruent, then the center is equidistant from the two chords.* *If the center is equidistant from two chords, then the two chords are congruent.*
	*Congruent chords*	*Congruent chords define central angles equal in measure.* *If two chords define central angles equal in measure, then they are congruent.*

Lesson Summary

Theorems about chords and diameters in a circle and their converses

- **If a diameter of a circle bisects a chord, then it must be perpendicular to the chord.**
- **If a diameter of a circle is perpendicular to a chord, then it bisects the chord.**
- **If two chords are congruent, then the center is equidistant from the two chords.**
- **If the center is equidistant from two chords, then the two chords are congruent.**
- **Congruent chords define central angles equal in measure.**
- **If two chords define central angles equal in measure, then they are congruent.**

Relevant Vocabulary

EQUIDISTANT: A point A is said to be *equidistant* from two different points B and C if $AB = AC$.

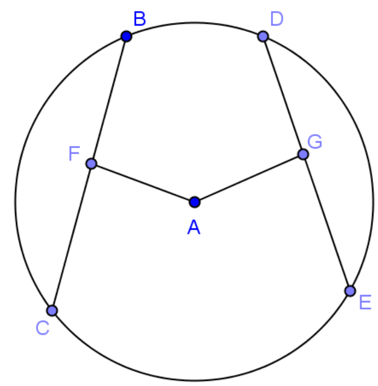
Exit Ticket (5 minutes)

Name ____ Date ________________

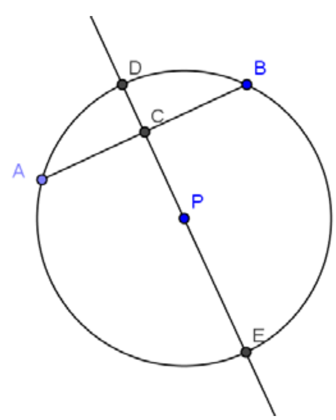
Lesson 2: Circles, Chords, Diameters, and Their Relationships

Exit Ticket

1. Given circle A shown, $AF = AG$ and $BC = 22$. Find DE.



2. In the figure, circle P has a radius of 10. $\overline{AB} \perp \overline{DE}$.
   a. If $AB = 8$, what is the length of AC?

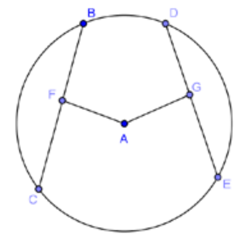


   b. If $DC = 2$, what is the length of AB?

Exit Ticket Sample Solutions

1. Given circle A shown, $AF = AG$ and $BC = 22$. Find DE.

   22



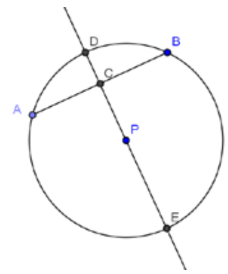
2. In the figure, circle P has a radius of 10. $\overline{AB} \perp \overline{DE}$.

   a. If $AB = 8$, what is the length of AC?

      4

   b. If $DC = 2$, what is the length of AB?

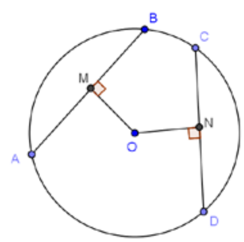
      12



Problem Set Sample Solutions

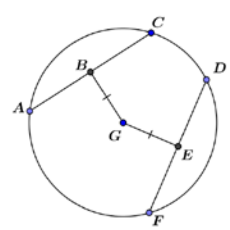
1. In this drawing, $AB = 30$, $OM = 20$, and $ON = 18$. What is CN?

   $2\sqrt{301}$ *(≈ 34.7)*



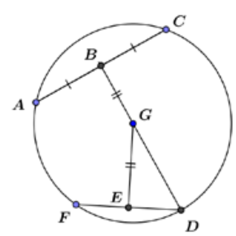
2. In the figure to the right, $\overline{AC} \perp \overline{BG}$ and $\overline{DF} \perp \overline{EG}$; $EF = 12$. Find AC.

   24

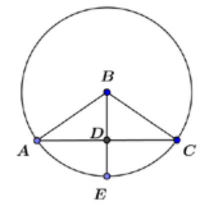


3. In the figure, $AC = 24$, and $DG = 13$. Find EG. Explain your work.

   5, $\triangle ABG$ is a right triangle with hypotenuse $=$ radius $= 13$ and $AB = 12$, so $BG = 5$ by Pythagorean theorem. $BG = GE = 5$.



4. In the figure, $AB = 10$, $AC = 16$. Find DE.



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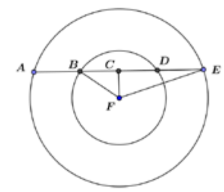
5. In the figure, $CF = 8$, and the two concentric circles have radii of 10 and 17. Find DE.



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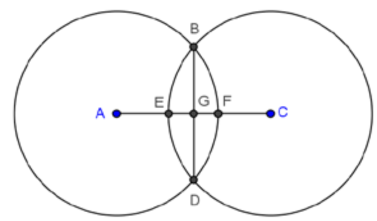
6. In the figure, the two circles have equal radii and intersect at points B and D. A and C are centers of the circles. $AC = 8$, and the radius of each circle is 5. $\overline{BD} \perp \overline{AC}$. Find BD. Explain your work.



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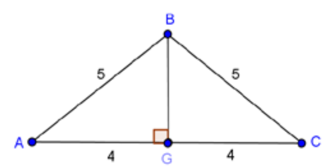
$BA = BC = 5$ *(radii)*

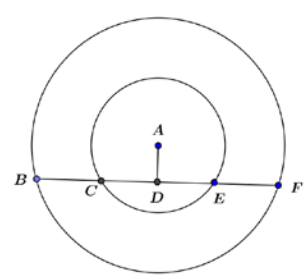
$AG = GC = 4$

$BG = 3$ *(Pythagorean theorem)*

$BD = 6$



7. In the figure, the two concentric circles have radii of 6 and 14. Chord $\overline{BF}$ of the larger circle intersects the smaller circle at C and E. $CE = 8$. $\overline{AD} \perp \overline{BF}$.



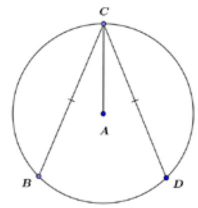
a. Find AD.

$2\sqrt{5}$

b. Find BF.

$8\sqrt{11}$

8. In the figure, A is the center of the circle, and $CB = CD$. Prove that $\overline{AC}$ bisects $\angle BCD$.

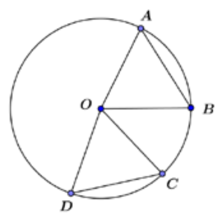


Let $\overline{AE}$ and $\overline{AF}$ be perpendiculars from A to $\overline{CB}$ and $\overline{CD}$, respectively.

$CB = CD$	*Given*
$AE = AF$	*If two chords are congruent, then the center is equidistant from the two chords.*
$AC = AC$	*Reflexive property*
$\angle CEA = \angle CFA = 90°$	*A line joining the center and the midpoint of a chord is perpendicular to the chord.*
$\triangle CEA \cong \triangle CFA$	*HL*
$\angle ECA = \angle FCA$	*Corresponding angles of congruent triangles are equal in measure.*
$\overline{AC}$ bisects $\angle BCD$	*Definition of angle bisector*

9. In class, we proved: *Congruent chords define central angles equal in measure*.

a. Give another proof of this theorem based on the properties of rotations. Use the figure from Exercise 5.



We are given that the two chords ($\overline{AB}$ and $\overline{CD}$) are congruent. Therefore, a rigid motion exists that carries $\overline{AB}$ to $\overline{CD}$. The same rotation that carries $\overline{AB}$ to $\overline{CD}$ also carries $\overline{AO}$ to $\overline{CO}$ and $\overline{BO}$ to $\overline{DO}$. The angle of rotation is the measure of $\angle AOC$, and the rotation is clockwise.

b. Give a rotation proof of the converse: *If two chords define central angles of the same measure, then they must be congruent.*

Using the same diagram, we are given that $\angle AOB \cong \angle COD$. Therefore, a rigid motion (a rotation) carries $\angle AOB$ to $\angle COD$. This same rotation carries $\overline{AO}$ to $\overline{CO}$ and $\overline{BO}$ to $\overline{DO}$. The angle of rotation is the measure of $\angle AOC$, and the rotation is clockwise.

Graphic Organizer on Circles

Diagram	Explanation of Diagram	Theorem or Relationship