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Probability and Statistics

[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Precalculus • Module 5

Probability and Statistics

OVERVIEW

In this module, students build on their understanding of probability developed in previous grades. In Topic A, the multiplication rule for independent events introduced in Grade 11 is generalized to a rule that can be used to calculate the probability of the intersection of two events in situations where the two events are not independent. In this topic, students are also introduced to three techniques for counting outcomes—the fundamental counting principle, permutations, and combinations. These techniques are then used to calculate probabilities, and these probabilities are interpreted in context (**S-CP.B.8**, **S-CP.B.9**).

In Topic B, students study probability distributions for discrete random variables (**S-MD.A.1**). They develop an understanding of the information that a probability distribution provides and interpret probabilities from the probability distribution of a discrete random variable in context (**S-MD.A.2**). For situations where the probabilities associated with a discrete random variable can be calculated given a description of the random variable, students determine the probability distribution (**S-MD.A.3**). Students also see how empirical data can be used to approximate the probability distribution of a discrete random variable (**S-MD.A.4**). This topic also introduces the idea of expected value, and students calculate and interpret the expected value of discrete random variables in context.

Topic C is a capstone topic for this module, where students use what they have learned about probability and expected value to analyze strategies and make decisions in a variety of contexts (**S-MD.B.5**, **S-MD.B.6**, **S-MD.B.7**). Students use probabilities to make a fair decision and explain how to make fair and "unfair" decisions. Students analyze simple games of chance as they calculate and interpret the expected payoff in context. They make decisions based on expected values in problems with business, medical, and other contexts. They also examine and interpret what it means for a game to be fair. Interpretation and explanations of expected values are important outcomes for Topic C.

Focus Standards

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

S-CP.B.8 (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.

S-CP.B.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Calculate expected values and use them to solve problems.

S-MD.A.1 (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

S-MD.A.2 (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

S-MD.A.3 (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.*

S-MD.A.4 (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in* 100 *randomly selected households?*

Use probability to evaluate outcomes of decisions.

S-MD.B.5 (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.*

b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*

S-MD.B.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

S-MD.B.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Foundational Standards

Understand independence and conditional probability and use them to interpret data.

S-CP.A.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

S-CP.A.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

S-CP.A.3 Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.

S-CP.A.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*

S-CP.A.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

S-CP.B.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.

S-CP.B.7 Apply the Addition Rule, $P(A \text{ or } B) – P(A) + P(B) – P(A \text{ and } B)$, and interpret the answer in terms of the model.

Focus Standards for Mathematical Practice

MP.2 **Reason abstractly and quantitatively**. Students interpret probabilities calculated using the addition rule and the general multiplication rule. They use permutations and combinations to calculate probabilities and interpret them in context. Students also explain the meaning of the expected value of a random variable as a long-run average and connect this interpretation to the given context.

MP.3 **Construct viable arguments and critique the reasoning of others**. Students construct arguments in distinguishing between situations involving combinations and those involving permutations. Students use permutations and combinations to calculate probabilities and evaluate decisions based on probabilities. Students also use expected values to analyze games of chance and to evaluate whether a game is "fair." Students design, compare, and evaluate games of chance that they construct, comparing their games to the games of other students based on probabilities and expected values. They analyze strategies based on probability. For example, students use expected value to explain which of two plans yields the largest earnings for an insurance company.

MP.4 **Model with mathematics**. Students develop a probability distribution for a random variable by finding the theoretical probabilities. Students model probability distributions by estimating probabilities empirically. They use probabilities to make and justify decisions. Throughout the module, students use statistical ideas to explain and solve real-world problems. For example, given the probability of finding a female egg in a nest, students determine a discrete probability distribution for the number of male eggs in the nest.

MP.5 **Use appropriate tools strategically**. Students use technology to carry out simulations in order to estimate probabilities empirically. For example, students use technology to simulate a dice-tossing game and generate random numbers to simulate the flavors in a pack of cough drops. They use technology to graph a probability distribution and to calculate expected values. Students come to view discrete probability distributions as tools that can be used to understand real-world situations and solve problems.

MP.8 **Look for and express regularity in repeated reasoning**. Students use simulations to observe the long-run behavior of a random variable, using the results of the simulations to estimate probabilities.

Terminology

New or Recently Introduced Terms

- **Combination of k items selected from a set of n distinct items** (an unordered set of k items selected from a set of n distinct items).
- **Continuous random variables** (a random variable for which the possible values form an entire interval along the number line).
- **Discrete random variables** (a random value for which the possible values are isolated points along the number line).
- **Empirical probability** (a probability that has been estimated by observing a large number of outcomes of a chancge experiment or values of a random variable).
- **Expected value of a random variable** (the long-run average value expected over a large number of observations of the value of a random variable).
- **Fundamental counting principle** (Let n_1 be the number of ways the first step or event can occur and n_2 be the number of ways the second step or event can occur. Continuing in this way, let n_k be the number of ways the k^{th} stage or event can occur. Then, based on the fundamental counting principle, the total number of different ways the process can occur is $n_1 * n_2 * n_3 * \ldots. * n_k$.)
- **General multiplication rule** (a probability rule for calculating the probability of the intersection of two events).
- **Long-run behavior of a random variable** (the behavior of the random variable over a very long sequence of observations).
- **Permutation of k items selected from a set of n distinct items** (an ordered sequence of k items selected from a set of n distinct items).
- **Probability distribution** (a table or graph that provides information about the long-run behavior of a random variable).
- **Probability distribution of a discrete random variable** (a table or graph that specifies the possible values of the random variable and the associated probabilities).
- **Random variable** (a variable whose possible values are based on the outcome of a random event).

- **Theoretical probability** (a probability calculated by assigning a probability to all possible outcomes in the sample space for a chance experiment).
- **Uniform probability model** (a probability distribution that assigns equal probability to each possible outcome of a chance experiment).

Familiar Terms and Symbols[2]

- Chance experiment
- Complement of an event
- Event
- Intersection of events
- Sample space
- Union of events

Suggested Tools and Representations

- Graphing calculator or graphing software
- Random number software
- Random number tables
- Two-way frequency tables

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	S-CP.B.8, S-CP.B.9, S-MD.A.1, S-MD.A.2, S-MD.A.3, S-MD.A.4
End-of-Module Assessment Task	After Topic C	Constructed response with rubric	S-CP.B.8, S-CP.B.9, S-MD.A.2, S-MD.A.3, S-MD.B.5, S-MD.B.6, S-MD.B.7

[2] These are terms and symbols students have seen previously.

Topic A:

Probability

S-CP.B.8, S-CP.B.9

Focus Standards:	S-CP.B.8	(+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B\|A) = P(B)P(A\|B)$, and interpret the answer in terms of the model.
	S-CP.B.9	(+) Use permutations and combinations to compute probabilities of compound events and solve problems.
Instructional Days:	4	
Lesson 1:	The General Multiplication Rule (P)[1]	
Lesson 2:	Counting Rules—The Fundamental Counting Principle and Permutations (P)	
Lesson 3:	Counting Rules—Combinations (P)	
Lesson 4:	Using Permutations and Combinations to Compute Probabilities (P)	

In this topic, students extend their understanding of probability, building on work from Grade 11. The multiplication rule for independent events introduced in Grade 11 is generalized to a rule that can be used to calculate the probability of the intersection of two events in situations where the two events are not independent (**S-CP.B.8**). Students are also introduced to three techniques for counting outcomes—the fundamental counting principle, permutations, and combinations (**S-CP.B.9**). Students consider the distinction between combinations and permutations and identify situations where it would be appropriate to use each of these methods. For example, in Lesson 4 students are presented with a scenario involving 20 singers auditioning for a high school musical and a director who must choose two singers for a duet, as well as two singers to perform lead and backup. To answer the question, students have to indicate whether or not the scenario involves a permutation or combination. In the final lesson of this topic, students use the fundamental counting principle, permutations, and combinations to calculate probabilities, and these probabilities are interpreted in context. In Lesson 4, students must explain how to determine the probability (either using a permutation or combination) that a 4-digit pass code could be 1234 if the digits in the code cannot repeat.

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lesson 1: The General Multiplication Rule

Student Outcomes

- Students use the general multiplication rule to calculate the probability of the intersection of two events.
- Students interpret probabilities in context.

Lesson Notes

Grade 7 introduced students to the basic ideas of probability. This was extended in Grades 9 and 11 where students learned two-way tables and conditional probability. In this lesson, the multiplication rule is developed and applied in several different contexts.

The first example asks students to list all the possible outcomes for an experiment that consists of two steps. You may wish to remind students that a tree diagram is one method to list all the possibilities. Tree diagrams can be useful in students' development of probabilistic ideas. Tree diagrams were analyzed from an algebraic perspective in Algebra II, Module 1, Lesson 7. Exercise 1 is designed to introduce the fundamental counting principle as a method for finding the total number of possibilities rather than listing them all.

Grade 9 introduced students to the intersection of two events by using two-way tables. Students may be familiar with the idea of finding $P(A \text{ and } B)$ and with the concepts of independent and dependent events. This lesson extends these ideas to a general multiplication rule for calculating $P(A \text{ and } B)$, and there is further discussion of independent and dependent events.

Classwork

Example 1 (3 minutes): Independent Events

Many cereal companies placed prizes or toys in their cereal boxes to generate interest in buying their cereal. Students may recall their own interest in getting a prize or toy in a cereal box. You might want to show some of the prizes available by doing an Internet search on cereal box prizes.

Discuss Example 1 with students.

Example 1: Independent Events

Do you remember when breakfast cereal companies placed prizes in boxes of cereal? Possibly you recall that when a certain prize or toy was particularly special to children, it increased their interest in trying to get that toy. How many boxes of cereal would a customer have to buy to get that toy? Companies used this strategy to sell their cereal.

One of these companies put one of the following toys in its cereal boxes: a block (B), a toy watch (W), a toy ring (R), and a toy airplane (A). A machine that placed the toy in the box was programmed to select a toy by drawing a random number of 1 to 4. If a 1 was selected, the block (or B) was placed in the box; if a 2 was selected, a watch (or W) was placed in the box; if a 3 was selected, a ring (or R) was placed in the box; and if a 4 was selected, an airplane (or A) was placed in the box. When this promotion was launched, young children were especially interested in getting the toy airplane.

Exercises 1–8 (15 minutes)

> *Scaffolding:*
>
> Teachers may consider doing a physical demonstration to accompany this explanation with a cereal box and sample prizes (which could just be slips of paper with B, W, R, and A written on them).

Before students begin work on the Exercises, give them an opportunity to think, talk, write, and try to solve an overall question such as the following:

- What's the probability of finding (choose a prize) in two boxes in a row? Three? Explain how you determined your answer.

Give students time to struggle with this a little and present their answers before working through the questions below.

Allow 10 minutes for students to work through Exercises 1–4 in small groups (2 or 3 students). After they have a chance to answer the Esxercises, discuss some of the explanations as a whole group.

Exercises 1–8

1. **If you bought one box of cereal, what is your estimate of the probability of getting the toy airplane? Explain how you got your answer.**

   The probability is $\frac{1}{4}$. I got this answer based on a sample space of $\{B, W, R, A\}$. Each outcome has the same chance of occurring. Therefore, the probability of getting the airplane is $\frac{1}{4}$.

2. **If you bought a second box of cereal, what is your estimate of the probability of getting the toy airplane in the second box? Explain how you got your answer.**

   The probability is again $\frac{1}{4}$. Since the machine that places a toy in the box picks a random number from 1 to 4, the probability that the second toy will be an airplane will again be $\frac{1}{4}$.

3. **If you bought two boxes of cereal, does your chance of getting at least one airplane increase or decrease? Explain your answer.**

   The probability of getting at least one airplane increases because the possible outcomes include the following an airplane in the first box but not the second box, an airplane in the second box but not the first box, and an airplane in both the first and second boxes.

4. **Do you think the probability of getting at least one airplane from two boxes is greater than 0.5? Again, explain your answer.**

   I think the probability is less than 0.5 as there are many more outcomes that I can describe that do not include the airplane. For example, you could get a watch and a block, a watch and a watch, a watch and ring, and so on.

Encourage each group to answer Exercise 5 without suggesting a specific strategy, as students could organize their lists in different ways. After students have formed their lists for Exercise 5, discuss the various strategies they used. If no group used a tree diagram, this would be a good time to review the design of a tree diagram for two events, which was developed in Grade 7. If students have not previously worked with a tree, this exercise provides an opportunity to present a tree as a way to organize the outcomes in a systemic list. A tree diagram provides a strategy to organize and analyze outcomes in other problems presented in this module.

5. List all of the possibilities of getting two toys from two boxes of cereals. (Hint: Think of the possible outcomes as ordered pairs. For example, *BA* would represent a block from the first box and an airplane from the second box.)

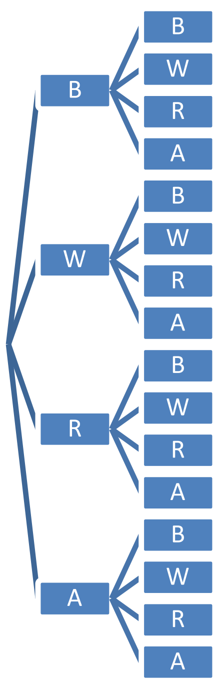
Consider the following ways students might create their lists using the notation:

B for block, W for watch, R for ring, and A for airplane.

The first letter represents the toy found in the first box, and the second letter represents the toy found in the second box. The first column represents getting the block in the first box, followed by each one of the other toys. The second column represents getting the watch in the first box, followed by each one of the other toys. The third column is developed with the ring in the first box, and the fourth column is developed with the airplane in the first box.

BB	*WB*	*RB*	*AB*
BW	*WW*	*RW*	*AW*
BR	*WR*	*RR*	*AR*
BA	*WA*	*RA*	*AA*

The following represents a tree diagram to form the lists:



Direct students to answer the following exercises using an organized list or the tree diagram.

6. **Based on the list you created, what do you think is the probability of each of the following outcomes if two cereal boxes are purchased?**

   a. **One (and only one) airplane**

      $\frac{6}{16}$ *or* $\frac{3}{8}$

   b. **At least one airplane**

      $\frac{7}{16}$

   c. **No airplanes**

      $\frac{9}{16}$

7. **Consider the purchase of two cereal boxes.**

   a. **What is the probability of getting an airplane in the first cereal box? Explain your answer.**

      The probability is $\frac{1}{4}$ *because there are* 4 *possible toys, and each is equally likely to be in the box.*

   b. **What is the probability of getting an airplane in the second cereal box?**

      Again, the probability is $\frac{1}{4}$ *because there are* 4 *possible toys, and each is equally likely to be in the box.*

   c. **What is the probability of getting airplanes in both cereal boxes?**

      The probability is $\frac{1}{16}$ *because there is* 1 *pair in the list (AA) out of the* 16 *that fits this description.*

Point out that the event of getting an airplane in the first box purchased and then getting an airplane in the second box purchased is an example of what are called *independent events*. Students worked with independent and dependent events in Algebra II. Review that two events are independent of each other if knowing that one event has occurred does not change the probability that the second event occurs. Point out that because of the way toys are placed in the boxes, knowing the type of toy placed in the first cereal box does not tell us anything about what toy will be found in the second cereal box. Discuss the following summary with students:

The probability of A and B is denoted as $P(A \text{ and } B)$. This is also called the probability of A intersect B or $P(A \text{ “intersect” } B)$. If A and B are independent events, then the multiplication rule for independent events applies:

$P(A \text{ and } B)$ is the probability that Events A and B *both* occur and is the probability of the intersection of A and B. The probability of the intersection of Events A and B is sometimes also denoted by $P(A \cap B)$.

Multiplication Rule for Independent Events

If A and B are independent events, $P(A \text{ and } B) = P(A) \cdot P(B)$.

This rule generalizes to more than two independent events, for example:

$P(A \text{ and } B \text{ and } C)$ or $P(A \text{ intersect } B \text{ intersect } C) = P(A) \cdot P(B) \cdot P(C)$.

Students were introduced to intersection using Venn diagrams in earlier grades. Revisiting a Venn diagram might be helpful in reviewing the intersection of two events. Use Exercise 8 to informally assess student understanding so far.

Scaffolding:

- In addition to using Venn diagrams, if students have trouble identifying the intersection of two events, point out real-world examples such as "intersecting streets" to connect students to the concept.
- For students above grade-level, pose the following: "What is the probability that you would *not* find an airplane in two boxes? Three? Four? Ten?" Explain how to determine these procedures.

8. **Based on the multiplication rule for independent events, what is the probability of getting an airplane in both boxes? Explain your answer.**

   The probability would be $\frac{1}{16}$ as I would multiply the probability of getting an airplane in the first box $\left(\frac{1}{4}\right)$ by the probability of getting an airplane in the second box, which is also $\frac{1}{4}$.

Example 2 (2 minutes): Dependent Events

Example 2 moves to a discussion of dependent events. As students discuss this example, ask them how it differs from the first example.

If students are not familiar with the movie *Forrest Gump*, you may wish to show a short clip of the video where Forrest says, "Life is like a box of chocolates." The main idea of this example is that as a piece of chocolate is chosen from a box, the piece is not replaced. Since the piece is not replaced, there is one less piece, so the number of choices for the second piece changes. The probability of getting a particular type of chocolate on the second selection is dependent on what type was chosen first.

Two events are dependent if knowing that one event occurring changes the probability that the other event occurs. This second probability is called a *conditional probability*. Students learned about conditional probabilities in several lessons in Algebra II. If this is their first time thinking about the conditional probabilities, use this example to indicate how the probability of the second selection is based on what the first selection was (or the *condition* of the first event).

Example 2: Dependent Events

Do you remember the famous line, "Life is like a box of chocolates," from the movie *Forrest Gump*? When you take a piece of chocolate from a box, you never quite know what the chocolate will be filled with. Suppose a box of chocolates contains 15 identical-looking pieces. The 15 are filled in this manner: 3 caramel, 2 cherry cream, 2 coconut, 4 chocolate whip, and 4 fudge.

Exercises 9–14 (10 minutes)

Students should work in small groups (2 or 3 students per group). Allow about five minutes for the students to complete Exercises 9 and 10. When they have finished, discuss the answers. Then introduce the definition of dependent events and the multiplication rule for dependent events. Emphasize the symbol $P(A|B)$ and its meaning.

At the end of the lesson, the rule will be referred to as the *general multiplication rule* because it will work for any event, independent or dependent (given $P(B|A) = P(B)$ if A and B are independent).

> *Scaffolding:*
>
> For struggling students, present a simpler problem and scaffold. In a box of 8 crayons of 8 different colors (red, orange, yellow, green, blue, purple, brown, black), what is the probablility that you pick a red crayon? What is the probability your neighbor will pick a yellow from the crayons left? Then go to a box of 16 crayons with 2 of each color and ask the same questions.

Exercises 9–14

9. **If you randomly select one of the pieces of chocolate from the box, what is the probability that the piece will be filled with fudge?**

   $\frac{4}{15} \approx 0.2667$

10. **If you randomly select a second piece of chocolate (after you have eaten the first one, which was filled with fudge), what is the probability that the piece will be filled with caramel?**

    $\frac{3}{14} \approx 0.2143$

Allow about five minutes for discussion of the multiplication rule for dependent events and for students to complete Exercises 11–14.

Before students begin the exercises, use the following as an opportunity to check for understanding at this phase:

- Ask students to restate the definition in their own words, either in writing or to a partner.
  - *Sample response: If two events are dependent, then I can find the probability of both the event A and B by multiplying the probability of event A by the probability of B given A.*

The events, *picking a fudge-filled piece on the first selection* and *picking a caramel-filled piece on the second selection*, are called dependent events.

Two events are dependent if knowing that one has occurred changes the probability that the other occurs.

Multiplication Rule for Dependent Events

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Recall from your previous work with probability in Algebra II that $P(B|A)$ is the conditional probability of event B given that event A occurred. If event A is *picking a fudge-filled piece on the first selection* and event B is *picking a caramel-filled piece on the second selection*, then $P(B|A)$ represents the probability of picking a caramel-filled piece second knowing that a fudge-filled piece was selected first.

11. **If $A1$ is the event *picking a fudge-filled piece on the first selection* and $B2$ is the event *picking a caramel-filled piece on the second selection*, what does $P(A1 \text{ and } B2)$ represent? Find $P(A1 \text{ and } B2)$.**

$P(A1 \text{ and } B2)$ represents the probability of picking a fudge-filled piece first and a caramel-filled piece second:

$$\frac{4}{15} \cdot \frac{3}{14} \approx 0.057$$

12. **What does $P(B1 \text{ and } A2)$ represent? Calculate this probability.**

$P(B1 \text{ and } A2)$ represents the probability of picking a caramel-filled piece first and a fudge-filled piece second:

$$\frac{3}{15} \cdot \frac{4}{14} \approx 0.057$$

13. **If C represents selecting a coconut-filled piece of chocolate, what does $P(A1 \text{ and } C2)$ represent? Find this probability.**

$P(A1 \text{ and } C2)$ represents the probability of picking a fudge-filled piece first and a coconut-filled piece second:

$$\frac{4}{15} \cdot \frac{2}{14} \approx 0.038$$

14. **Find the probability that both the first and second pieces selected are filled with chocolate whip.**

$$\frac{4}{15} \cdot \frac{3}{14} \approx 0.057$$

Exercises 15–17 (8 minutes)

MP.2

Students should work in small groups (2 or 3 students per group). Allow them about five minutes to complete Exercises 15–17. When students have finished, discuss the answers. Students reason abstractly and quantitatively as these exercises provide practice in deciding if events are independent or dependent and practice in calculating probability. If students struggle, discuss that selections which occur with replacement are independent events, and selections which occur without replacement are dependent events.

Exercises 15–17

15. **For each of the following, write the probability as the intersection of two events. Then, indicate whether the two events are independent or dependent, and calculate the probability of the intersection of the two events occurring.**

a. **The probability of selecting a 6 from the first draw and a 7 on the second draw when two balls are selected without replacement from a container with 10 balls numbered 1 to 10.**

Dependent

$$P(6 \text{ first and } 7 \text{ second}) = P(6 \text{ first}) \cdot P(7 \text{ second} | 6 \text{ first}) = \frac{1}{10} \cdot \frac{1}{9} = \frac{1}{90} \approx 0.011$$

b. **The probability of selecting a 6 on the first draw and a 7 on the second draw when two balls are selected with replacement from a container with 10 balls numbered 1 to 10.**

Independent

$$P(6 \text{ first and } 7 \text{ second}) = P(6 \text{ first}) \cdot P(7 \text{ second}) = \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{100} = 0.01$$

c. The probability that two people selected at random in a shopping mall on a very busy Saturday both have a birthday in the month of June. Assume that all 365 birthdays are equally likely, and ignore the possibility of a February 29 leap-year birthday.

Independent

$$P(\text{first person June birthday and second person June birthday}) =$$
$$P(\text{first person June birthday}) \cdot P(\text{second person June birthday}) =$$
$$\frac{30}{365} \cdot \frac{30}{365} = 0.0068$$

d. The probability that two socks selected at random from a drawer containing 10 black socks and 6 white socks will both be black.

Dependent

$$P(\text{first sock black and second sock black}) =$$
$$P(\text{first sock black}) \cdot P(\text{second sock black }|\text{first sock black})$$
$$= \frac{10}{16} \cdot \frac{9}{15}$$
$$= 0.375$$

16. A gumball machine has gumballs of 4 different flavors: sour apple (A), grape (G), orange (O), and cherry (C). There are six gumballs of each flavor. When 50¢ is put into the machine, two random gumballs come out. The event $C1$ means a cherry gumball came out first, the event $C2$ means a cherry gumball came out second, the event $A1$ means sour apple gumball came out first, and the event $G2$ means a grape gumball came out second.

a. What does $P(C2|C1)$ mean in this context?

The probability of the second gumball being cherry knowing the first gumball was cherry

b. Find $P(C1 \text{ and } C2)$.

$$\frac{6}{24} \cdot \frac{5}{23} \approx 0.054$$

c. Find $P(A1 \text{ and } G2)$.

$$\frac{6}{24} \cdot \frac{6}{23} \approx 0.0652$$

17. Below are the approximate percentages of the different blood types for people in the United States.

Type O 44%

Type A 42%

Type B 10%

Type AB 4%

Consider a group of 100 people with a distribution of blood types consistent with these percentages. If two people are randomly selected with replacement from this group, what is the probability that

a. both people have type O blood?

$$0.44 \cdot 0.44 = 0.1936$$

b. **the first person has type A blood and the second person has type AB blood?**

$\mathbf{0.42 \cdot 0.04 = 0.0168}$

Closing (2 minutes)

- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

Lesson Summary

- Two events are independent if knowing that one occurs does not change the probability that the other occurs.
- Two events are dependent if knowing that one occurs changes the probability that the other occurs.
- **GENERAL MULTIPLICATION RULE:**

  $P(A \text{ and } B) = P(A) \cdot P(B|A)$

  If A and B are independent events then $P(B|A) = P(B)$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 1: The General Multiplication Rule

Exit Ticket

Serena is in a math class of 20 students. Each day for a week (Monday to Friday), a student in Serena's class is randomly selected by the teacher to explain a homework problem. Once a student's name is selected, that student is not eligible to be selected again that week.

1.
   a. What is the probability that Serena is selected on Monday?

   b. What is the probability that Serena is selected on Tuesday given that she was not selected on Monday?

   c. What is the probability that she will be selected on Friday given that she was not selected on any of the other days?

2. Suppose A represents Serena being selected, and B represents Dominic (another student in class) being selected. The event $A1$ means Serena was selected on Monday, and the event $B2$ means Dominic was selected on Tuesday. The event $B1$ means Dominic was selected on Monday, and the event $A2$ means Serena was selected on Tuesday.
   a. Explain in words what $P(A1 \text{ and } B2)$ represents, and then calculate this probability.

   b. Explain in words what $P(B1 \text{ and } A2)$ represents, and then calculate this probability.

Exit Ticket Sample Solutions

Serena is in a math class of 20 students. Each day for a week (Monday to Friday), a student in Serena's class is randomly selected by the teacher to explain a homework problem. Once a student's name is selected, that student is not eligible to be selected again that week.

1.

a. What is the probability that Serena is selected on Monday?

$$\frac{1}{20} = 0.05$$

b. What is the probability that Serena is selected on Tuesday given that she was not selected on Monday?

$$\frac{1}{19} \approx 0.0526$$

c. What is the probability that she will be selected on Friday given that she was not selected on any of the other days?

$$\frac{1}{16} \approx 0.0625$$

2. Suppose A represents Serena being selected, and B represents Dominic (another student in class) being selected. The event $A1$ means Serena was selected on Monday, and the event $B2$ means Dominic was selected on Tuesday. The event $B1$ means Dominic was selected on Monday, and the event $A2$ means Serena was selected on Tuesday.

a. Explain in words what $P(A1 \text{ and } B2)$ represents, and then calculate this probability.

It is the probability that Serena is seleced on Monday and Dominic is selected on Tuesday.

$$P(A1 \text{ and } B2) = \frac{1}{20} \cdot \frac{1}{19} \approx 0.0026$$

b. Explain in words what $P(B1 \text{ and } A2)$ represents, and then calculate this probability.

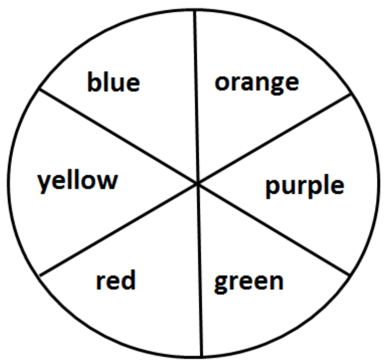
It is the probability that Dominic is selected on Monday and Serena is selected on Tuesday.

$$P(B1 \text{ and } A2) = \frac{1}{20} \cdot \frac{1}{19} \approx 0.0026$$

Problem Set Sample Solutions

Use this space to describe any specific details about the problem set for teacher reference.

1. **In a game using the spinner below, a participant spins the spinner twice. If the spinner lands on red both times, the participant is a winner.**



   a. **The event *participant is a winner* can be thought of as the intersection of two events. List the two events.**

   First spin lands on red and second spin lands on red.

   b. **Are the two events independent? Explain.**

   Independent—knowing the first spin landed on red does not change the probability of the second spin landing on red.

   c. **Find the probability that a participant wins the game.**

   $\frac{1}{6} \cdot \frac{1}{6} \approx 0.0278$

2. **The overall probability of winning a prize in a weekly lottery is $\frac{1}{32}$. What is the probability of winning a prize in this lottery three weeks in a row?**

   $\frac{1}{32} \cdot \frac{1}{32} \cdot \frac{1}{32} \approx 0.00003$

3. **A Gallup poll reported that 28% of adults (age 18 and older) eat at a fast food restaurant about once a week. Find the probability that two randomly selected adults would both say they eat at a fast food restaurant about once a week.**

   $0.28 \cdot 0.28 = 0.0784$

4. In the game *Scrabble*, there are a total of 100 tiles. Of the 100 tiles, 42 tiles have the vowels A, E, I, O, and U printed on them, 56 tiles have the consonants printed on them, and 2 tiles are left blank.

   a. If tiles are selected at random, what is the probability that the first tile drawn from the pile of 100 tiles is a vowel?

   $\frac{42}{100} = 0.42$

   b. If tiles drawn are not replaced, what is the probability that the first two tiles selected are both vowels?

   $\frac{42}{100} \cdot \frac{41}{99} \approx 0.174$

   c. Event A is *drawing a vowel*, event B is *drawing a consonant*, and event C is *drawing a blank tile*. $A1$ means a vowel is drawn on the first selection, $B2$ means a consonant is drawn on the second selection, and $C2$ means a blank tile is drawn on the second selection. Tiles are selected at random and without replacement.

      i. Find $P(A1 \text{ and } B2)$ $= \frac{42}{100} \cdot \frac{56}{99} \approx 0.238$

      ii. Find $P(A1 \text{ and } C2)$ $= \frac{42}{100} \cdot \frac{2}{99} \approx 0.008$

      iii. Find $P(B1 \text{ and } C2)$ $= \frac{56}{100} \cdot \frac{2}{99} \approx 0.011$

5. To prevent a flooded basement, a homeowner has installed two special pumps that work automatically and independently to pump water if the water level gets too high. One pump is rather old and does not work 28% of the time, and the second pump is newer and does not work 9% of the time. Find the probability that both pumps will fail to work at the same time.

   $0.28 \cdot 0.09 \approx 0.025$

6. According to a recent survey, approximately 77% of Americans get to work by driving alone. Other methods for getting to work are listed in the table below.

Method of getting to work	Percent of Americans using this method
Taxi	0.1%
Motorcycle	0.2%
Bicycle	0.4%
Walk	2.5%
Public Transportation	4.7%
Car Pool	10.7%
Drive Alone	77%
Work at Home	3.7%
Other	0.7%

   a. What is the probability that a randomly selected worker drives to work alone?

   0.77

   b. What is the probability that two workers selected at random with replacement both drive to work alone?

   Assume independent $0.77 \cdot 0.77 \approx 0.593$

7. A bag of M&Ms contains the following distribution of colors:

   9 blue

   6 orange

   5 brown

   5 green

   4 red

   3 yellow

   Three M&Ms are randomly selected without replacement. Find the probabilities of the following events.

   a. All three are blue.

   $$\frac{9}{32}\cdot\frac{8}{31}\cdot\frac{7}{30}\approx 0.017$$

   b. The first one selected is blue, the second one selected is orange, and the third one selected is red.

   $$\frac{9}{32}\cdot\frac{6}{31}\cdot\frac{4}{30}\approx 0.007$$

   c. The first two selected are red, and the third one selected is yellow.

   $$\frac{4}{32}\cdot\frac{3}{31}\cdot\frac{3}{30}\approx 0.001$$

8. Suppose in a certain breed of dog, the color of fur can either be tan or black. Eighty-five percent of the time, a puppy will be born with tan fur, while 15% of the time, the puppy will have black fur. Suppose in a future litter, six puppies will be born.

   a. Are the events *having tan fur* and *having black fur* independent? Explain.

   Yes, knowing the color of fur for one puppy doesn't affect the probability of fur color for another puppy.

   b. What is the probability that one puppy in the litter will have black fur and another puppy will have tan fur?

   $0.15 \cdot 0.85 = 0.1275$

   c. What is the probability that all six puppies will have tan fur?

   $0.85 \cdot 0.85 \cdot 0.85 \cdot 0.85 \cdot 0.85 \cdot 0.85 \approx 0.377$

   d. Is it likely for three out of the six puppies to be born with black fur? Justify mathematically.

   No, the probability of three puppies being born with black fur is $0.15 \cdot 0.15 \cdot 0.15 = 0.003375$. *This is not likely to happen.*

9. Suppose that in the litter of six puppies from Exercise 8, five puppies are born with tan fur, and one puppy is born with black fur.

a. You randomly pick up one puppy. What is the probability that puppy will have black fur?

$\frac{1}{6} \approx 0.167$

b. You randomly pick up one puppy, put it down, and randomly pick up a puppy again. What is the probability that both puppies will have black fur?

$\frac{1}{6} \cdot \frac{1}{6} \approx 0.028$

c. You randomly pick up two puppies, one in each hand. What is the probability that both puppies will have black fur?

0; *this outcome can never happen since there is only one black puppy.*

d. You randomly pick up two puppies, one in each hand. What is the probability that both puppies will have tan fur?

$\frac{5}{6} \cdot \frac{4}{5} \approx 0.667$

Lesson 2: Counting Rules—The Fundamental Counting Principle and Permutations

Student Outcomes

- Students use the fundamental counting principle to determine the number of different possible outcomes for a chance experiment consisting of a sequence of steps.
- Students calculate the number of different permutations of a set of n distinct items.
- Students calculate the number of different permutations of k items from a set of n distinct items.

Lesson Notes

A formal definition of permutation as an ordered arrangement is presented. Several contexts are developed in which students apply the multiplication rule to calculate the number of different permutations of a set of n distinct items. The next lesson introduces combinations, which are unordered collections. In the examples and exercises of this lesson, point out that different orderings of the same set of items are considered different outcomes.

Classwork

Example 1 (3 minutes): Fundamental Counting Principle

Similar to examples in the previous lesson, this example and the first exercise ask students to make a list. Suggest an organized list or a tree diagram.

As you introduce this first example, you may wish to illustrate an example or two of a possible three-course dinner. For example: salad, burger, and cheesecake would make up one of the 24 possible fixed dinner choices.

Scaffolding:

- If students struggle with the presented scenario, consider using a parallel example closer to student experience. For example, find the number of outfits that can be generated from three pair of pants, four shirts, and two pairs of shoes.
- Then break the task into parts by posing the following questions:
  - How many different possibilities are there for the pants?
  - How many different possibilities of pants and shirts are there?
  - How many different possibilities of pants, shirts, and shoes are there?

Example 1: Fundamental Counting Principle

A restaurant offers a fixed-price dinner menu for $\$30$. The dinner consists of three courses, and the diner chooses one item for each course.

The menu is shown below:

First Course	Second Course	Third Course
Salad	Burger	Cheesecake
Tomato Soup	Grilled Shrimp	Ice Cream Sundae
French Onion Soup	Mushroom Risotto	
	Ravioli	

Exercises 1–4 (12 minutes)

MP.3

Students should work in small groups (2 or 3 students per group). Allow about 5 minutes for them to complete Exercise 1; then, discuss as a class the strategies that groups used to list all the possibilities. Ask students to explain how they know that they have listed all the possibilities and compare/critique the methods that are shared. Then pose the following question to the class. Allow about 5 minutes for discussion and possible presentation of additional examples as needed.

- In general, how can you find the number of possibilities in situations like this?
  - *I found that the number of possibilities is the same as the product of the choices for each course (or clothing option). So in general, I can multiply the number of choices that each option can occur.*

Convey to students that this generalization is known as the *fundamental counting principle*. Instead of making a list or a tree diagram, another method for finding the total number of possibilities is to use the fundamental counting principle. Suppose that a process involves a sequence of *steps* or *events*. Let n_1 be the number of ways the first step or event can occur and n_2 be the number of ways the second step or event can occur. Continuing in this way, let n_k be the number of ways the k^{th} stage or event can occur. Then the total number of different ways the process can occur is: $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.

In the fixed-price dinner example, there are 3 choices for the first course (step 1), 4 choices for the second course (step 2), and 2 choices for the third course (step 3). Using the fundamental counting principle, there are $3 \cdot 4 \cdot 2$, or 24 total choices. For the fixed-dinner example, the students could draw three boxes and label the boxes as shown. Then, identify the number of choices for each box.

First Course (3 choices)	Second Course (4 choices)	Third Course (2 choices)
$\square$	$\square$	$\square$

Students can continue working in small groups to complete Exercises 2–4. Confirm answers as a class.

Scaffolding:

- If students struggle to articulate the principle, consider presenting additional examples and then asking them again to generalize the rule:
- You are ordering lunch. The only choice for an entrée is a hamburger. You need to choose chips or fries. The drink options are bottled water, lemonade, or apple juice. Find the number of possibilities for your lunch.
- You are planning activities for the weekend. On Saturday you can either go to the movies or to the mall. On Sunday you can choose to participate in one of the following sports events: basketball, soccer, flag football, or lacrosse. Find the number of possibilities for weekend activities.

Exercises 1–4

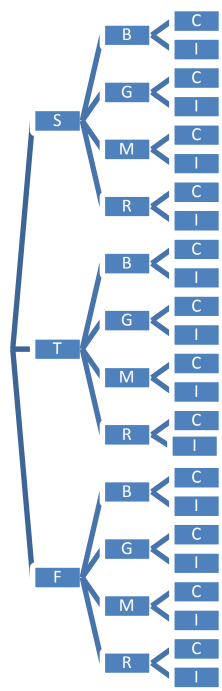
1. **Make a list of all of the different dinner fixed-price meals that are possible. How many different meals are possible?**

   Note: The list below shows all the possibilities using abbreviations of the items.

   They are listed in order, first course choice, second course choice, and third course choice. For example, SBC would represent salad, burger, and cheesecake.

   There are 24 different dinner fixed-price meals possible.

SBC	*TBC*	*FBC*
SBI	*TBI*	*FBI*
SGC	*TGC*	*FGC*
SGI	*TGI*	*FGI*
SMC	*TMC*	*FMC*
SMI	*TMI*	*FMI*
SRC	*TRC*	*FRC*
SRI	*TRI*	*FRI*



2. **For many computer tablets, the owner can set a 4-digit pass code to lock the device.**

   a. **How many digits could you choose from for the first number of the pass code?**

      10

b. **How many digits could you choose from for the second number of the pass code? Assume that the numbers can be repeated.**

10

c. **How many different 4-digit pass codes are possible? Explain how you got your answer.**

$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$

There are 10 choices for the first digit, 10 choices for the second digit, 10 choices for the third digit, and 10 choices for the fourth digit. I used the fundamental counting principle and multiplied the number of choices for each digit together to get the number of possible pass codes.

d. **How long (in hours) would it take someone to try every possible code if it takes three seconds to enter each possible code?**

It would take $30{,}000$ seconds, which is $8\frac{1}{3}$ hr.

3. **The store at your school wants to stock sweatshirts that come in four sizes (small, medium, large, xlarge) and in two colors (red and white). How many different types of sweatshirts will the store have to stock?**

$4 \cdot 2 = 8$

4. **The call letters for all radio stations in the United States start with either a W (east of the Mississippi river) or a K (west of the Mississippi River) followed by three other letters that can be repeated. How many different call letters are possible?**

$2 \cdot 26 \cdot 26 \cdot 26 = 35{,}152$

Scaffolding:

- The word *stock* has multiple meanings and may confuse English language learners.
- Point out that in this context, *stock* refers to a supply of goods or items kept on hand for sale to customers by a merchant.
- Consider displaying an image of clothing on store shelves to reinforce the meaning of the word.

Example 2 (3 minutes): Permutations

This example introduces the definition of *permutations*. Combinations will be introduced in the next lesson, so the focus of this example is that the order matters for a permutation. The permutation formula is not introduced until Example 3. Give students a moment to read through the example, and ask them to share ideas among their group for how to determine the number of pass codes.

If students struggle, break the task into parts by posing the following questions:

- How many digits can you choose from for the first digit? 10
- How many digits can you choose from for the second digit? (Remember–no repeats) 9
- How many digits can you choose from for the third digit? 8
- How many digits can you choose from for the fourth digit? 7

Encourage students to write out the number choices in the diagram, show the multiplication, and make the connection to the fundamental counting principle.

- How many different 4-digit pass codes are possible if digits cannot be repeated?
  - $10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$

- Explain how the fundamental counting principle allows you to make this calculation.
  - *The process of choosing the four digits for the pass code involves a sequence of events. There are* 10 *choices for the first digit,* 9 *choices for the second digit, and so on. So, I can multiply the number of choices that each digit in the pass code can occur.*

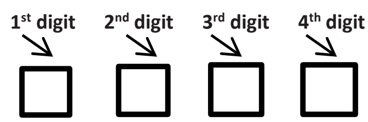
Now introduce the notation for permutations. Explain that finding the number of ordered arrangements of the digits in the pass code is an example of the number of permutations of 10 things taken 4 at a time. This can be written: ${}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$.

Example 2: Permutations

Suppose that the 4-digit pass code a computer tablet owner uses to lock the device *cannot* have any digits that repeat. For example, 1234 is a valid pass code. However, 1123 is not a valid pass code since the digit "1" is repeated.

An arrangement of four digits with no repeats is an example of a permutation. A permutation is an arrangement in a certain order (a sequence).

How many different 4-digit pass codes are possible if digits cannot be repeated?



Exercises 5–9 (8 minutes)

Students should work in small groups (2 or 3 students per group). Allow about 8 minutes for them to complete Exercises 5–9. Encourage students to write the permutation notation and to write out the multiplication. At this point, students should use a calculator to perform the multiplication rather than using the permutation option on the calculator.

Point out to students that when they simplify this formula, the result is the same as using the fundamental counting principle when numbers cannot be repeated. For example: If a password requires three distinct letters, how many different passwords are possible? Students could solve as $26 \cdot 25 \cdot 24$.

When students have finished answering the questions, discuss the answers.

Exercises 5–9

5. **Suppose a password requires three distinct letters. Find the number of permutations for the three letters in the code, if the letters may not be repeated.**

   $${}_{26}P_3 = 26 \cdot 25 \cdot 24 = 15{,}600$$

6. **The high school track has 8 lanes. In the 100 meter dash, there is a runner in each lane. Find the number of ways that 3 out of the 8 runners can finish first, second, and third.**

   $${}_{8}P_3 = 8 \cdot 7 \cdot 6 = 336$$

7. **There are 12 singers auditioning for the school musical. In how many ways can the director choose first a lead singer and then a stand-in for the lead singer?**

$_{12}P_2 = 12 \cdot 11 = 132$

8. **A home security system has a pad with 9 digits (1 to 9). Find the number of possible 5-digit pass codes:**

a. **if digits can be repeated.**

$9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 59,049$

b. **if digits cannot be repeated.**

$_{9}P_5 = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15,120$

MP.8

9. **Based on the patterns observed in Exercises 5–8, describe a general formula that can be used to find the number of permutations of n things taken r at a time, or $n\ Pr$**

Based on the answers to the exercises, a permutation of n things take r at a time can be found using the formula:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1).$$

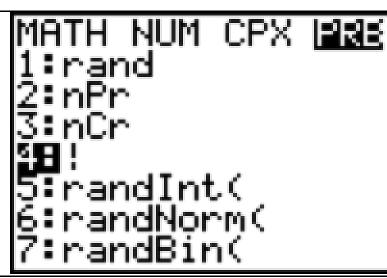
Before moving to the next example, use the following as an opportunity to informally assess student understanding.

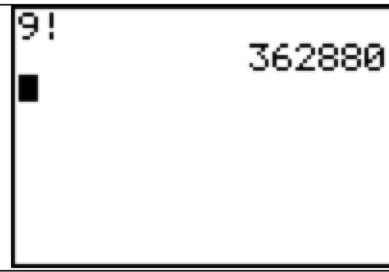
- Explain to your neighbor how to find the number of permutations of n things taken r at a time.

Example 3 (5 minutes): Factorials and Permutations

Factorial notation is introduced in this example. It is also important to tell students that $0!$ is defined to be equal to 1 (see scaffolding note). After defining factorial notation, show students how to find factorial on their calculator. Use $9!$ as an example.

Note there are a variety of calculators and software that can be used to find factorials. The steps displayed here refer to the TI-84 graphing calculator and are meant to serve only as a quick reference for teachers:

1. Enter the integer (9) on the Home screen.	
2. Select the MATH menu.	
3. Scroll to PRB.	
4. Select option 4: !	

5. Press ENTER	

Now discuss the permutation formula and show students all the steps prior to using the permutation option on the calculator. For example, ${}_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 9\cdot8\cdot7\cdot6$. Showing these steps reinforces the connection to the fundamental counting principle.

Example 3: Factorials and Permutations

You have purchased a new album with 12 music tracks and loaded it onto your MP3 player. You set the MP3 player to play the 12 tracks in a random order (no repeats). How many different orders could the songs be played in?

This is the permutation of 12 things taken 12 at a time, or
$\mathbf{12\ \mathit{P}12 = 12\cdot\ 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 =\ 479,001,600}$.

The notation $12!$ is read "12 factorial" and equals $\mathbf{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$.

Factorials and Permutations

The factorial of a non-negative integer n is

$$\boldsymbol{n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot 1}.$$

Note: $0!$ is defined to equal 1.

The number of permutations can also be found using factorials. The number of permutations of n things taken r at a time is

$$\mathbf{{}_nP_r} = \frac{\boldsymbol{n!}}{\boldsymbol{(n-r)!}}.$$

Scaffolding:

- Give students an opportunity to explain why $0!$ is equal to 1.
- Consider using the following to demonstrate why $0!$ is equal to 1:
  $5! = 5\cdot(4\cdot3\cdot2\cdot1)$ or $5! = 5\cdot 4!$
  By rearranging the equation, we get $4! = \frac{5!}{5}$, $4! = 4\cdot(3\cdot2\cdot1)$, or $4! = 4\cdot 3!$
  By rearranging the equation, we get $3! = \frac{4!}{4}$.
  Therefore, $0! = \frac{1!}{1} = 1$.

Now have students compare the general formula they described in Exercise 9 with the permutation formula that was just introduced.

- Rewrite the permutation formula by expanding the factorial notation.
  - *Permutation:*

$$_nP_r = \frac{n!}{(n-r)!}$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdot \ldots \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 1}$$

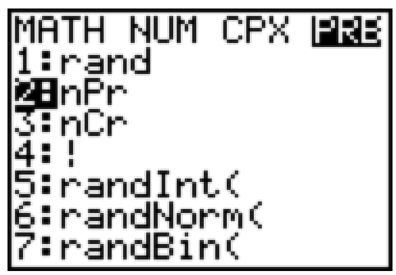
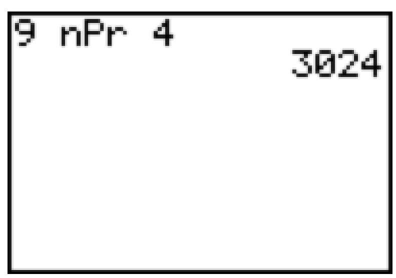
$$= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$$

- How does this formula compare with the general formula you described in Exercise 9?
  - *The formulas are the same. My formula from Exercise 9 was*

$$_nP_r = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1).$$

The permutation formula is just another version of the fundamental counting principle.

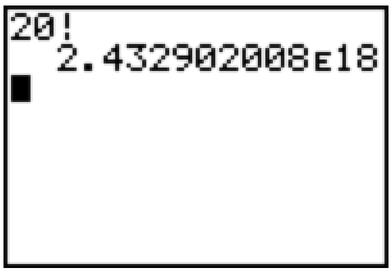
Show students how to find permutations using technology. The following are steps to find $9\text{ P}4$ using the permutation option on the TI-84 graphing calculator:

<table>
<tr><td>1. Enter the integer (9) on the Home screen.</td><td></td></tr>
<tr><td>2. Select the MATH menu.</td><td></td></tr>
<tr><td>3. Scroll to PRB.</td><td></td></tr>
<tr><td>4. Select option 2: nPr.</td><td>
</td></tr>
<tr><td>5. Enter 4 and press ENTER.</td><td>
</td></tr>
</table>

Exercises 10–15 (7 minutes)

MP.2

Students should work in small groups (2 or 3 students per group). Allow about 7 minutes for them to complete Exercises 10–15. Exercise 13 requires students to reason abstractly and quantitatively as they determine how repetition affects the number of outcomes. Encourage students to write the permutation formula and show the substitution. Some of the answers require the use of scientific notation. You many need to remind students of the notation and how the calculator displays the results. One common student error is that they do not see the E displayed for scientific notation. The screen shot below shows the results for 20!. Some students may record the answer as 2.43 instead of $2.43\ x\ 1018$.



When students have finished answering the questions, discuss the answers.

Exercises 10–15

10. If $9!$ is $362,880$, find $10!$.

$3,628,800$

11. How many different ways can the 16 numbered pool balls be placed in a line on the pool table?

$16!\ or\ {}_{16}P_{16} = 2 \cdot 10^{13}$

12. Ms. Smith keeps eight different cookbooks on a shelf in one of her kitchen cabinets. How many ways can the eight cookbooks be arranged on the shelf?

$8!$ *or* ${}_{8}P_{8} = 40,320$

13. How many distinct 4-letter groupings can be made with the letters from the word *champion* if letters may not be repeated?

$${}_{8}P_{4} = \frac{8!}{(8-4)!} = 1,680$$

14. There are 12 different rides at an amusement park. You buy five tickets that allow you to ride on five different rides. In how many different orders can you ride the five rides? How would your answer change if you could repeat a ride?

Different order: ${}_{12}P_{3} = \frac{12!}{(12-5)!} = 95,040$

Repeat rides: $12^5 = 248,832$

15. In the summer Olympics, 12 divers advance to the finals of the 3-meter springboard diving event. How many different ways can the divers finish 1st, 2nd, or 3rd?

$${}_{12}P_3 = \frac{12!}{(12-3)!} = 1,32$$

Closing (2 minutes)

- How do permutations relate to the fundamental counting principle?
  - *Sample response: Permutations are just another version of the fundamental counting principle.*
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from this lesson.

Lesson Summary

- **Let n_1 be the number of ways the first step or event can occur and n_2 be the number of ways the second step or event can occur. Continuing in this way, let n_k be the number of ways the k^{th} stage or event can occur. Then based on fundamental counting principle, the total number of different ways the process can occur is: $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.**
- **The factorial of a non-negative integer n is**
  $$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1.$$
  Note: $0!$ is defined to equal 1.
- **The number of permutations of n things taken r at a time is**
  $${}_nP_r = \frac{n!}{(n-r)!}.$$

Exit Ticket (5 minutes)

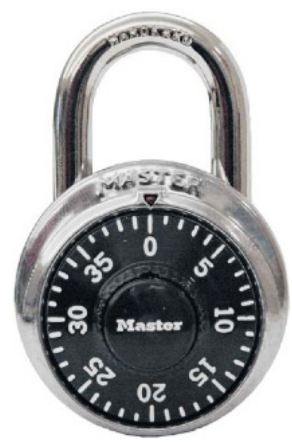
Name ______________________________ Date ______________

Lesson 2: Counting Rules—The Fundamental Counting Principle and Permutations

Exit Ticket

1. The combination for the lock shown below consists of three numbers.

   a. If the numbers can be repeated, how many different combinations are there? Explain your answer.



   b. If the numbers cannot be repeated, how many different combinations are there? Explain your answer.

2. Jacqui is putting together sets of greeting cards for a school fundraiser. There are four different card options, two different colored envelopes, and four different sticker designs. A greeting card set consists of one type of card, one color for the envelopes, and one sticker design. How many different ways can Jacqui arrange the greeting card sets? Explain how you determined your answer.

Exit Ticket Sample Solutions

1. **The combination for the lock shown below consists of three numbers.**

   a. **If the numbers can be repeated, how many different combinations are there? Explain your answer.**

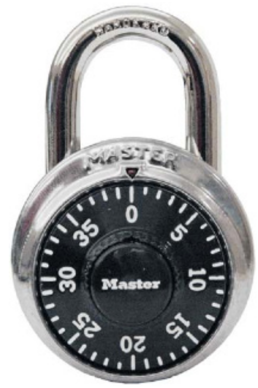
   $40^3 = 64,000$

   Because numbers can be repeated, there are 40 choices for each of the three digits. Therefore, I applied the fundamental counting principle.

   b. **If the numbers cannot be repeated, how many different combinations are there? Explain your answer.**

   ${}_{40}P_3 = 59,280$

   Since numbers cannot be repeated, this is an example of a permutation.



2. **Jacqui is putting together sets of greeting cards for a school fundraiser. There are four different card options, two different colored envelopes, and four different sticker designs. A greeting card set consists of one type of card, one color for the envelopes, and one sticker design. How many different ways can Jacqui arrange the greeting card sets?**

   By using the fundamental counting principle, there are $4 \cdot 2 \cdot 4 = 32$ *ways to arrange the sets.*

Problem Set Sample Solutions

Use this space to describe any specific details about the problem set for teacher reference.

1. **For each of the following, show the substitution in the permutation formula and find the answer.**

   a. ${}_4P_4$

   $$\frac{4!}{(4-4)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{0!} = 24$$

   b. ${}_{10}P_2$

   $$\frac{10!}{(10-2)!} = \frac{10!}{8!} = 10 \cdot 9 = 90$$

   c.

   $$\frac{5!}{(5-1)!} = \frac{5!}{4!} = 5$$

2. **A serial number for a TV begins with three letters, is followed by six numbers, and ends in one letter. How many different serial numbers are possible? Assume the letters and numbers can be repeated.**

   $26 \cdot 26 \cdot 26 \cdot\ 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 26 = 4.6 \cdot 10^{11}$

3. In a particular area code, how many phone numbers (###-####) are possible? The first digit cannot be a zero and assume digits can be repeated.

   $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 9,000,000$

4. There are four NFL teams in the AFC east: Bills, Jets, Dolphins, and Patriots. How many different ways can two of the teams finish first and second?

   ${}_4P_2 = 12$

5. How many ways can 3 of 10 students come in first, second, and third place in a spelling contest, if there are no ties?

   ${}_{10}P_3 = 720$

6. In how many ways can a president, a treasurer, and a secretary be chosen from among nine candidates if no person can hold more than one position?

   ${}_9P_3 = 504$

7. How many different ways can a class of 22 second graders line up to go to lunch?

   $22! = {}_{22}P_{22} = 1.1 \cdot 10^{21}$

8. Describe a situation that could be modeled by using ${}_5P_2$.

   Answers will vary. Suppose that there are five members of a family living at home. The first one home has to take out the garbage, and the second one home has to walk the dog. ${}_5P_2$ can be used to model the number of ways the family members can be assigned the different tasks.

9. To order books from an online site, the buyer must open an account. The buyer needs a username and a password.
   a. If the username needs to be eight letters, how many different usernames are possible:
      i. if letters can be repeated?

         $26^8 = 2 \cdot 10^{11}$

      ii. if the letters cannot be repeated?

         ${}_{26}P_8 = 6.3 \cdot 10^{10}$

   b. If the password must be eight characters, which can be any of the 26 letters, 10 digits, and 12 special keyboard characters, how many passwords are possible:
      i. if characters can be repeated?

         $48^8 = 2.8 \cdot 10^{13}$

      ii. if characters cannot be repeated?

         ${}_{48}P_8 = 1.5 \cdot 10^{13}$

c. **How would your answers to part (b) change if the password is case-sensitive? (In other words, *Password* and *password* are considered different because the letter *p* is in uppercase and lowercase.)**

The answers would change because the number of letters that can be used will double to 52, which means the number of characters that can be used is now 74.

So, if characters can be repeated, the answer will be $72^8 = 7.2 \cdot 10^{14}$.

If characters cannot be repeated, the answer will be ${}_{72}P_8 = 4.8 \cdot 10^{14}$.

10. **Create a scenario to explain why ${}_3P_3 = 3!$.**

Suppose three friends are running in a race. ${}_3P_3$ can be used to model the order in which the three friends finish in 1st, 2nd, and 3rd place. There are three choices for 1st place and two choices for 2nd place, which only leaves one choice for 3rd place. So, there are $3 \cdot 2 \cdot 1 = 3!$, or 6 ways for the friends to finish the race.

11. **Explain why ${}_nP_n = n!$ for all positive integers n.**

Using the permutation formula:

$${}_nP_r = \frac{n!}{(n-r)!}.$$

In this case $r = n$, therefore:

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!.$$

Lesson 3: Counting Rules—Combinations

Student Outcomes

- Students calculate the number of different combinations of k items selected from a set of n distinct items.

Lesson Notes

This lesson continues counting rules by introducing the number of combinations of n things taken k at a time. A formal definition of *combination* is presented. Several applications will be developed in which students calculate the number of different combinations of k things selected from a set of n distinct items.

Classwork

Example 1 (8 minutes): Combinations

This lesson begins with an example that makes the distinction between situations in which order is important and those in which order is not important. Work through this example with the class. The definition of combinations as a subset of k items selected from a set of n distinct items is formally introduced. Focus on the set of distinct items and that the sequence or order is not important.

Example 1

Seven speed skaters are competing in an Olympic race. The first-place skater earns the gold medal, the second-place skater earns the silver medal, and the third-place skater earns the bronze medal. In how many different ways could the gold, silver, and bronze medals be awarded? The letters A, B, C, D, E, F, and G will be used to represent these seven skaters.

How can we determine the number of different possible outcomes? How many are there?

We can use the permutations formula from the previous lesson. Because each outcome is a way of forming an ordered arrangement of 3 things from a set of 7, the total number of possible outcomes is

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210.$$

Now consider a slightly different situation. Seven speed skaters are competing in an Olympic race. The top three skaters move on to the next round of races. How many different "top three" groups can be selected?

How is this situation different from the first situation? Would you expect more or fewer possibilities in this situation? Why?

The outcomes in the first situation are medals based on order–first place gets gold, second place gets silver, and third place gets bronze. The outcomes in this situation are not based on order. The top three finishers move on; the others do not. I would expect more possibilities in the first situation because each skater can be first, second, or third and still advance (i.e., there are three different possible finishing positions each skater can attain to earn advancement). In the first situation, there is only one position to earn gold, one for silver, and one for bronze.

Scaffolding:

Below are two questions to ask students who may be struggling with how to begin answering the questions.

- What are two examples of possible outcomes?

  Example: BCF and ADC

- Should the outcome CDA be considered as a different outcome than ADC, even though it includes the same three skaters?

  Yes, because the order in which skaters finish matters.

Would you consider the outcome where skaters B, C, and A advance to the final to be a different outcome than A, B, and C advancing?

No, this is the same outcome—the same three skaters are advancing to the final competition.

A permutation is an ordered arrangement (a sequence) of k items from a set of n distinct items.

In contrast, a combination is an unordered collection (a set) of k items from a set of n distinct items.

When we wanted to know how many ways there are for seven skaters to finish first, second, and third, order was important. This is an example of a permutation of 3 selected from a set of 7. If we want to know how many possibilities there are for which three skaters will advance to the finals, order is not important. This is an example of a combination of 3 selected from a set of 7.

After discussing the two situations and the difference between a permutation and a combination, provide an opportunity to check for understanding.

- Turn and talk to your neighbor: What is the difference between a permutation and a combination? With your partner, come up with an example of each.

Encourage students to elaborate if they provide answers such as, "Permutation means order matters, and combination means order doesn't matter." The order of what? Students should understand that in a permutation, the order of possible outcomes of a situation matters, or that outcomes need to happen in a specific order.. In a combination, the order of possible outcomes does not matter, or the set of specific outcomes needs to happen in no specific order.

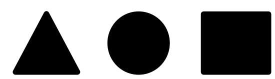
Exercises 1–4 (8 minutes)

MP.3

Students should work in small groups (2 or 3 students per group). Allow about 8 minutes for them to complete Exercises 1–4. Encourage students to write and/or talk to a partner about how they might answer Exercise 1. To assist students in making their conjectures, it may be beneficial to provide a diagram for them. This could be a circle with four points on it drawn on a whiteboard. When students have finished answering the four questions, discuss the answers. The main point is that if order was important, then segment *DC* would be different from *CD*. Since we are only interested in the number of segments, the order of the segment labels is not important. The purpose of this activity is to, again, distinguish between permutations and combinations.

Scaffolding:

- English language learners may have difficulty with the term *distinct,* so repeated and visual examples of the meaning should be considered.
- To aid in visualizing the meaning of *distinct* (different in a way that you can see, hear, smell, feel, etc.), put three different shapes in a bag, have students reach into the bag to feel each shape, and ask them to identify the distinct shapes and how they know they are different.



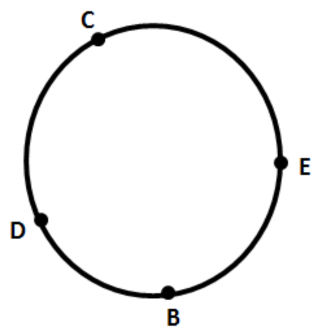
- Use seven students from the class to represent each of the seven skaters from the example. Each is distinct. While they are representing a group of skaters, each student is an individual and can be identified by a distinct name such as John or Claire (or your students' names).

Exercises 1–4

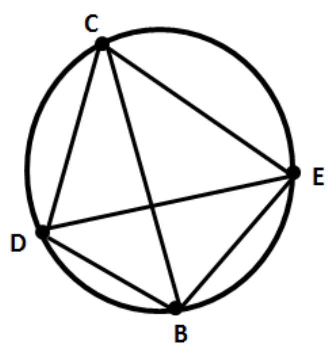
1. **Given four points on a circle, how many different line segments connecting these points do you think could be drawn? Explain your answer.**

   Answers will vary. Expect answers to range from 4 to 42

2. **Draw a circle and place four points on it. Label the points as shown. Draw segments (chords) to connect all the pairs of points. How many segments did you draw? List each of the segments that you drew. How does the number of segments compare to your answer in Exercise 1?**



Answer:



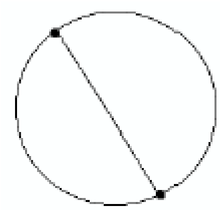
6 segments: $\overline{DC}, \overline{DE}, \overline{DB}, \overline{CB}, \overline{CE}$, *and* $\overline{BE}$

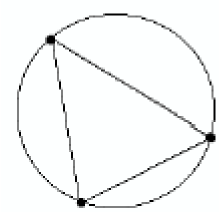
You can think of each segment as being identified by a subset of two of the four points on the circle. Chord ED is the same as chord DE. The order of the segment labels is not important. When you count the number of segments (chords), you are counting combinations of two points chosen from a set of four points.

Scaffolding:

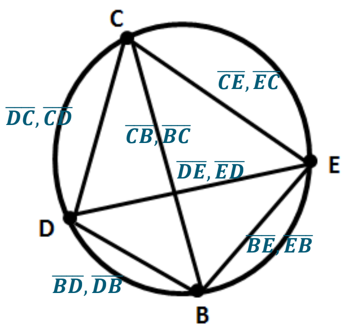
For students who struggle, pose the following questions:

- Given two points on a circle, how many different line segments connecting these points do you think could be drawn?
- It may be necessary to begin with two or three points on a circle and build up to four using the diagrams below.





3. **Find the number of permutations of two points from a set of four points. How does this answer compare to the number of segments you were able to draw?**



The diagram is the same, but this answer is double the number of combinations. In a permutation, the segment $\overline{DC}$ is counted as different from segment $\overline{CD}$. In this case, there are six combinations, and each segment can be represented two ways. $6 \times 2 = 12$ permutations.

4.

a. **If you add a fifth point to the circle, how many segments (chords) can you draw?**

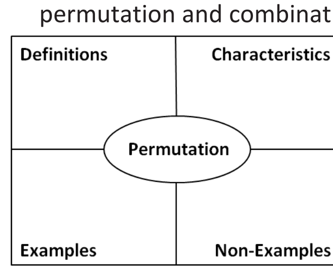
10 ***segments***

b. **If you add a sixth point, how many segments (chords) can you draw?**

15 ***segments***

Scaffolding:

- Students may associate the term *combination* with a combination of a lock.
- Clarify that in this context, *combination* refers to a selection of items from a group where order does not matter.
- The combination of a lock is a collection of numbers or letters where order *does* matter. (This is an example of a permutation.)
- Consider having students employ Frayer diagrams, such as the one below, to help distinguish between permutation and combination.



Example 2: Combinations Formula (7 minutes)

Briefly summarize the four examples from the lesson. Students should work independently to answer the question.

Example 2

Let's look closely at the four examples we have studied so far.

Choosing gold, silver, and bronze medal skaters	**Choosing groups of the top three skaters**
Finding the number of segments that can be drawn connecting two points out of four points on a circle	**Finding the number of unique segments that can be drawn connecting two points out of four points on a circle**

What do you notice about the way these are grouped?

Sample responses: The examples on the left are permutations, and the examples on the right are combinations. In the left examples, the order of the outcomes matter, while in those on the right, the order does not matter.

Now discuss as a class the combinations formula:

> **The number of combinations of k items selected from a set of n distinct items is**
>
> $${}_nC_k = \frac{{}_nP_k}{k!} \text{ or } \quad {}_nC_k = \frac{n!}{k!(n-k)!}.$$

The number of permutations of three skaters from the seven is found in Example 1 to be

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210.$$

This means that there are 210 different ordered arrangements of three skaters from a set of seven skaters.

There are fewer combinations because when order is not important, we do not want to count different orders of the same three skaters as different outcomes.

Consider the set of skaters A, B, and C. There are $3 \cdot 2 \cdot 1 = 3! = 6$ different ordered arrangements of these three skaters, each of which is counted with the permutations formula. So, if we want combinations of 3 from a set of 7, you would need to divide the number of permutations by 3!. Then

$${}_7C_3 = \frac{{}_7P_3}{3!} = \frac{7!}{3!\,(7-3)!} = \frac{7!}{3!\,4!} = 35.$$

In general

$${}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!\,(n-k)!}.$$

For example, ${}_9C_4$

$$\begin{aligned}
{}_9C_4 &= \frac{9!}{4!\,(9-4)!} \\
&= \frac{9!}{4! \cdot 5!} \\
&= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} \\
&= \frac{9 \cdot 8 \cdot 7 \cdot 6}{(4 \cdot 3 \cdot 2 \cdot 1)} \\
&= \frac{3024}{24} \\
&= 126
\end{aligned}$$

Be sure to discuss the connection between the combination formula and the permutation formula. If your students have access to technology (a graphing calculator or computer software), show them how to use technology to calculate the number of permutations and the number of combinations.

- Ask students, with a partner, to explain how to use combinations and permutations to explain the circle segments and point permutations examples. Use this as an opportunity to informally assess student learning.
  - The number of segments (chords) that can be drawn with four points on a circle is the number of combinations of two points selected from a set of four points.

$$_4C_2 = \frac{_4P_2}{2!} = 6 \text{ or } _4C_2 = \frac{4!}{2!\,(4-2)!} = 6$$

Exercises 5–11 (15 minutes)

MP.5

Students should work in small groups (2 or 3 students per group). Allow about 15 minutes for them to complete Exercises 4–11. Students may use a calculator, but encourage them to write the combination formula and show the substitution prior to using the calculator. In Exercise 7, students need to think carefully about whether to use the formula for permutations or for combinations in solving problems. Also encourage students to think about whether or not order is important in each problem. After students have completed the problems, review the answers.

Exercises 5–11

5. Find the value of each of the following:

a. $_9C_2$ — ***36***

b. $_7C_7$ — ***1***

c. $_8C_0$ — ***1***

d. $_{15}C_1$ — ***15***

6. Find the number of segments (chords) that can be drawn for each of the following:

a. 5 points on a circle — $_5C_2 = 10$

b. 6 points on a circle — $_6C_2 = 15$

c. 20 points on a circle — $_{20}C_2 = 190$

d. n points on a circle — $_nC_2 = \frac{_nP_2}{2!} = \frac{n!}{2!\,(n-2)!}$

7. For each of the following questions, indicate whether the question posed involves permutations or combinations. Then provide an answer to the question with an explanation for your choice.

a. A student club has 20 members. How many ways are there for the club to choose a president and a vice-president?

Permutations, 380. The role of president is different than that of vice-president. The order of outcomes matters.

b. A football team of 50 players will choose two co-captains. How many different ways are there to choose the two co-captains?

Combinations, $1,225$. Regardless of order, two players will attain the same outcome of co-captain.

c. **There are seven people who meet for the first time at a meeting. They shake hands with each other and introduce themselves. How many handshakes have been exchanged?**

Combinations, 21 People only shake hands with those they have not greeted yet. Each greeting is unique. Person 1 shaking hands with person 2 is the same event as person 2 shaking hands with person 1.

d. **At a particular restaurant, you must choose two different side dishes to accompany your meal. If there are eight side dishes to choose from, how many different possibilities are there?**

Combinations, 28 The order of dishes chosen does not matter. Choosing veggies and mac and cheese is the same as choosing mac and cheese and veggies.

e. **How many different four-letter sequences can be made using the letters A, B, C, D, E, and F if letters may not be repeated?**

Permutations, 360 Each four-letter sequence is unique. ABCD is different from BACD even though they contain identical letters.

8. **How many ways can a committee of 5 students be chosen from a student council of 30 students? Is the order in which the members of the committee are chosen important?**

No, ${}_{30}C_5 = 142,506$.

9. **Brett has ten distinct t-shirts. He is planning on going on a short weekend trip to visit his brother in college. He has enough room in his bag to pack four t-shirts. How many different ways can he choose four t-shirts for his trip?**

${}_{10}C_4 = 210$

MP.3

10. **How many three-topping pizzas can be ordered from the list of toppings below? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?**

Pizza Toppings

sausage	pepperoni	meatball	onions	olives	spinach
pineapple	ham	green peppers	mushrooms	bacon	hot peppers

${}_{12}C_3 = 220$; I used combinations because the order of toppings chosen does not matter. My pizza is the same if I order pepperoni, olives, and mushrooms or olives, pepperoni, and mushrooms.

Calculated number of combinations—we are choosing 3 items from the 12 distinct toppings, and the order that the 3 toppings are chosen is not important.

11. **Write a few sentences explaining how you can distinguish a question about permutations from a question about combinations.**

Answers will vary but should address whether order is important or not.

Closing (2 minutes)

- Ask students to explain the difference between a permutation and combination. If they struggle with the articulating the difference, ask them to provide an example for each instead.
  - *If order does matter, it is a permutation. A phone number.*
  - *If order doesn't matter, it is a combination. Picking five friends from an entire class for a dodge-ball team.*

- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

> **Lesson Summary**
>
> **A combination is a subset of k items selected from a set of n distinct items.**
> **The number of combinations of k items selected from a set of n distinct items is**
>
> $$_nC_k = \frac{{}_nP_k}{k!} \quad \text{or} \quad {}_nC_k = \frac{n!}{k!(n-k)!}.$$

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 3: Counting Rules—Combinations

Exit Ticket

1. Timika is a counselor at a summer camp for young children. She wants to take 20 campers on a hike and wants to choose a pair of students to lead the way. In how many ways can Timika choose this pair of children?

2. Sean has 56 songs on his MP3 player. He wants to randomly select 6 of the songs to use in a school project. How many different groups of 6 songs could Sean select? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?

3. A fast food restaurant has the following options for toppings on their hamburgers: mustard, ketchup, mayo, onions, pickles, lettuce, and tomato. In how many ways could a customer choose four different toppings from these options?

4. Seven colored balls (red, blue, yellow, black, brown, white, and orange) are in a bag. A sample of three balls is selected without replacement. How many different samples are possible?

Exit Ticket Sample Solutions

1. Timika is a counselor at a summer camp for young children. She wants to take 20 campers on a hike and wants to choose a pair of students to lead the way. In how many ways can Timika choose this pair of children?

   ${}_{20}C_2 = 190$

2. Sean has 56 songs on his MP3 player. He wants to randomly select 6 of the songs to use in a school project. How many different groups of 6 songs could Sean select? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?

   ${}_{56}C_6 = 32,468,436$

   Calculated number of combinations—choosing 6 songs from 56 distinct songs, and order is not important

3. A fast food restaurant has the following options for toppings on their hamburgers: mustard, ketchup, mayo, onions, pickles, lettuce, and tomato. In how many ways could a customer choose 4 different toppings from these options?

   $${}_{7}C_4 = 35$$

4. Seven colored balls (red, blue, yellow, black, brown, white, and orange) are in a bag. A sample of three balls is selected without replacement. How many different samples are possible?

   $${}_{7}C_3 = 35$$

Problem Set Sample Solutions

1. Find the value of each of the following:

   a. ${}_{9}C_8$ 9

   b. ${}_{9}C_1$ 9

   c. ${}_{9}C_9$ 1

2. Explain why ${}_{6}C_4$ is the same value as ${}_{6}C_2$.

   ${}_{6}C_4 = \frac{6!}{4! \cdot 2!}$ ${}_{6}C_2 = \frac{6!}{2! \cdot 4!}$

   The denominators are the same. This is because the number of ways to choose four from a set of six is the same as the number of ways to select which two to exclude.

3. Pat has 12 books he plans to read during the school year. He decides to take 4 of these books with him while on winter break vacation. He decides to take *Harry Potter and the Sorcerer's Stone* as one of the books. In how many ways can he select the remaining 3 books?

   ${}_{11}C_3 = 165$

4. In a basketball conference of 10 schools, how many conference basketball games are played during the season if the teams all play each other exactly once?

   ${}_{10}C_2 = 45$

5. Which scenario(s) below is represented by ${}_9C_3$? *B and C*
   a. Number of ways 3 of 9 people can sit in a row of 3 chairs.
   b. Number of ways to pick 3 students out of 9 students to attend an art workshop.
   c. Number of ways to pick 3 different entrees from a buffet line of 9 different entrees.

6. Explain why ${}_{10}C_3$ would not be used to solve the following problem:

   There are 10 runners in a race. How many different possibilities are there for the runners to finish first, second, and third?

   This is an example of a permutation. The order of how the runners finish is important.

7. In a lottery, players must match five numbers plus a bonus number. Five white balls are chosen from 59 white balls numbered from 1 to 59, and one red ball (the bonus number) is chosen from 35 red balls numbered 1 to 35. How many different results are possible?

   white ball: ${}_{59}C_5 = 5,006,386$

   red ball: ${}_{35}C_1 = 35$

   Number of possible results $= 5,006,386 \cdot 35 = 175,223,510$

8. In many courts, 12 jurors are chosen from a pool of 30 perspective jurors.
   a. In how many ways can 12 jurors be chosen from the pool of 30 perspective jurors?

   $${}_{30}C_{12} = 86,493,225$$

   b. Once the 12 jurors are selected, 2 alternates are selected. The order of the alternates is specified. If a selected juror cannot complete the trial, the first alternate is called on to fill that jury spot. In how many ways can the 2 alternates be chosen after the 12 jury members have been chosen?

   $${}_{18}P_2 = 306$$

9. A band director wants to form a committee of 4 parents from a list of 45 band parents.
   a. How many different groups of 4 parents can the band director select?

   $${}_{45}C_4 = 148,995$$

   b. How many different ways can the band director select 4 parents to serve in the band parents' association as president, vice-president, treasurer, and secretary?

   ${}_{45}P_4 = 3,575,880$

   c. Explain the difference between parts (a) and (b) in terms of how you decided to solve each part.

   Part (a) is an example of finding the number of combinations—how many ways can 4 parents be chosen from 45 distinct parents. The order is not important.

   Part (b) is an example of finding the number of permutations. The order of how the parents are selected is important.

10. A cube has faces numbered 1 to 6. If you roll this cube 4 times, how many different outcomes are possible?

    Using the fundamental counting principle: $6^4 = 1,296$

11. Write a problem involving students that has an answer of ${}_6C_3$.

Answers will vary. One example follows:

There are six seniors on the principal's advisory committee on school improvement. The principal would like to select three of the students to attend a workshop on school improvement. How many ways can the principal select three students out of the six students on the advisory committee?

12. Suppose that a combination lock is opened by entering a three-digit code. Each digit can be any integer between 0 and 9, but digits may not be repeated in the code. How many different codes are possible? Is this question answered by considering permutations or combinations? Explain.

There are 720 possible codes. This question is answered by considering *permutations because in a code, the order of the digits is important.*

13. Six musicians will play in a recital. Three will perform before intermission, and three will perform after intermission. How many different ways are there to choose which three musicians will play before intermission? Is this question answered by considering permutations or combinations? Explain.

There are 20 possible ways to choose the musicians. *This questions is answered by considering combinations because order is not important if we just care about which group of three is before the intermission.*

14. In a game show, contestants must guess the price of a product. A contestant is given nine cards with the numbers 1 to 9 written on them (each card has a different number). The contestant must then choose three cards and arrange them to produce a price in dollars. How many different prices can be formed using these cards? Is this question answered by considering permutations or combinations? Explain.

There are 504 possible prices. The questions is answered by considering *permutations because the order of the digits is important. $\$123$ is different than $\$312$.*

15. a. Using the formula for combinations, show that the number of ways of selecting 2 items from a group of 3 items is the same as the number of ways to select 1 item from a group of 3.

$${}_3C_2 = \frac{3!}{2!\cdot 1!} \qquad = \frac{3!}{1!\cdot 2!}$$

b. Show that ${}_nC_k$ and ${}_nC_{n-k}$ are equal. Explain why this makes sense.

$${}_nC_k = \frac{n!}{k!\cdot(n-k)!} \qquad {}_nC_{n-k} = \frac{n!}{(n-k)!\cdot k!}$$

The denominators are the same. This is because the number of ways to choose k from a set of n is the same as the number of ways to select which $(n-k)$ to exclude.

Lesson 4: Using Permutations and Combinations to Compute Probabilities

Student Outcomes

- Students distinguish between situations involving combinations and situations involving permutations.
- Students use permutations and combinations to calculate probabilities.
- Students interpret probabilities in context.

Lesson Notes

This lesson ties together the previous three lessons. It builds on the concepts of permutations and combinations by presenting applications in which students decide whether the application is a permutation problem or a combination problem. Based on these decisions, students calculate probabilities.

Classwork

Exercises 1–6 (12 minutes)

The first example is a review of the previous two lessons. Discuss the difference between a permutation and a combination. Allow students a few minutes to answer Exercises 1–2. Discuss the answers to each question.

Students should work in small groups (2 or 3 students per group) on Exercises 3–6. Allow about 8 minutes for them to complete the exercises. When students have finished, discuss the answers as a class. Allow students to use a calculator to determine the answer for each. While students are working, teachers should circulate around the room to assess their progress on these concepts.

Scaffolding:

- Consider writing the formulas for ${}_nP_k$ and the formula for ${}_nC_k$ on the board.
- For advanced learners, consider the following extension activity:

  "Describe a situation that could be described by ${}_5P_3$ and another one that could be described by ${}_5C_3$. How are they similar? How are they different?"

Exercises 1–6

1. **A high school is planning to put on the musical *West Side Story*. There are 20 singers auditioning for the musical. The director is looking for two singers who could sing a good duet. In how many ways can the director choose two singers from the 20 singers?**

   Indicate if this question involves a permutation or a combination. Give a reason for your answer.

   Combination; the order that the singers are selected is not important.

2. **The director is also interested in the number of ways to choose a lead singer and a backup singer. In how many ways can the director choose a lead singer and then a backup singer?**

   Indicate if this question involves a permutation or a combination. Give a reason for your answer.

   Permutation; the order is important; the lead is chosenfirst, then the backup singer is chosen.

3. **For each of the following, indicate if it is a problem involving permutations, combinations, or neither, and then answer the question posed. Explain your reasoning.**

   a. **How many groups of five songs can be chosen from a list of 35 songs?**

      Combination; the order of the songs is not important; $_{35}C_5 = 324,632$

   b. **How many ways can a person choose three different desserts from a dessert tray of eight desserts?**

      Combination; the order of the desserts is not important; $_8C_3 = 56$

   c. **How many ways can a manager of a baseball team choose the lead-off batter and second batter from a baseball team of nine players?**

      Permutation; the order of the batters is important; $_9P_2 = 72$

   d. **How many ways are there to place seven distinct pieces of art in a row?**

      Permutation; the order of each piece of art is important; $_7P_7$ *or* $7! = 5,040$

   e. **How many ways are there to randomly select four balls without replacement from a container of 15 balls numbered 1 to 15?**

      Combination; the order of the balls is not important; $_{15}C_4 = 1,365$

4. **The manager of a large store that sells TV sets wants to set up a display of all the different TV sets that they sell. The manager has seven different TVs that have screen sizes between 37 and 43 inches, nine that have screen sizes between 46 and 52 inches, and twelve that have screen sizes of 55 inches or greater.**

   a. **In how many ways can the manager arrange the 37–43 inch TV sets?**

      $7! = 5,040$

   b. **In how many ways can the manager arrange the 55-inch or greater TV sets?**

      $12! = 479,001,600$

   c. **In how many ways can the manager arrange all the TV sets if he is concerned about the order they were placed in?**

      $28!\ or\ _{28}P_{28} \approx 3.8 * 10^{29}$

Consider demonstrating Exercise 5. Place seven slips of paper in a jar or bag numbered 1 to 7. Draw out two slips, recording the order of the digits drawn. Discuss with students possible outcomes of selecting the slips. Before proceeding with the rest of the exercise, also make sure that students can suggest a way to organize the outcomes. It is not necessary to have an organized list of the outcomes to answer the questions. It is important, however, that students are able to describe a way to generate the list. If students have problems explaining a process, review with them some of the exercises in the previous lessons in which they organized the outcomes using strategic lists or trees.

5. Seven slips of paper with the digits 1 to 7 are placed in a large jar. After thoroughly mixing the slips of paper, two slips are picked without replacement,

   a. Explain the difference between ${}_7P_2$ and ${}_7C_2$ in terms of the digits selected.

   ${}_7P_2$ is the number of permutations of picking two slips from the seven slips of paper. The value of ${}_7P_2$ is the number of ways that the two slips can be picked from the seven slips of paper in which the order of the digits is important. For example, if the digit "2" is picked first and the digit "1" is picked second, then "21" is considered a different outcome than if "1" is picked first and "2" is picked second, or "12". ${}_7C_2$ is the combination of picking two slips from the seven slips. It represents the total number of ways two digits can be selected in which order does not matter. Therefore, "21" and "12" are not counted as different outcomes.

   b. Describe a situation in which ${}_7P_2$ is the total number of outcomes.

   Answers will vary. One example could be similar to the following: Seven students from your school are eligible to participate in a tennis competition; however, only one student can compete, with one other student designated as a backup. (The backup will compete if the first student is injured or unable to attend.) ${}_7P_2$ represents how many different pairings of the seven students could be selected for the competition.

   c. Describe a situation in which ${}_7C_2$ is the total number of outcomes.

   Answers will vary. One example that could be developed by students is the following: Two students from seven eligible students will receive a prize. Each person is assigned a number from 1 to 7 (with no duplicates). Their numbers are placed in the jar. Two slips are drawn. The students assigned to the selected digits are the winners and will receive the prizes. ${}_7C_2$ represents the number of ways two people from the group of seven students could win the prizes.

   d. What is the relationship between ${}_7P_2$ and ${}_7C_2$?

   $${}_7C_2 = \left(\frac{{}_7P_2}{2!}\right)$$

6. If you know ${}_nC_k$, and you also know the value of n and k, how could you find the value of ${}_nP_k$? Explain your answer.

   The following steps show the relation to be the value of the combination and the permutation:

   Recall that ${}_nC_k = \frac{n!}{k!(n-k)!}$.

   Multiply each side of the above equation by $k!$, or

   $$k!\,{}_nC_k = \frac{n!}{(n-k)!}.$$

   $\frac{n!}{(n-k)!}$ is ${}_nP_k$, therefore:

   $$k!\,{}_nC_k = {}_nP_k.$$

Example 1 (6 minutes): Calculating Probabilities

MP.3 In this example, students calculate probabilities using combinations and permutations. Students engage in problem solving as a form of argument and need to understand the question asked and how to distinguish between a combination and a permutation as they analyze the situation. Consider beginning the example by asking students to attempt to solve the problem on their own before encouraging them to collaborate; then, discuss answers as a class.

Example 1: Calcualting Probabilites

In a high school there are 10 math teachers. The principal wants to form a committee by selecting three math teachers at random. If Mr. H, Ms. B, and Ms. J are among the group of 10 math teachers, what is the probability that all three of them will be on the committee?

Because every different committee of 3 is equally likely,

$$P(\text{these three math teachers will be on the commitee}) = \frac{\text{number of ways Mr. H, Ms. B, and Ms. J can be selected}}{\text{total number of 3 math teacher committees that can be formed}}$$

The total number of possible committees is the number of ways that three math teachers can be chosen from 10 math teachers, which is the number of combinations of 10 math teachers taken 3 at a time or ${}_{10}C_3 = 120$. Mr. H, Ms. B, and Ms. J form one of these selections. The probability that the committee will consist of Mr. H, Ms. B, and Ms. J is $\frac{1}{120}$.

Before students begin work on the next set of exercises, have them discuss the following with a partner:

- How did combinations or permutations help us to answer the question?
  - *The probability of this event is based on equally likely outcomes. We needed to determine how many ways three teachers could be selected from ten teachers. Since order did not matter, we could use a combination to help us determine the number of ways.*

Exercises 7–9 (7 minutes)

MP.1 Students should work in small groups (2 or 3 students per group). Allow about 7 minutes for students to complete the exercises. Exercise 7 references the problem introduced in Exercise 1. In these exercises, students must make sense of the situation presented and determine whether a combination or permutation is needed in order to calculate probabilities. When students have finished answering the questions, discuss the answers. You may need to remind students how the calculator expresses an answer in scientific notation.

Exercises 7–9

7. **A high school is planning to put on the musical *West Side Story*. There are 20 singers auditioning for the musical. The director is looking for two singers who could sing a good duet.**

   a. **What is the probability that Alicia and Juan are the two singers who are selected by the director? How did you get your answer?**

$$\frac{1}{\left({}_{20}C_2\right)} \approx 0.005$$

This question involves a combination because the order of the two students selected does not matter. The probability of one of the selections (Alicia and Juan) would be 1 divided by the combination.

   b. **The director is also interested in the number of ways to choose a lead singer and a backup singer. What is the probability that Alicia is selected the lead singer and Juan is selected the backup singer? How did you get your answer?**

$$\frac{1}{{}_{20}P_2} \approx 0.0026$$

This question involves a permutation because the order of the two singers matters. The probability of one of these selections (Alicia as the lead singer and Juan as the backup) would be 1 divided by the permutation.

8. **For many computer tablets, the owner can set a 4-digit pass code to lock the device.**

   a. **How many different 4-digit pass codes are possible if the digits cannot be repeated? How did you get your answer?**

$${}_{10}P_4 = 5{,}040$$

I used a permutation because order matters.

   b. **If the digits of a pass code are chosen at random and without replacement from the digits $0, 1, \ldots, 9$, what is the probability that the pass code is 1234? How did you get your answer?**

$$\frac{1}{5{,}040} \approx 1.98 * 10^{-4} = 0.000198$$

The pass code 1234 is 1 out of the total number of possible pass codes. Therefore, the probability would be 1 divided by the permutation representing the total number of pass codes.

   c. **What is the probability that two people, who both chose a pass code by selecting digits at random and without replacement, both have a pass code of 1234? Explain your answer.**

$$\frac{1}{5040} * \frac{1}{5040} \approx 3.9 * 10^{-8} = 0.000000039$$

I multiplied the probability of the first person getting this pass code by the probability of a second person getting this pass code.

9. A chili recipe calls for ground beef, beans, green pepper, onion, chili powder, crushed tomatoes, salt, and pepper. You have lost the directions about the order in which to add the ingredients, so you decide to add them in a random order.

   a. How many different ways are there to add the ingredients? How did you get this answer?

   $${}_8P_8 \text{ or } 8! = 40,320$$

   This problem indicates that the order of adding the ingredients is important. The total number of ways of adding the eight ingredients in which order is important is the permutation indicated.

   b. What is the probability that the first ingredient that you add is crushed tomatoes? How did you get your answer?

   $$\frac{1}{8} = 0.125$$

   There are eight ingredients to pick for my first pick. The probability of selecting crushed tomatoes would be the probability of selecting one of the eight ingredients, or $\frac{1}{8}$.

   c. What is the probability that the ingredients are added in the exact order listed above? How did you get your answer?

   $$\frac{1}{40,320} \approx 2.5 * 10^{-5} = 0.000025$$

   The exact order of adding the ingredients represents 1 *of the total number of permutations of the eight ingredients. The probability of selecting this* 1 *selection would be* 1 *divided by the total number of permutations.*

Example 2 (5 minutes): Probability and Combinations

This example and the next exercise set are extensions of Example 1 and Exercises 7–9. Example 2 and Exercises 10–11 can be considered optional.

Example 2: Probabiltiy and Combinations

A math class consists of 14 girls and 15 boys. The teacher likes to have the students come to the board to demonstrate how to solve some of the math problems. During a lesson the teacher randomly selects 6 of the students to show their work. What is the probability that all 6 of the students selected are girls?

$$P(\text{all 6 students are girls}) = \frac{\text{number of ways to select 6 girls out of 14}}{\text{number of groups of 6 from the whole class}}$$

The number of ways to select 6 girls from the 14 girls is the number of combinations of 6 from 14, which is ${}_{14}C_6 = 3,003$. The total number of groups of 6 is ${}_{29}C_6 = 475,020$.

The probability that all 6 students are girls is

$$P(\text{all 6 students are girls}) = ({}_{14}C_6)/({}_{29}C_6) = \frac{3,003}{475,020} = 0.006$$

Scaffolding:

Discuss this example by presenting it in three parts.

- First, write the probability formula on the board.
- Second, find the number of groups of 6 from the whole class.
- Third, find the number of ways to select 6 girls out of the 14. Then, substitute into the probability formula to complete the problem.

Exercises 10–11 (8 minutes)

If Example 2 was not discussed, Exercises 10–11 should not be assigned.

Students should work in small groups (2 or 3 students per group). Allow about 8 minutes for them to complete the questions. When students have finished, discuss the answers as a class.

Exercises 10–11

10. **There are nine golf balls numbered from 1 to 9 in a bag. Three balls are randomly selected without replacement to form a 3-digit number.**

    a. **How many 3-digit numbers can be formed? Explain your answer.**

    $${}_9P_3 = 504$$

    Order is important. As a result, I formed the permutation of selecting three golf balls from the nine golf balls.

    b. **How many 3-digit numbers start with the digit 1? Explain how you got your answer.**

    $$1 \cdot {}_8P_2 = 1 \cdot 8 \cdot 7 = 56$$

    If I select the digit 1 first, then there are 8 digits left for the other two digits. The number of ways of picking two of the remaining eight digits would be how many 3-digit numbers formed with the digit 1 in the first position.

    c. **What is the probability that the 3-digit number formed is less than 200? Explain your answer.**

    $$\frac{{}_8P_2}{{}_9P_3} \approx 0.111$$

    The probability of a 3-digit number formed that is less than 200 would be the probability formed by the number of 3-digit numbers that start with 1 (my answer to part (b)) divided by the total number of 3-digit numbers (my answer to part (a)).

11. **There are eleven seniors and five juniors who are sprinters on the high school track team. The coach must select four sprinters to run the 800-meter relay race.**

    a. **How many 4-sprinter relay teams can be formed from the group of 16 sprinters?**

    $${}_{16}C_4 = 1{,}820$$

    b. **In how many ways can two seniors be chosen to be part of the relay team?**

    $${}_{11}C_2 = 55$$

    c. **In how many ways can two juniors be chosen to be part of the relay team?**

    $${}_5C_2 = 10$$

    d. **In how many ways can two seniors and two juniors be chosen to be part of the relay team?**

    $${}_{11}C_2 * {}_5C_2 = 550$$

    e. **What is the probability that two seniors and two juniors will be chosen for the relay team?**

    $$\frac{{}_{11}C_2 * {}_5C_2}{{}_{16}C_4} = \frac{550}{1820} \approx 0.302$$

Closing (2 minutes)

- Explain how permutations and combinations can aid in calculating probabilities.
  - *Permutations and combinations can help us determine how many outcomes there are for an event, which can be used to calculate the probability.*
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

Lesson Summary

- **The number of permutations of n things taken k at a time is**

$$_nP_k = \frac{n!}{(n-k)!}.$$

- **The number of combinations of k items selected from a set of n distinct items is**

$$_nC_k = \frac{_nP_k}{k!} \text{ or } {_nC_k} = \frac{n!}{k!(n-k)!}.$$

- **Permutations and combinations can be used to calculate probabilities.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 4: Using Permutations and Combinations to Compute Probabilities

Exit Ticket

1. An ice-cream shop has 25 different flavors of ice cream. For each of the following, indicate whether it is a problem that involves permutations, combinations, or neither.
   a. What is the number of different 3-scoop ice-cream cones that are possible if all three scoops are different flavors, and a cone with vanilla, strawberry, and chocolate is different from a cone with vanilla, chocolate, and strawberry?

   b. What is the number of different 3-scoop ice-cream cones that are possible if all three scoops are different flavors, and a cone with vanilla, strawberry, and chocolate is considered the same as a cone with vanilla, chocolate, and strawberry?

   c. What is the number of different ice-cream cones if all three scoops could be the same, and the order of the flavors is important?

2. A train consists of an engine at the front, a caboose at the rear, and 27 boxcars that are numbered from 1 to 27.
   a. How many different orders are there for cars that make up the train?

   b. If the cars are attached to the train in a random order, what is the probability that the boxcars are in numerical order from 1 to 27?

3. The dance club at school has 22 members. The dance coach wants to send four members to a special training on new dance routines.
   a. The dance coach will select four dancers to go to the special training. Is the number of ways to select four dancers a permutation, a combination, or neither? Explain your answer.

   b. If the dance coach chooses at random, how would you determine the probability of selecting dancers Laura, Matthew, Lakiesha, and Santos?

Exit Ticket Sample Solutions

Note: The Exit Ticket does not contain problems similar to Example 2 or Exercises 10–11.

1. **An ice-cream shop has 25 different flavors of ice cream. For each of the following, indicate whether it is a problem that involves permutations, combinations, or neither.**
   a. **What is the number of different 3-scoop ice-cream cones that are possible if all three scoops are different flavors, and a cone with vanilla, strawberry, and chocolate is different from a cone with vanilla, chocolate, and strawberry.**

      Permutation

   b. **What is the number of different 3-scoop ice-cream cones that are possible if all three scoops are different flavors, and a cone with vanilla, strawberry, and chocolate is considered the same as a cone with vanilla, chocolate, and strawberry.**

      Combination

   c. **What is the number of different ice-cream cones if all three scoops could be the same, and the order of the flavors is important?**

      Neither; $25^3 = 15,625$

2. **A train consists of an engine at the front, a caboose at the rear, and 27 boxcars that are numbered from 1 to 27.**
   a. **How many different orders are there for cars that make up the train?**

      $1 \cdot 27! \cdot 1 \approx 1 \cdot 10^{28}$

   b. **If the cars are attached to the train in a random order, what is the probability that the boxcars are in numerical order from 1 to 27?**

      $\frac{1}{27!} \approx 9.2 \cdot 10^{-29}$

3. **The dance club at school has 22 members. The dance coach wants to send four members to a special training on new dance routines.**
   a. **The dance coach will select four dancers to go to the special training. Is the number of ways to select four dancers a permutation, a combination, or neither? Explain your answer.**

      The order of the dancers is not important; therefore, the number of ways of selecting four dancers would be a combination of four dancers from 22 possible dancers.

   b. **If the dance coach chooses at random, how would you determine the probability of selecting dancers Laura, Matthew, Lakiesha, and Santos?**

      The probability of selecting one of the combinations would be 1 divided by the total number of combinations.

Problem Set Sample Solutions

If Example 2 and Exercises 10–11 were not discussed, Problem 6 part (c), 7 part (d), 8 parts (d) and (e), and 9 should not be assigned.

1. **For each of the following, indicate whether it is a question that involves permutations, combinations, or neither, and then answer the question posed. Explain your reasoning.**

   a. **How many ways can a coach choose two co-captains from 16 players in the basketball team?**

      Combination; order does not matter; $_{16}C_2 = 120$

   b. **In how many ways can seven questions out of ten be chosen on an examination?**

      Combination; the order of the questions does not matter; $_{10}C_7 = 120$

   c. **Find the number of ways that 10 women in the finals of the skateboard street competition can finish first, second, and third in the X Games final.**

      Permutation; the order of the women is important in determining first, second, and third place; $_{10}P_3 = 720$

   d. **A postal zip code contains five digits. How many different zip codes can be made with the digits 0–9? Assume a digit can be repeated.**

      Neither; the digits can repeat; $10^5 = 100,000$

2. **Four pieces of candy are drawn at random from a bag containing five orange pieces and seven brown pieces.**

   a. **How many different ways can four pieces be selected from the 12 colored pieces?**

      $_{12}C_4 = 495$

   b. **How many different ways can two orange pieces be selected from five orange pieces?**

      $_5C_2 = 10$

   c. **How many different ways can two brown pieces be selected from seven brown pieces?**

      $_7C_2 = 21$

3. **Consider the following:**

   a. **A game was advertised as having a probability of 0.4 of winning. You know that the game involved five cards with a different digit on each card. Describe a possible game involving the cards that would have a probability of 0.4 of winning.**

      Answers will vary. One possible answer would be to have the numbers $1, 2, 3, 4, 5$ *on the cards. A card is selected at random. If the number is even, you win the game.*

b. A second game involving the same five cards was advertised as having a winning probability of 0.05. Describe a possible game that would have a probability of 0.05 or close to 0.05 of winning.

Answers will again vary. Given that this probability is considerably less than the probability in Exercise 3 part (a), students would be expected to consider games in which more than one card is randomly selected, and the numbers formed from the selections would be involved in winning the game. If they randomly select two cards from the five possible cards, the probability of picking one of the possible 2-digit numbers without replacement is $\frac{1}{{}_7P_2}$, or 0.05. Encourage students to experiment with their suggestions by approximating the winning probabilities.

4. You have five people who are your friends on a certain social network. You are related to two of the people, but you do not recall who of the five people are your relatives. You are going to invite two of the five people to a special meeting. If you randomly select two of the five people to invite, explain how you would derive the probability of inviting your relatives to this meeting?

The number of ways of picking two people does not involve order. There is only one way I could pick the two relatives, and that would be if my pick involved both relatives. I would divide 1 by the total number of ways I could pick two people from five people, or the combination of picking two out of five.

5. Charlotte is picking out her class ring. She can select from a ruby, an emerald, or an opal stone, and she can also select silver or gold for the metal.

a. How many different combinations of one stone and one type of metal can she choose? Explain how you got your answer.

6 *I multiplied the number of different stones by the number of different metals.*

b. If Charlotte selects a stone and a metal at random, what is the probability that she would select a ring with a ruby stone and gold metal?

$$\frac{1}{6}$$

6. In a lottery, three numbers are chosen from 0 to 9. You win if the three numbers you pick match the three numbers selected by the lottery machine.

a. What is the probability of winning this lottery if the numbers cannot be repeated?

$$\frac{1}{{}_{10}C_3} \approx 0.008$$

b. What is the probability of winning this lottery if the numbers can be repeated?

$$\frac{1}{10^3} = 0.001$$

c. What is the probability of winning this lottery if you must match the exact order that the lottery machine picked the numbers?

$$\frac{1}{{}_{10}P_3} \approx 0.0014$$

7. **The store at your school wants to stock t-shirts that come in five sizes (small, medium, large, XL, XXL) and in two colors (orange and black).**

   a. **How many different type t-shirts will the store have to stock?**

   10

   b. **At the next basketball game, the cheerleaders plan to have a t-shirt toss. If they have one t-shirt of each type in a box and select a shirt at random, what is the probability that the first randomly selected t-shirt is a large orange t-shirt?**

   $\frac{1}{10}$

8. **There are 10 balls in a bag numbered from 1 to 10. Three balls are selected at random without replacement.**

   a. **How many different ways are there of selecting the three balls?**

   ${}_{10}C_3 = 120$

   b. **What is the probability that one of the balls selected is the number 5?**

   $\frac{{}_{9}C_2}{{}_{10}C_3} = 0.3$

9. **There are nine slips of paper in a bag numbered from 1 to 9 in a bag. Four slips are randomly selected without replacement to form a 4-digit number.**

   a. **How many 4-digit numbers can be formed?**

   ${}_{9}P_4 = 3,024$

   b. **How many 4-digit numbers start with the digit 1?**

   ${}_{8}P_3 = 336$

   c. **What is the probability that the 2-digit number formed is less than 20?**

   $\frac{{}_{8}P_3}{{}_{9}P_4} = \frac{336}{3024} \approx 0.111$

10. **There are fourteen juniors and twenty-three seniors in the Service Club. The club is to send four representatives to the State Conference.**

    a. **How many different ways are there to select a group of four students to attend the conference from the 37 Service Club members?**

    ${}_{37}C_4 = 66,045$

    b. **How many are there ways to select exactly two juniors?**

    ${}_{14}C_2 = 91$

    c. **How many ways are there to select exactly two seniors?**

    ${}_{23}C_2 = 253$

d. **If the members of the club decide to send two juniors and two seniors, how many different groupings are possible?**

$${}_{14}C_2 \cdot {}_{23}C_2 = 23{,}023$$

e. **What is the probability that two juniors and two seniors are selected to attend the conference?**

$$\frac{{}_{14}C_2 \cdot {}_{23}C_2}{{}_{37}C_4} = \frac{23{,}023}{66{,}045} \approx 0.349$$

11. **A basketball team of 16 players consists of 6 guards, 7 forwards, and 3 centers. Coach decides to randomly select 5 players to start the game. What is probability of 2 guards, 2 forwards, and 1 center starting the game?**

$$\frac{{}_{6}C_2 \cdot {}_{7}C_2 \cdot {}_{3}C_1}{{}_{16}C_5} \approx 0.216$$

12. **A research study was conducted to estimate the number of white perch (a type of fish) in a Midwestern lake. 300 perch were captured and tagged. After they were tagged, the perch were released back into the lake. A scientist involved in the research estimates there are $1{,}000$ perch in this lake. Several days after tagging and releasing the fish, the scientist caught 50 perch of which 20 were tagged. If this scientist's estimate about the number of fish in the lake is correct, do you think it was likely to get 20 perch out of 50 with a tag? Explain your answer.**

Assume the total number of fish was $1{,}000$ perch. The probability of getting 20 of 50 perch tagged would be based on the number of ways to get 20 perch from the 300 tagged fish, multiplied by the number of ways of getting 30 perch from the 700 fish that are not tagged, divided by the number of ways of picking 50 fish from $1{,}000$ fish. This result is the requested probability, or

$$\frac{({}_{300}C_{20})({}_{700}C_{30})}{{}_{1000}C_{50}}.$$

The above probability is approximately 0.04. This is a small probability and not likely to occur.

Topic B:

Random Variables and Discrete Probability Distributions

S-MD.A.1, S-MD.A.2, S-MD.A.3, S-MD.A.4

Focus Standards:	S-MD.A.1	(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.
	S-MD.A.2	(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.
	S-MD.A.3	(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.*
	S-MD.A.4	(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in* 100 *randomly selected households?*
Instructional Days:	8	
Lesson 5:	Discrete Random Variables (P)[1]	
Lesson 6:	Probability Distribution of a Discrete Random Variable (P)	
Lesson 7:	Expected Value of a Discrete Random Variable (E)	
Lesson 8:	Interpreting Expected Value (E)	
Lessons 9–10:	Determining Discrete Probability Distributions (P)	
Lessons 11–12:	Estimating Probability Distributions Empirically (P,E)	

In this topic, students first distinguish between discrete and continuous random variables and then focus on probability distributions for discrete random variables. In the early lessons of this topic, students develop an

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S-**Socratic Lesson

understanding of the information that a probability distribution provides and interpret probabilities from the probability distribution of a discrete random variable in context (**S-MD.A.1**). Students work with different representations of probability distributions, using both tables and graphs to represent the probability distribution of a discrete random variable. Lessons 7 and 8 introduce the concept of expected value, and students calculate and interpret the expected value of discrete random variables in context. For example, in Lesson 7, students explore the concept of expected value by playing a game called Six Up, where the first player to roll 15 sixes wins. Students play the game, create a probability distribution, and use the data to calculate the mean of the distribution or expected value. In Lesson 8, students are given a probability distribution for the results of a donation drive for a cancer charity and use the distribution to calculate the expected value for the amount of money donated and interpret the value in context (**S-MD.A.2**).

Once students have developed an understanding of what the probability distribution of a discrete random variable is and what information it provides, they see in Lessons 9 and 10 that probabilities associated with a discrete random variable can be calculated given a description of the random variable (**S-MD.A.3**).

In the final lessons of this topic, students also see how empirical data can be used to approximate the probability distribution of a discrete random variable (**S-MD.A.4**). For example, in Lesson 12 students play a game where they toss two dice, find the absolute difference of the numbers on the two faces, and move the same number of places on a number line. The first player to move past 20 wins the game. Students carry out simulations of the game and use the estimated probabilities to create a probability distribution which is then used to determine expected value.

Lesson 5: Discrete Random Variables

Student Outcomes

- Students distinguish between discrete random variables and continuous random variables.
- Given the probability distribution for a discrete random variable in table or graphical form, students identify possible values for the variable and associated probabilities.
- Students interpret probabilities in context.

Lesson Notes

This lesson introduces students to the definition of a discrete random variable as a function that assigns a numerical value to each outcome of a sample space. You do not need to use the function terminology, but if you do, discuss the domain of the function as representing the sample space and the range of the function as representing the set of possible values that the random variable might take on. In the opening example, students work with cards that describe features of ten different apartments. The domain or sample space is the ten apartments, and the range or random variable could be the number of bedrooms in the apartments. Students are then introduced to the probability distribution of a discrete random variable. They look at graphical and tabular representations of a probability distribution and recognize that the sum of the probabilities of individual outcomes is 1 and that the probability of any individual outcome is a number between and including 0 and 1.

Classwork

Example 1 (2 minutes): Types of Data

Distribute the set of ten apartment cards to each student. A template for these cards can be found at the end of this lesson. Copy this template and cut out the cards, or have students cut out the cards.

Example 1: Types of Data

Recall that the sample space of a chance experiment is the set of all possible outcomes for the experiment. For example, the sample space of the chance experiment that consists of randomly selecting one of ten apartments in a small building would be a set consisting of the ten different apartments that might have been selected. Suppose that the apartments are numbered from 1 to 10. The sample space for this experiment is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Cards with information about these ten apartments will be provided by your teacher. Mix the cards and then select one. Record the following information for the apartment you selected.

Number of bedrooms: | **Floor number:**

Size (sq. ft.): | **Distance to elevator:**

Color of walls: | **Floor type:**

Exercise 1 (3 minutes)

Let students work through the exercise individually and share their responses with a neighbor. Discuss the three categories as a class and confirm the features that are random variables. While all students may not sort the features the same way, direct attention and conversation toward the sample responses shown below. Then, ask students to share the information they recorded for the following:

- If we wanted to study the random variable *number of bedrooms*, what are the possible values that might be observed?
  - *Multiple responses should confirm the values* 1, 2, *and* 3.
- If we wanted to study the random variable *size*, what are the possible values that might be observed?
  - *Multiple responses should confirm the values from the apartment cards.*

Exercise 1

1. Sort the features of each apartment into three categories:

Number of bedrooms *Floor number*	*Size (sq. ft.)* *Distance to elevator*	*Color of walls* *Floor type*

a. Describe how the features listed in each category are similar.

The number of bedrooms and floor numbers are integers.

The size of the apartment and distance to the elevator are not all integers. and none of the values are the same.

The color of the walls and floor type are verbal descriptions, not numbers.

b. A random variable associates a number with each outcome of a chance experiment. Which of the features are random variables? Explain.

Example 2 (2 minutes): Random Variables

Introduce the definitions of discrete and continuous random variables. Ask students to label the random variables in the table from Example 1 as discrete or continuous.

- Which features are discrete random variables?
  - *Number of bedrooms and floor number*
- Which features are continuous random variables?
  - *Size and distance to elevator*
- Why are color and floor type not random variables?
  - *A number is not associated with the outcome.*

Scaffolding:

- For English language learners, use a Frayer model, visuals, or choral repetition to help with understanding the difference between discrete and continuous.
- For students who struggle with the concept, provide additional examples of random variables and have them classify as discrete or continuous:
  - Number of times it takes a teenager to pass a driver's test
  - Temperature
  - Height of person
  - Number of people living in a house
  - Speed of a car
  - Number of languages a person speaks

Example 2: Random Variables

One way you might have sorted these variables is whether they are based on counting (such as the number of languages spoken) or based on measuring (such as the length of a leaf). Random variables are classified into two main types: *discrete* and *continuous.*

- **A *discrete random variable* is one that has possible values that are isolated points along the number line. Often, discrete random variables involve counting.**
- **A *continuous random variable* is one that has possible values that form an entire interval along the number line. Often, continuous random variables involve measuring.**

Exercises 2–3 (5 minutes)

Students should work individually on Exercises 2 and 3. Then, discuss answers as a class. Teachers should use this as an opportunity to informally assess student understanding of discrete and continuous random variables.

Exercises 2–3

2. For each of the six variables listed in Exercise 1, give a specific example of a possible value the variable might have taken on, and identify the variable as discrete or continuous.

Responses will vary.

The number of bedrooms is a discrete random variable; a possible value of this variable is 3. The distance to the elevator could be 100 ft., and it is a continuous variable because it could be a little more or less than 100 ft. depending on where your starting point is within the apartment. The discrete random variables are number of bedrooms, floor number, color, and floor type.

3. Suppose you were collecting data about dogs. Give at least two examples of discrete and two examples of continuous data you might collect.

Responses will vary.

Continuous data: length of tail, length of ears, height, weight

Discrete data: number of legs, typical number of puppies in the litter; whether ears point up, down, or break in the middle and flop

Exercises 4–8: Music Genres (15 minutes)

Students should work individually on Exercises 4–8. Then, discuss answers as a class.

Exercises 4–8: Music Genres

People like different genres of music: country, rock, hip hop, jazz, etc. Suppose you were to give a survey to people asking them how many different music genres they like.

4. What do you think the possible responses might be?

Possible answer: $0, 1, 2$, etc.

5. **The table below shows $11,565$ responses to the survey question: How many music genres do you like listening to?**

Table 1: Number of music genres survey responders like listening to

Number of music genres	0	1	2	3	4	5	6	7	8
Number of responses	568	$2,012$	$1,483$	654	749	$1,321$	$1,233$	608	$2,937$

Find the relative frequency for each possible response (each possible value for number of music genres), rounded to the nearest hundredth. (The relative frequency is the proportion of the observations that take on a particular value. For example, the relative frequency for 0 is $\frac{568}{11565}$.)

Answer:

Number of music genres	0	1	2	3	4	5	6	7	8
Relative frequency	0.05	0.17	0.13	0.06	0.07	0.11	0.11	0.05	0.25

Note: Due to rounding, values may not always add up to exactly 1.

6. **Consider the chance experiment of selecting a person at random from the people who responded to this survey. The table you generated in Exercise 5 displays the *probability distribution* for the random variable number of music genres liked. Your table shows the different possible values of this variable and the probability of observing each value.**

   a. **Is the random variable discrete or continuous?**

   The random variable is discrete because the possible values are $0, 1, \ldots, 8$, and these are isolated points along the number line.

   b. **What is the probability that a randomly selected person who responded to the survey said that they like 3 different music genres?**

   0.06

   c. **Which of the possible values of this variable has the greatest probability of being observed?**

   The greatest probability 8 *genres, which has a probability of* 0.25

   d. **What is the probability that a randomly selected person who responded to the survey said that they liked 1 or fewer different genres?**

   0.22

   e. **What is the sum of the probabilities of all of the possible outcomes? Explain why your answer is reasonable for the situation.**

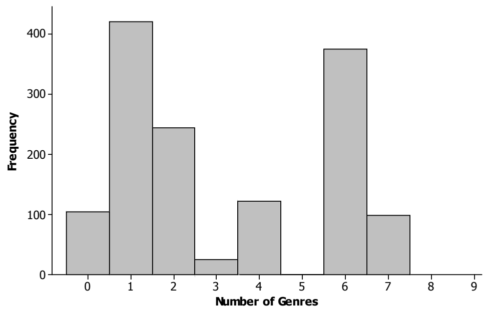
   1.00 *or close to* 1.00. *The probabilities of all the possible values should add to up to* 1 *because they represent everything that might possibly occur. However, due to rounding, values may not always add up to exacly 1.*

Scaffolding:

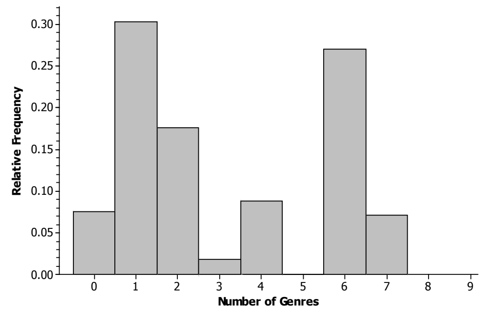
- The word *relative* may be familiar to English language learners. Point out that in this context, *relative* refers to something considered in relation to something else.
- Those new to the curriculum may not understand the term *relative frequency*. Explain that it can be found by dividing a certain number of observations by the total number of observations.
- For students above grade level, give an example of a continuous random variable (building height, for example). Ask students to answer the following:
  What is difficult about describing its probability distribution? How could we represent its probability distribution?

7. The survey data for people age 60 or older are displayed in the graphs below.

Probability Distribution of Frequency of Responses



Probability Distribution of Relative Frequency of Responses



a. What is the difference between the two graphs?

The graph on the left shows the total number of people (the frequency) for each possible value of the random variable number of music genres liked. The graph on the right shows the relative frequency for each possible value.

b. What is the probability that a randomly selected person from this group of people age 60 or older chose 4 music genres?

0.08

c. Which of the possible values of this variable has the greatest probability of occurring?

One genre with a probability of 0.30

d. What is the probability that a randomly selected person from this group of people age 60 or older chose 5 different genres?

0

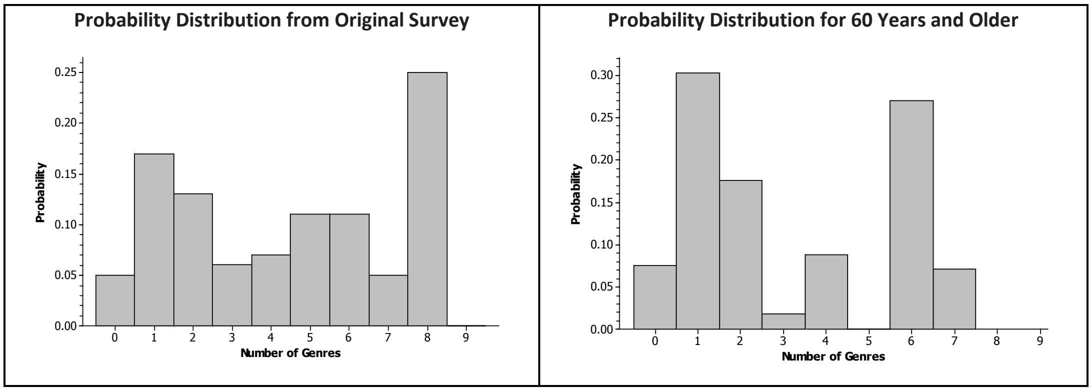
MP.3

e. Make a conjecture about the sum of the relative frequencies. Then check your conjecture using the values in the table.

Responses will vary.

The sum should be 1 *because that would be the total of the probabilities of all of the outcomes:* $0.07 + 0.30 + 0.17 + 0.02 + 0.09 + 0.27 + 0.07 = 0.99$*, which is not quite* 1*, but there is probably some rounding error. Note that students might not get the exact values when reading off the graph, but their answers should be close.*

8. **Below are graphs of the probability distribution based on responses to the original survey and based on responses from those age 60 and older.**



Identify which of the statements are true and which are false. Give a reason for each claim.

a. **The probability that a randomly selected person chooses 0 genres is greater for those age 60 and older than for the group that responded to the original survey.**

True: Overall, the probability is about 0.05, and for those 60 and older, it is about 0.07.

b. **The probability that a randomly selected person chooses fewer than 3 genres is smaller for those age 60 and older than for the group that responded to the original survey.**

False: Overall, the probability is 0.35, and for those 60 and older, it is 0.54.

c. **The sum of the probabilities for all of the possible outcomes is larger for those age 60 and older than for the group that responded to the original survey.**

False: In both cases, the sum of the probabilities is 1.

Exercises 9–11: Family Sizes (12 minutes)

In this set of exercises, students interpret the probabilities in terms of the number of people living in a household and compare probability distributions. Introduce the data in the example. Let students work individually on Exercises 9–11. Then, discuss answers as a class.

Exercises 9–11: Family Sizes

The table below displays the distribution of the number of people living in a household according to a recent U.S. Census. This table can be thought of as the probability distribution for the random variable that consists of recording the number of people living in a randomly selected U.S. household. Notice that the table specifies the possible values of the variable, and the relative frequencies can be interpreted as the probability of each of the possible values.

Table 2: Relative frequency of the number of people living in a household

Number of People	Relative Frequency
1	0.24
2	0.32
3	0.17
4	0.16
5	0.07
6	0.02
7 or more	0.02

9. **What is the random variable, and is it continuous or discrete? What values can it take on?**

   The random variable is the number of people in a household, and it is discrete. The possible values are 1, 2, 3, 4, 5, 6, 7, or more.

10. **Use the table to answer each of the following.**

    a. **What is the probability that a randomly selected household would have 5 or more people living there?**

    $$0.07 + 0.02 + 0.02 = 0.11$$

    b. **What is the probability that 1 or more people live in a household? How does the table support your answer?**

    Common sense says that 100% of the households should have 1 or more people living in them. If you add up the relative frequencies for the different numbers of people per household, you get 1.00.

MP.2

    c. **What is the probability that a randomly selected household would have fewer than 6 people living there? Find your answer in two different ways.**

    By adding the probabilities for 1, 2, 3, 4, and 5 people in a household, the answer would be 0.96.
    By adding the probabilities for 6 and 7 or more people living in a household, then subtracting the sum from 1, the answer would be $1 - 0.04 = 0.96$.

11. **The probability distributions for the number of people per household in 1790, 1890, and 1990 are below.**

Number of people per household	1	2	3	4	5	6	7 or more
1790: Probability	0.03	0.08	0.12	0.14	0.14	0.13	0.36
1890: Probability	0.04	0.13	0.17	0.17	0.15	0.12	0.23
1990: Probability	0.24	0.32	0.17	0.16	0.07	0.02	0.01

Source: U.S. Census Bureau (www.census.gov)

a. **Describe the change in the probability distribution of the number of people living in a randomly selected household over the years.**

Responses will vary.

In 1790 and 1890, the largest percentage of people were living in households of 7 or more people. In 1990, most people lived in houses with 1 or 2 people.

b. **What are some factors that might explain the shift?**

Responses will vary.

The shift might be because more people lived in urban areas instead of rural areas in the 1990s; more extended families with parents and grandparents lived in the same household in the 1790s and 1890s; more children lived in the same household per family in the earlier years.

Closing (3 minutes)

- What is the distinction between discrete and continuous random variables?
  - *The answer basically depends on the possible values that the random variable can take. Discrete random variables have possible outcomes that are isolated points along the number line. The possible values of a continuous random variables form an entire interval on the number line.*
- What information about a discrete random variable is included in the probability distribution?
  - *The possible values of the variable and the probability associated with each of the possible values*
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

Lesson Summary

- **Random variables can be classified into two types: discrete and continuous.**
- **A discrete random variable is one that has possible values that are isolated points along the number line. Often, discrete random variables involve counting.**
- **A continuous random variable is one that has possible values that form an entire interval along the number line. Often, continuous random variables involve measuring.**
- **Each of the possible values can be assigned a probability, and the sum of those probabilities is 1.**
- **Discrete probability distributions can be displayed graphically or in a table.**

Exit Ticket (3 minutes)

Name ______________________________ Date ________________

Lesson 5: Discrete Random Variables

Exit Ticket

1. Create a table that illustrates the probability distribution of a discrete random variable with four outcomes.

Random variable	1	2	3	4
Probability				

2. Which of the following variables are discrete and which are continuous? Explain your answer.

   Number of items purchased by a customer at a grocery store

   Time required to solve a puzzle

   Length of a piece of lumber

   Number out of 10 customers who pay with a credit card

Exit Ticket Sample Solutions

1. **Create a table that illustrates the probability distribution of a discrete random variable with four outcomes.**

   Answers will vary.

   One possible answer:

Random variable	1	2	3	4
Probability	0.25	0.15	0.5	0.1

   Check to make sure that all probabilities are between 0 and 1 and that the probabilities add to 1.

2. **Which of the following variables are discrete and which are continuous? Explain your answer.**

   Number of items purchased by a customer at a grocery store

   Time required to solve a puzzle

   Length of a piece of lumber

   Number out of 10 customers who pay with a credit card

   Discrete: Number of items purchased by a customer at a grocery store; Number out of 10 customers who pay with a credit card; they can each be described by an integer value

   Continuous: Time required to solve a puzzle; Length of a piece of lumber; the values need to be measured and involve intervals along a number line

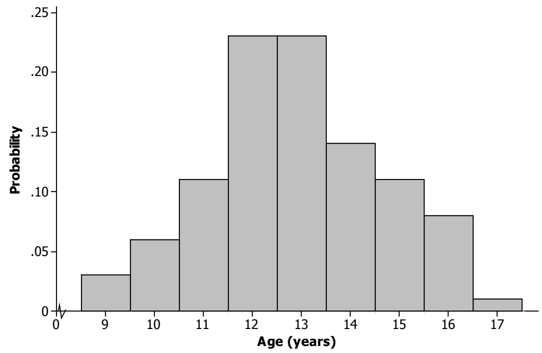
Problem Set Sample Solutions

1. **Each person in a large group of children with cell phones was asked, "How old were you when you first received a cell phone?"**

   The responses are summarized in the table below.

Age in years	Probability
9	0.03
10	0.06
11	0.11
12	0.23
13	0.23
14	0.14
15	0.11
16	0.08
17	0.01

a. **Make a graph of the probability distribution.**



b. **The bar centered at 12 in your graph represents the probability that a randomly selected person in this group first received a cell phone at age 12. What is the area of the bar representing age 12? How does this compare to the probability corresponding to 12 in the table?**

The base of the rectangle is 1 and the height is 0.23, so the area should be 0.23. This is the same as the probability for 12 in the table.

c. **What do you think the sum of the areas of all of the bars will be? Explain your reasoning.**

The sum of all the areas should be 1 because the sum of all probabilities in the probability distribution of a discrete random variable is always 1 or very close to 1 due to rounding.

d. **What is the probability that a randomly selected person from this group first received a cell phone at age 12 or 13?**

0.46

e. **Is the probability that a randomly selected person from this group first received a cell phone at an age older than 15 greater than or less than the probability that a randomly selected person from this group first received a cell phone at an age younger than 12?**

$P(older\ than\ 15) = 0.09$; $p(< 12) = 0.20$; *the probability for over 15 is less than the probability for under 12.*

2. **The following table represents a discrete probability distribution for a random variable. Fill in the missing values so that the results make sense; then, answer the questions.**

Possible value	4	5	10	12	15
Probability	0.08	$???$	0.32	0.27	$???$

Responses will vary.

The two missing values can be any two positive numbers whose sum adds to 0.33. For example, the probability for 5 could be 0.03, and the probability for 15 could be 0.3.

a. **What is the probability that this random variable takes on a value of 4 or 5?**

Responses will vary.

Possible answer: $0.08 + 0.03 = 0.11$

b. What is the probability that the value of the random variable is not 15?

Responses will vary.

Possible answer: $1 - 0.3 = 0.7$

c. Which possible value is least likely?

Responses will vary.

Possible answer: 5 *would be the least likely as it has the smallest probability.*

3. Identify the following as true or false. For those that are false, explain why they are false.

a. The probability of any possible value in a discrete random probability distribution is always greater than or equal to 0 and less than or equal to 1.

True

b. The sum of the probabilities in a discrete random probability distribution varies from distribution to distribution.

False; the sum of the probabilities is always equal to 1 *or very close to* 1 *due to rounding.*

c. The total number of times someone has moved is a discrete random variable.

True

4. Suppose you plan to collect data on your classmates. Identify three discrete random variables and three continuous random variables you might observe.

Responses will vary. Possible responses are shown below.

Discrete: how many siblings; how many courses they are taking; how many pets they have in their home; how many cars in their family; how many movies they saw last month

Continuous: height; hand-span; time it takes to get to school; time per week playing video games

5. Which of the following are not possible for the probability distribution of a discrete random variable? For each one you identify, explain why it is not a legitimate probability distribution.

Possible value	1	2	3	4	5
Probability	0.1	0.4	0.3	0.2	0.2

Possible value	1	2	3	4
Probability	0.8	0.2	0.3	-0.2

Possible value	1	2	3	4	5
Probability	0.2	0.2	0.2	0.2	0.2

The first distribution cannot be a probability distribution because the given probabilities add to more than 1. The second distribution cannot be a probability distribution because there is a negative probability given, and probabilities cannot be negative.

6. Suppose that a fair coin is tossed 2 times, and the result of each toss (H or T) is recorded.

   a. What is the sample space for this chance experiment?

   $\{HH, HT, TH, TT\}$

   b. For this chance experiment, give the probability distribution for the random variable of total number of heads observed.

Possible value	0	1	2
Probability	0.25	0.50	0.25

7. Suppose that a fair coin is tossed 3 times.

   a. How are the possible values of the random variable of total number of heads observed different from the possible values in the probability distribution of Problem 6(b)?

   Possible values are now 0, 1, 2, and 3.

   b. Is the probability of observing a total of 2 heads greater when the coin is tossed 2 times or when the coin is tossed 3 times? Justify your answer.

   The probability of 2 heads is greater when the coin is tossed 3 times. The probability distribution of the number of heads for 3 tosses is

Possible value	0	1	2	3
Probability	0.125	0.375	0.375	0.125

   The probability for the possible value of 2 is 0.375 for 3 tosses and only 0.25 for 2 tosses.

Template for apartment cards used in Example 1

Apartment number: 1 Number of bedrooms: 1 Size (sq. ft.): 1,102 Color of walls: white Floor number: 1 Distance to elevator: 5 ft. Floor type: carpet	Apartment number: 2 Number of bedrooms: 2 Size (sq. ft.): 975.5 Color of walls: white Floor number: 6 Distance to elevator: 30 ft. Floor type: carpet
Apartment number: 3 Number or bedrooms: 3 Size (sq. ft.): 892.25 Color of walls: green Floor number: 2 Distance to elevator: 20 ft. Floor type: tile	Apartment number: 4 Number or bedrooms: 3 Size (sq. ft.): 639 Color of walls: white Floor number: 2 Distance to elevator: 20.5 ft. Floor type: tile
Apartment number: 5 Number or bedrooms: 2 Size (sq. ft.): 2,015 Color of walls: white Floor number: 3 Distance to elevator: 45.25 ft. Floor type: carpet	Apartment number: 6 Number or bedrooms: 2 Size (sq. ft.): 415 Color of walls: white Floor number: 1 Distance to elevator: 40 ft. Floor type: carpet
Apartment number: 7 Number or bedrooms: 1 Size (sq. ft.): 1,304 Color of walls: white Floor number: 4 Distance to elevator: 15.75 ft. Floor type: carpet	Apartment number: 8 Number or bedrooms: 2 Size (sq. ft.): 1,500 Color of walls: green Floor number: 3 Distance to elevator: 60.75 ft. Floor type: carpet
Apartment number: 9 Number or bedrooms: 2 Size (sq. ft.): 2,349.75 Color of walls: white Floor number: 5 Distance to elevator: 100 ft. Floor type: carpet	Apartment number: 10 Number or bedrooms: 3 Size (sq. ft.): 750 Color of walls: green Floor number: 1 Distance to elevator: 10.5 ft. Floor type: tile

Lesson 6: Probability Distribution of a Discrete Random Variable

Student Outcomes

- Given the probability distribution of a discrete random variable in table or graphical form, students describe the long-run behavior of the random variable.
- Given a discrete probability distribution in table form, students construct a graph of the probability distribution.

Lesson Notes

This lesson builds on the prior lesson about discrete probability distributions by asking students to describe the long-run behavior of a random variable. Students should note that there is variability in the long-run behavior. For example, $P(\text{success}) = 0.5$ does not mean that in 100 trials, there will be exactly 50 successes. Understanding the behavior of a random variable provides a sense of what is likely or very unlikely in the long run. For example, if the probability of a seal pup being female is 60%, that does not mean that in every litter there will be 60% females; however, over a long period of time and many samples, the probability will approach 60%. The focus here is not on exactly what is likely to be observed, but on developing an understanding that some values describing long-run behavior of a random variable for a given probability distribution seem reasonable, while others do not.

You may want to discuss the context for the exercises to make sure students understand the situations.

Classwork

Exercises 1–3 (15 minutes): Credit Cards

Students can work through the first three exercises individually or in small groups. Consider providing 3 to 5 minutes for students to first work through the exercises individually. After this time, ask them to listen and share their answers with another student. Discuss the exercises as a class after each group has discussed the work.

MP.1 In the exercises, students create a histogram given a relative frequency table, and use the graph to make sense of given scenarios. Use the histogram to review several previous topics students studied. For example, the histogram indicates a skewed distribution of this data. As a result, the median number of credit cards is the most appropriate measure of center to describe this data. The measure of center for a distribution, often used to indicate a typical value of the data, was an important topic for students in Grades 6 and 9. The histogram students create is a relative frequency histogram that was also developed in Grades 6 and 9. Point out to students that the relative frequencies that summarize this sample have a sum of 1. This result indicates that all of the adults in this sample had between 0 and 10 credit cards. Relative frequencies are used to interpret probability distributions.

Exercises 1–3: Credit Cards

Credit bureau data from a random sample of adults indicating the number of credit cards is summarized in the table below.

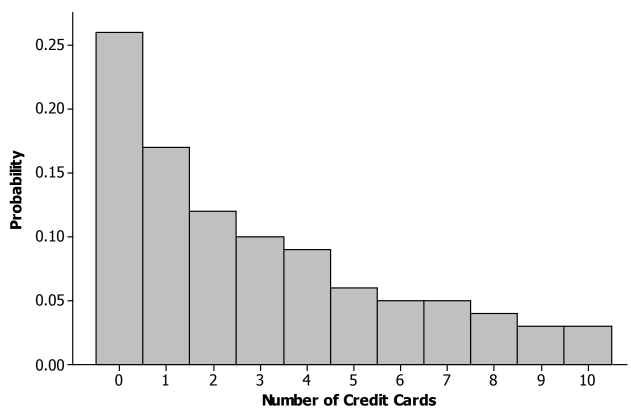
Scaffolding:
- Have students who may be below grade-level model the creation of the histogram or analyze a completed histogram that you provide.
- Have students who are above grade-level construct their own hypothetical histogram and explain the meaning of the probabilities in context.

Table 1: Number of credit cards carried by adults

Number of Credit Cards	Relative Frequency
0	0.26
1	0.17
2	0.12
3	0.10
4	0.09
5	0.06
6	0.05
7	0.05
8	0.04
9	0.03
10	0.03

1. **Consider the chance experiment of selecting an adult at random from the sample. The number of credit cards is a discrete random variable. The table above sets up the probability distribution of this variable as a relative frequency. Make a histogram of the probability distribution of the number of credit cards per person based on the relative frequencies.**

   Answer:



2. **Answer the following questions based on the probability distribution.**
   a. **Describe the distribution.**

      Responses will vary.

      The distribution is skewed right with a peak at 0. The mean number of cards (the balance point of the distribution) is about 3 or 4. The median number of cards is between 2 or 3 cards. Adults carry anywhere from 0 to 10 credit cards.

   b. **Is a randomly selected adult more likely to have 0 credit cards or 7 or more credit cards?**

      The probability that an adult has no credit cards is 0.26, while the probability of having 7 or more credit cards is about 0.15, so the probability of having no credit cards is larger.

c. **Find the area of the bar representing 0 credit cards.**

The area is $0.26 \cdot 1 = 0.26$.

d. **What is the area of all of the bars in the histogram? Explain your reasoning.**

The total area is 1 because the area of each bar represents the probability of one of the possible values of the random variable, and the sum of all of the possible values is 1.

3. **Suppose you asked each person in a random sample of 500 people how many credit cards he or she has. Would the following surprise you? Explain why or why not in each case.**

a. **Everyone in the sample owned at least one credit card.**

This would be surprising because 28% of adults do not own a credit card. It would be unlikely that in our sample of 500, no one had zero credit cards.

b. **65 people had 2 credit cards.**

This would not be surprising because 12% (0.12) of adults own two credit cards, so we would expect somewhere around 60 out of the 500 people to have two credit cards. 65 is close to 60.

c. **300 people had at least 3 credit cards.**

Based on the probability distribution, about 45% of adults have at least 3 credit cards, which would be about 225. 300 is greater than 225, but not enough to be surprising.

d. **150 people had more than 7 credit cards.**

This would be surprising because about 10% of adults own more than 7 credit cards. In this sample, $\frac{150}{500}$ or about 30% (three times as many as the proportion in the population) own more than 7 credit cards.

Before moving on to the next exercise set, check for understanding by asking students to answer the following:

- Explain to your neighbor how the histogram enabled us to answer questions about probabilities.

Exercises 4–7 (22 minutes): Male and Female Pups

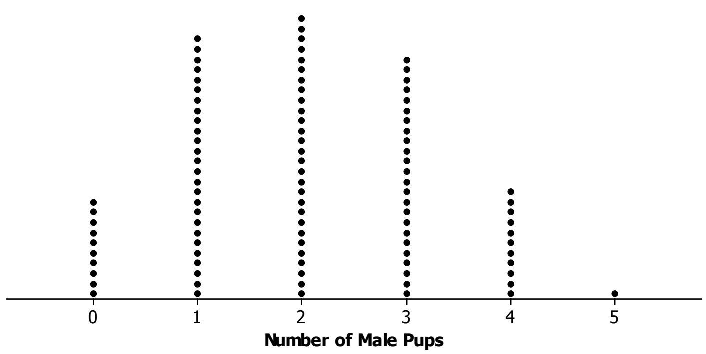
Depending on the class, these exercises might be done as a whole class discussion or with students working in small groups. Exercise 4 is intended to remind students of their earlier work with probability. This exercise describes a scenario involving animals, some species of seals, for example, that have biased sex ratios in their offspring. Biased sex ratios are ratios that are different from the expected probability due to conditions in the population studied. The reasons for this, and the reasons for studying the sex ratios of certain animals, are based on understanding the survival of the animal. Ask students to think of reasons why the probability of a female might be greater than the probability of a male for certain animals in order for the species to survive. Consider encouraging students to do independent research on the probability of a male or a female at birth for seals, elephants, or other endangered species of animals.

To help students think about some of the questions related to the scenario described in Exercise 4, you might have students look at data from a simulation. Conducting a simulation that records the number of males in a litter of the seals described in Exercise 4 will indicate variability, as well as how the long-run behavior follows the pattern of the probability distribution. For this lesson, generate 100 random samples of size 6 from the set $\{1, 0, 0\}$. Selecting a 1 from this sample set represents the birth of a male; selecting a 0 represents selecting a female. Have students work in groups to create random selections of 6 (with replacement) from this set. The 6 random selections represent a litter. Consider using the random features of graphing calculators or other probability software to simulate 100 litters of 6 pups. Organize students to record the number of males from the simulated litter. You might also consider having students record the simulated number of males in a litter on a whiteboard or class poster, or have students post their results to a class dot plot.

Simulated results of the number of male pups in a litter of 6 was conducted and summarized below. Use this data distribution if there is limited time to conduct a simulation.

1	2	2	4	1	2	2	2	2	1
2	1	1	1	2	3	1	3	3	3
4	0	3	0	1	1	4	2	2	1
3	2	4	1	2	3	3	0	3	4
2	3	1	2	3	2	0	2	1	3
4	1	2	3	2	1	3	4	3	4
2	4	5	2	3	3	2	0	1	3
3	1	3	4	1	2	0	3	2	4
1	3	2	1	0	0	1	2	1	0
0	1	3	2	1	1	3	1	2	2

A graph of this distribution is also provided below and will help students connect the results to the probability distribution. Provide this graph or as a class develop the dot plot for the simulated 100 litters of seals.



The graph will help students answer several of the questions in the exercises. For example, in thinking about Exercise 6, part (a), students could look at the first five litters in the simulated distribution, the second five, and so on to see about how many of the litters had fewer males than females. They could count the number of times one male showed up in every two litters for Exercise 6, part (b). See table below.

Exercises 4–7: Male and Female Pups

4. The probability that certain animals will give birth to a male or a female is generally estimated to be equal, or approximately 0.50. This estimate, however, is not always the case. Data are used to estimate the probability that the offspring of certain animals will be a male or a female. Scientists are particularly interested about the probability that an offspring will be a male or a female for animals that are at a high risk of survival. In a certain species of seals, two females are born for every male. The typical litter size for this species of seals is six pups.

   a. What are some statistical questions you might want to consider about these seals?

   Statistical questions to consider include the number of females and males in a typical litter, or the total number of males or females over time. (The question about the number of males in a typical litter will be explored in the exercises.)

   b. What is the probability that a pup will be a female? A male? Explain your answer.

   Out of every three animals that are born, one is male and two are female.

   $\frac{1}{3}$ *probability of a male,* $\frac{2}{3}$ *probability of a female*

   c. Assuming that births are independent, which of the following can be used to find the probability that the first two pups born in a litter will be male? Explain your reasoning.

   i. $\frac{1}{3}+\frac{1}{3}$

   ii. $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$

   iii. $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$

   iv. $2\left(\frac{1}{3}\right)$

   The two probabilities are multiplied if the events are independent; these are independent events, so the probability of the first two pups being male is $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)=\frac{1}{9}$.

5. The probability distribution for the number of males in a litter of six pups is given below.

Table 2: Probability distribution of number of male pups per litter*

Number of male pups	Probability
0	0.088
1	0.243
2	0.330
3	0.220
4	0.075
5	0.018
6	0.001

*The sum of the probabilities in the table is not equal to 1 due to rounding.

Use the probability distribution to answer the following questions.

   a. How many male pups will typically be in a litter?

   The most common will be two males and four females in a litter, but it would also be likely to have one to three male pups in a litter.

b. **Is a litter more likely to have six male pups or no male pups?**

It is more likely to have no male pups (probability of no males is 0.088) than the probability of all male pups (0.001).

6. **Based on the probability distribution of the number of male pups in a litter of six given above, indicate whether you would be surprised in each of the situations. Explain why or why not.**

a. **In every one of a female's five litters of pups, there were fewer males than females.**

Responses will vary.

This seems like it would not be surprising because in each litter, the chance of having more females than males is $0.33 + 0.243 + 0.088 = 0.621$. The probability that this would happen in five consecutive litters would be $(0.621)^5$, which is much smaller than 0.09, but which might be considered not too unlikely.

Scaffolding:

The word *litter* may be familiar to English language learners in terms of trash. Point out that in this context, *litter* refers to a group of young animals born at the same time to the same animal.

b. **A female had only one male in two litters of pups.**

Responses will vary.

The probability of no males in a litter is about 0.088 and of one male in a litter is 0.243, for a probability of 0.33 for either case (the two litters are independent of each other, so you can add the probabilities). While having only one male in two litters might be somewhat unusual, the probability of happening twice in a row would be 0.11, which is not too surprising if it happened twice in a row.

c. **A female had two litters of pups that were all males.**

Responses will vary.

This would be surprising because the chance of having all males is 0.001, and to have two litters with all males would be unusual (0.000001).

d. **In a certain region of the world, scientists found that in 100 litters born to different females, 25 of them had four male pups.**

Responses will vary.

This would be surprising because it shows a shift to about $\frac{1}{4}$ of the litters having four males, which is quite a bit larger than the probability of 0.075 given by the distribution for four males per litter where the probability of a male is $\frac{1}{3}$.

MP.2

7. **How would the probability distribution change if the focus was the number of females rather than the number of males?**

Responses will vary.

The probabilities would be in reverse order. The probability of 0 females (all males) would now be the probability of 6 females (no males), the probability of 1 female would be the same as the probability of 5 males, and so on.

Closing (3 minutes)

- Students should understand that probabilities are interpreted as long-run relative frequencies but that in a finite number of observations, the observed proportions will vary slightly from those predicted by a probability distribution.
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

> Lesson Summary
>
> **The probability distribution of a discrete random variable in table or graphical form describes the long-run behavior of a random variable.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 6: Probability Distribution of a Discrete Random Variable

Exit Ticket

The following statements refer to a discrete probability distribution for the number of songs a randomly selected high school student downloads in a week, according to an online music library.

Probability distribution of number of songs downloaded by high school students in a week:

Number of songs	0	1	2	3	4	5	6	7	8	9
Probability	0.06	0.14	0.22	0.25	0.15	0.09	0.05	0.024	0.011	0.005

Which of the following statements seem reasonable to you based on a random sample of 200 students? Explain your reasoning, particularly for those that are unreasonable.

a. 25 students downloaded 3 songs a week.

b. More students downloaded 4 or more songs than downloaded 3 songs.

c. 30 students in the sample downloaded 9 or more songs per week.

Exit Ticket Sample Solutions

The following statements refer to a discrete probability distribution for the number of songs a randomly selected high school student downloads in a week, according to an online music library.

Probability distribution of number of songs downloaded by high school students in a week:

Number of songs	0	1	2	3	4	5	6	7	8	9
Probability	0.06	0.14	0.22	0.25	0.15	0.09	0.05	0.024	0.011	0.005

Which of the following statements seem reasonable to you based on a random sample of 200 students? Explain your reasoning, particularly for those that are unreasonable.

a. 25 students downloaded 3 songs a week.

Responses will vary.

If the probability is 0.25 that a student will download 3 songs, then in 200 students, it would seem reasonable to have around 50 students downloading 3 songs. To have only 25 does not seem reasonable.

b. More students downloaded 4 or more songs than downloaded 3 songs.

Responses will vary.

The probability of 4 or more songs is $0.15 + 0.09 + 0.05 + 0.024 + 0.011 + 0.005 = 0.33$, which is larger than 0.25, so it would seem reasonable to have more students downloading 4 or more songs.

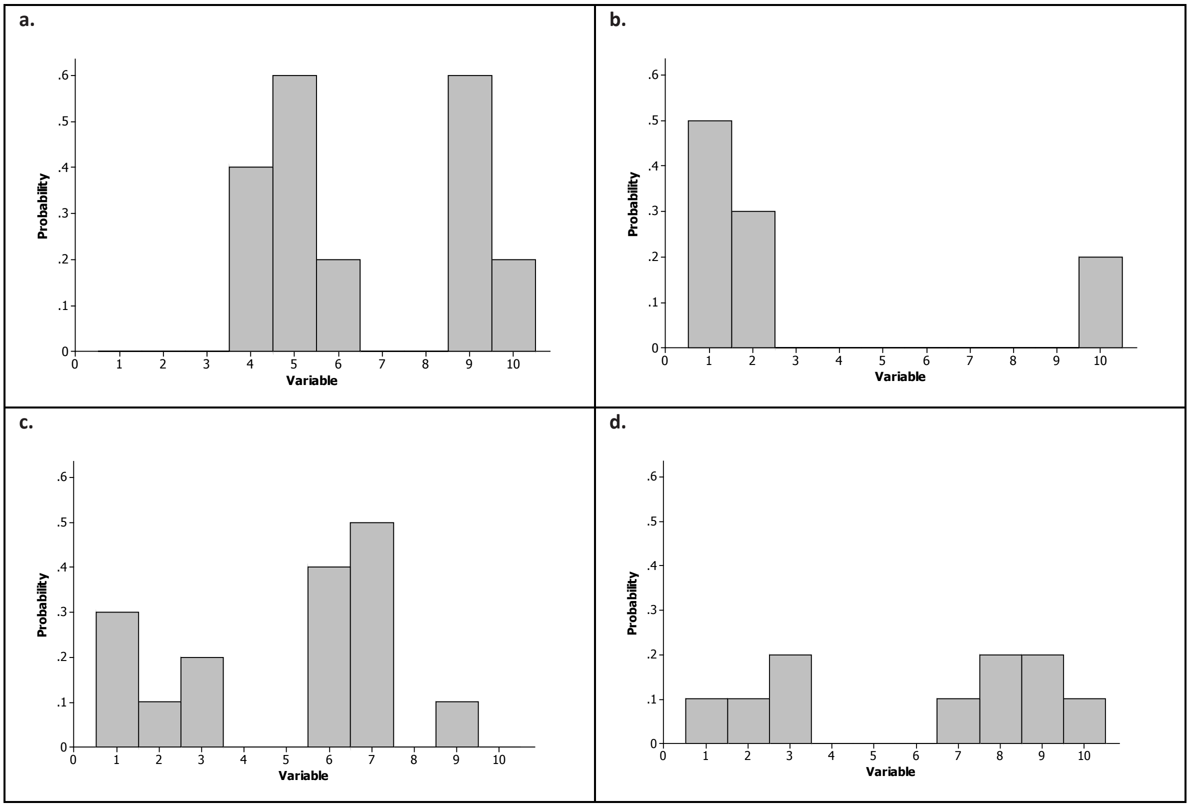
c. 30 students in the sample downloaded 9 or more songs per week.

Responses will vary.

The probability that a student would download 9 or more songs per week is 0.005 or 0.5%; 30 out of 200 is about 15%. This does not seem reasonable.

Problem Set Sample Solutions

1. **Which of the following could be graphs of a probability distribution? Explain your reasoning in each case.**



Graphs (b) and (d) are probability distributions of discrete random variables because the outcomes are discrete numbers from 1 to 10, the probability of every outcome is less than 1, and the sum of the probabilities of all the outcomes is 1. The data distribution for graphs (a) and (c) are not probability distributions of discrete random variables because the sum of the probabilities is greater than 1.

2. **Consider randomly selecting a student from New York City schools and recording the value of the random variable number of languages in which the student can carry on a conversation. A random sample of $1,000$ students produced the following data.**

Table 3: Number of languages spoken by random sample of students in New York City

Number of languages	1	2	3	4	5	6	7
Number of students	542	280	71	40	34	28	5

a. **Create a probability distribution of the relative frequencies of the number of languages students can use to carry on a conversation.**

Answer:

Table 4: Number of languages spoken by random sample of students in New York City

Number of languages	1	2	3	4	5	6	7
Probability	0.542	0.280	0.071	0.040	0.034	0.028	0.005

b. **If you took a random sample of 650 students, would it be likely that 350 of them only spoke one language? Why or why not?**

About 54% of all the students speak only one language; $\frac{350}{650}$ is about 54%, so it seems likely to have 350 students in the sample who could carry on a conversation in only one language.

c. **If you took a random sample of 650 students, would you be surprised if 100 of them spoke exactly 3 languages? Why or why not?**

$\frac{100}{650}$ is about 15%. The data suggest that only about 7% of all students speak exactly 3 languages. This could happen but does not seem too likely.

d. **Would you be surprised if 448 students spoke more than two languages? Why or why not?**

$\frac{448}{650}$ is about 69%, which seems to be a lot more than the 18% of all students who speak more than two languages, so I would be surprised.

3. **Suppose someone created a special six-sided die. The probability distribution for the number of spots on the top face when the die is rolled is given in the table.**

Table 5: Probability distribution of the top face when rolling a die

Face	1	2	3	4	5	6
Probability	$\frac{1-x}{6}$	$\frac{1-x}{6}$	$\frac{1-x}{6}$	$\frac{1+x}{6}$	$\frac{1+x}{6}$	$\frac{1+x}{6}$

a. **If x is an integer, what does x have to be in order for this to be a valid probability distribution?**

The probabilities have to add to 1, so we need $6-3x+3x=6$. However, this is true for any value of x. Since the probabilities also have to be greater than or equal to 0, x can only be 1 or 0.

b. **Find the probability of getting a 4.**

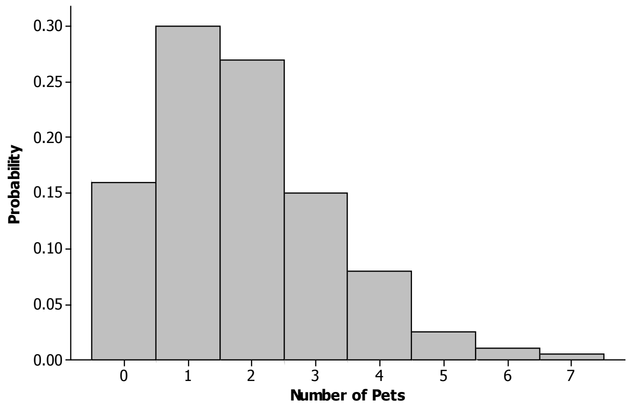
If $x = 0$, $P(4) = \frac{1}{6}$; if $x = 1$, then $P(4) = \frac{1}{3}$.

c. **What is the probability of rolling an even number?**

If $x = 0$, then $P(\text{even}) = \frac{3}{6} = \frac{1}{2}$.

If $x = 1$, then $P(\text{even}) = \frac{4}{6} = \frac{2}{3}$.

4. **The graph shows the relative frequencies of the number of pets for households in a particular community.**



a. **If a household in the community is selected at random, what is the probability that a household would have at least 1 pet?**

0.84

b. **Do you think it would be likely to have 25 households with 4 pets in a random sample of 225 households? Why or why not?**

$\frac{25}{225}$ is about 11%. The probability distribution indicates about 8% of the households would have 4 pets. These are fairly close, so it would seem reasonable to have 25 households with 4 pets.

c. **Suppose the results of a survey of 350 households in a section of a city found 175 of them did not have any pets. What comments might you make?**

Responses will vary.

Based on this probability distribution, there should be about 16% or 56 of the 350 households without pets. To have 175 houses without pets suggests that there is something different; perhaps the survey was not random, and it included areas where pets were not allowed.

Lesson 7: Expected Value of a Discrete Random Variable

Student Outcomes

- Students calculate the expected value of a discrete random variable.

Lesson Notes

This lesson, which is divided into two parts, develops the concept of the expected value of a discrete random variable as the mean of the distribution of the discrete random variable. In Exploratory Challenge 1, the method for computing the expected value of a discrete random variable is developed. In Exploratory Challenge 2, this lesson relates the method for computing the expected value of a discrete random variable to previous work with vectors, i.e., the computation of the dot product of two vectors. This lesson is designed for students to work in pairs. Each student will need a die. Either the teacher or each pair of students will need a timer.

Classwork

Exploratory Challenge 1/Exercises 1–5 (5 minutes)

Read the instructions for the new game, Six Up. Be sure that students understand the rules for the game. Allow them to discuss and answer Exercises 1–5. When students are ready, discuss the answers.

> *Scaffolding:*
>
> Teachers may choose to use a simpler version of this question. For example:
>
> If you flip a coin 10 times, how many times would you expect the coin to land on heads? What about if you flipped it 20 times?

Exploratory Challenge1/Exercises 1–5

A new game, Six Up, involves two players. Each player rolls his or her die, counting the number of times a "six" is rolled. The players roll their die for up to one minute. The first person to roll 15 sixes wins. If no player rolls 15 sixes in the one-minute time limit, then the player who rolls the greatest number of sixes wins that round. The player who wins the most rounds wins the game.

Suppose that your class will play this game. Your teacher poses the following question:

How many sixes would you expect to roll in one round?

MP.3

1. **How would you answer this question?**

   Answers will vary. Anticipate a wide range of answers to this question.

2. **What discrete random variable should you investigate to answer this question?**

   The discrete random variable is the number of sixes rolled in a round.

3. **What are the possible values for this discrete random variable?**

   The values are $0, 1, 2, 3, 4, \ldots, 15$.

4. **Do you think these possible values are all equally likely to be observed?**

   No, some outcomes are more likely than others. For example, it would be unlikely to see only 0 or 1 six rolled in a round.

5. **What might you do to estimate the probability of observing each of the different possible values?**

   We could play many rounds of the game to see how often each number occurs. This would allow us to estimate the probabilities.

Exploratory Challenge 1/Exercise 6–8 (12 minutes)

Be sure that students understand the rules of the game. Students should roll the die as quickly as possible to try to be the first to roll 15 sixes (or to roll the greatest number of sixes). You may time each minute, giving signals to begin and end the round. (The pairs of students may be allowed to time their own minutes, if preferred.)

Exploratory Challenge 1/Exercises 6–8

You and your partner will play the Six Up game. Roll the die until the end of one minute or until you or your partner rolls 15 sixes. Count the number of sixes you rolled. Remember to stop rolling if either you or your partner rolls 15 sixes before the end of one minute.

6. **Play five rounds. After each round, record the number of sixes that you rolled.**

   Student answers will vary. Values range from 0 to 15. One possible answer is shown.

   The number of sixes rolled in each round is

Round 1	Round 2	Round 3	Round 4	Round 5
5	9	14	11	15

7. **On the board, put a tally mark for the number of sixes rolled in each round.**

   Student answers will vary. Values range from 0 to 15. One possible answer is shown.

   The frequencies for the number of sixes rolled by the class are shown below:

Number of sixes rolled	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Frequency	0	0	0	0	3	3	3	3	6	3	5	10	9	4	2	9

8. **Using the data summarized in the frequency chart on the board, find the mean number of sixes rolled in a round.**

Answers will vary. For the example given in Exercise 6:

The total number of observations is 60.

The mean is

$$\frac{0(0)+1(0)+2(0)+3(0)+4(3)+5(3)+6(3)+7(3)+8(6)+9(3)+10(5)+11(10)+12(9)+13(4)+14(2)+15(9)}{60}$$

$= 10.4$ *sixes.*

Exploratory Challenge 1/Exercises 9–13 (8 minutes)

Students should answer Exercises 9–13. They may work with their partner. Be sure that students understand that the relative frequency is a proportion.

Exploratory Challenge 1/Exercises 9–13

9. **Calculate the relative frequency (*proportion*) for each value of the discrete random variable (i.e., the number of sixes rolled) by dividing the frequency for each possible value of the number of sixes rolled by the total number of rounds (the total number of tally marks). (The relative frequencies can be interpreted as estimates of the probabilities of observing the different possible values of the discrete random variable.)**

Student answers will vary. For the above example

The proportions are displayed in the table below. The observed number for each x*-value is divided by* 60*, the total number of observations of the variable.*

Number of sixes rolled	0	1	2	3	4	5	6	7
Relative frequency	$\frac{0}{60}=0$	$\frac{0}{60}=0$	$\frac{0}{60}=0$	$\frac{0}{60}=0$	$\frac{3}{60}=0.05$	$\frac{3}{60}=0.05$	$\frac{3}{60}=0.05$	$\frac{3}{60}=0.05$

Number of sixes rolled	8	9	10	11	12	13	14	15
Relative frequency	$\frac{6}{60}=0.1$	$\frac{3}{60}=0.05$	$\frac{5}{60}=0.083$	$\frac{10}{60}=0.167$	$\frac{9}{60}=0.15$	$\frac{4}{60}=0.067$	$\frac{2}{60}=0.033$	$\frac{9}{60}=0.15$

10. **Multiply each possible value for the number of sixes rolled by the corresponding probability (relative frequency).**

    Answers will vary. The products for the example given above are the following:

    $0(0) = 0$ $1(0) = 0$ $2(0) = 0$ $3(0) = 0$ $4(0.05) = 0.2$ $5(0.05) = 0.25$

    $6(0.05) = 0.3$ $7(0.05) = 0.35$ $8(0.1) = 0.8$ $9(0.05) = 0.45$ $10(0.083) = 0.83$

    $11(0.167) = 1.837$ $12(0.15) = 1.8$ $13(0.067) = 0.871$ $14(0.033) = 0.462$ $15(0.15) = 2.25$

11. **Find the sum of the calculated values in Exercise 10. This number is called the *expected value* of the discrete random variable.**

    Student answers will vary. For the above example:

    The expected value is

    $0 + 0 + 0 + 0 + 0.2 + 0.25 + 0.3 + 0.35 + 0.8 + 0.45 + 0.83 + 1.837 + 1.8 + 0.871 + 0.462 + 2.25$

    $= 10.4$ *sixes.*

12. **What do you notice about the sum in Exercise 11 and the mean that you calculated in Exercise 8?**

    The two values are the same. The procedures in Exercises 9, 10, and, 11 combine to create a procedure for calculating the expected value of a discrete random variable.

13. **The *expected value* of a random variable, x, is also called the *mean* of the distribution of that random variable. Why do you think it is called the mean?**

    Because it is like calculating a long-run average value for the random variable.

Before moving on to the next exercise, check for understanding by asking students to answer the following:

- What is expected value and what did it represent in this example?
  - *It is the mean of the distribution of the random variable "the number of sixes rolled." In this case, the expected value was* 10.4 *sixes.*

Exploratory Challenge 1/Exercise 14 (3 minutes)

Explain the formula presented in the text.

Be sure to discuss symbols in the formula:

$$\text{expected value} = \sum xp$$

- The mathematical symbol, Σ, denotes the word *summation*.
- This equation indicates to find the sum of the products of each individual x-value times its corresponding probability.

Read the question and have students answer it.

Scaffolding:

- The word *sigma* may need rehearsal.
- The symbol denotes the word *summation*.
- Students may need to be shown how to use the formula with the probability distribution. Consider using the following as an example:

X	0	1	2
Probability	0.3	0.3	0.4

Exploratory Challenge 1/Exercise 14

The expected value for a discrete random variable is computed using the following equation:

$$\text{expected value} = \sum(\text{each value of the random variable } (x)) \times (\text{the corresponding probability } (p))$$

or

$$\text{expected value} = \sum xp$$

where x is a possible value of the random variable, and p is the corresponding probability.

The following table provides the probability distribution for the number of heads occurring when two coins are flipped.

Number of heads	0	1	2
Probability	0.25	0.5	0.25

14. If two coins are flipped many times, how many heads would you expect to occur, on average?

$0(0.25) + 1(0.5) + 2(0.25) = 1$ *head*

You would expect 1 head to occur when two coins are flipped.

Scaffolding

- Students might need to think of two different coins, for example a penny and a nickel.
- This will help students understand that there are two ways to get one head. The penny can land on heads and the nickel lands tails, *or* the penny lands on tails and the nickel lands heads.
- Have students who are above grade level calculate the expected value of a more complicated random variable:

Number of Cars	0	1	2	3	4
Probability	0.07	0.44	0.24	0.13	0.12

Exploratory Challenge 1/Exercises 15–16 (5 minutes)

These exercises focus on the interpretation of expected value. Make sure that students understand that expected value is interpreted as a long-run average. Then have students work in pairs to complete Exercises 15 and 16. You may want to have some students share their answers to Exercise 16 with the class.

Exploratory Challenge 1/Exercises 15–16

15. The estimated expected value for the number of sixes rolled in one round of the Six Up game was 10.4. Write a sentence interpreting this value.

If a player were to play many rounds of the Six Up game, the average number of sixes rolled by that player would be about 10.4 sixes per round.

16. Suppose that you plan to change the rules of the Six Up game by increasing the one-minute time limit for a round. You would like to set the time so that most rounds will end by a player reaching 15 sixes. Considering the estimated expected number of sixes rolled in a one minute round, what would you recommend for the new time limit. Explain your choice.

Answers will vary. Look for a justification based on the expected value. For example, a student might say 1.5 minutes, reasoning that if the average number of sixes rolled in one minute is about 10, increasing the time by 10% to 1.5 minutes should be enough time because the average would then be around 15. Or a student might say two minutes should be enough time because if the average number of sixes in one minute is around 19, in two minutes, it would be very likely that one of the players would get to 15 sixes.

Exploratory Challenge 2/Exercises 17–19 (5 minutes)

Read the description of the two vectors. You might ask students to recall how to find a dot product. Have students answer the questions and then discuss answers as a class.

Exploratory Challenge 2/Exercises 17–19

Suppose that we convert to two vectors the above table displaying the discrete distribution for the number of heads occurring when two coins are flipped.

Let vector A be the number of heads occurring.

Let vector B be the corresponding probabilities.

$$A = \langle 0, 1, 2 \rangle \qquad B = \langle 0.25, 0.5, 0.25 \rangle$$

17. Find the dot product of these two vectors.

$0(0.25) + 1(0.5) + 2(0.25) = 1$

18. Explain how the dot product computed in Exercise 17 compares to the expected value computed in Exercise 14.

The dot product and the expected value are equal.

MP.2

19. How do these two processes, finding the expected value of a discrete random variable and finding the dot product of two vectors, compare?

The two processes are the same.

Closing (2 minutes)

Students should understand that:

- The long-run relative frequency or proportion of each possible value of a discrete random variable can be interpreted as an estimate of the probability of the values of observing that value.
- The expected value of a random variable is the mean of the distribution of that random variable.
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

Lesson Summary

The *expected value* of a random variable is the *mean* of the distribution of that random variable.

The expected value of a discrete random variable is the *sum* of the *products* of each possible value (x) and the corresponding probability.

The process of computing the expected value of a discrete random variable is similar to the process of computing the dot product of two vectors.

Exit Ticket (5 minutes)

Name ________________________________ Date ____________________

Lesson 7: Expected Value of a Discrete Random Variable

Exit Ticket

At a carnival, one game costs $1 to play. The contestant gets one shot in an attempt to bust a balloon. Each balloon contains a slip of paper with one of the following messages.

- Sorry, you do not win, but you get your dollar back. (The contestant has not lost the $1 cost.)
- Congratulations, you win $2. (The contestant has won $1.)
- Congratulations, you win $5. (The contestant has won $4.)
- Congratulations, you win $10. (The contestant has won $9.)

If the contestant does not bust a balloon, then the $1 cost is forfeited. The table below displays the probability distribution of the discrete random variable, or ***net*** winnings for this game.

Net winnings	-1	0	1	4	9
Probability	0.25	?	0.3	0.08	0.02

1. What is the sum of the probabilities in a discrete probability distribution? Why?

2. What is the probability that a contestant will bust a balloon and receive the message, "Sorry, you do not win, but you get your dollar back"?

3. What is the net amount that a contestant should expect to win per game if the game were to be played many times?

Exit Ticket Sample Solutions

At a carnival, one game costs $\$1$ to play. The contestant gets one shot in an attempt to bust a balloon. Each balloon contains a slip of paper with one of the following messages.

- **Sorry, you do not win, but you get your dollar back. (The contestant has not lost the $\$1$ cost.)**
- **Congratulations, you win $\$2$. (The contestant has won $\$1$.)**
- **Congratulations, you win $\$5$. (The contestant has won $\$4$.)**
- **Congratulations, you win $\$10$. (The contestant has won $\$9$.)**

If the contestant does not bust a balloon, then the $\$1$ cost is forfeited. The table below displays the probability distribution of the discrete random variable, or *net* winnings for this game.

Net winnings	-1	0	1	4	9
Probability	0.25	$?$	0.3	0.08	0.02

1. **What is the sum of the probabilities in a discrete probability distribution? Why?**

   The sum of the probabilities in a discrete probability distribution is one. In a discrete distribution, every possible x-value of the random variable is listed. Thus, the sum of the corresponding probabilities must equal one.

2. **What is the probability that a contestant will bust a balloon and receive the message, "Sorry, you do not win, but you get your dollar back"?**

   If you receive the message, "Sorry, you do not win, but you get your dollar back," then your net winnings is $\$0$. The probability of winning $\$0$ is

   $$1 - (0.25 + 0.3 + 0.08 + 0.02) = 0.35$$

3. **What is the net amount that a contestant should expect to win per game if the game were to be played many times?**

   The net amount that a contestant should expect to win is the expected value of the probability distribution.

   $$-1(0.25) + 0(0.35) + 1(0.3) + 4(0.08) + 9(0.02) = \$0.55$$

Problem Set Sample Solutions

1. **The number of defects observed in the paint of a newly manufactured car is a discrete random variable. The probability distribution of this random variable is shown in the table below.**

Number of defects	0	1	2	3	4	5
Probability	0.02	0.15	0.40	0.35	0.05	0.03

If large numbers of cars were inspected, what would you expect to see for the average number of defects per car?

The expected number is

$$0(0.02) + 1(0.15) + 2(0.40) + 3(0.35) + 4(0.05) + 5(0.03) = 2.35 \textit{ defects.}$$

Note: Since the expected number of defects is the mean of the probability distribution, we do not round to a whole number.

2.
   a. **Interpret the expected value calculated in Problem 1. Be sure to give your interpretation in context.**

   If many cars were inspected and the number of defects was observed for each car, the average number of defects per car would be about 2.35.

   b. **Explain why it is not reasonable to say that every car will have the expected number of defects.**

   The expected value is an average. It represents a typical value for a random variable, but individual values may be greater than or less than the expected value.

3. **Students at a large high school were asked how many books they read over the summer. The number of books read is a discrete random variable. The probability distribution of this random variable is shown in the table below.**

Number of books read	0	1	2	3
Probability	0.12	0.33	0.48	0.07

If a large number of students were asked how many books they read over the summer, what would you expect to see for the average number of books read?

The expected number is

$$0(0.12) + 1(0.33) + 2(0.48) + 3(0.07) = 1.5 \textit{ books.}$$

4. **Suppose two dice are rolled. The sum of the two numbers showing is a discrete random variable. The following table displays the probability distribution of this random variable.**

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

If you rolled two dice a large number of times, what would you expect the average of the sum of the two numbers showing to be?

The expected sum of the two rolled dice is

$$2\left(\frac{1}{36}\right)+3\left(\frac{1}{18}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{1}{9}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{1}{6}\right)+8\left(\frac{5}{36}\right)+9\left(\frac{1}{9}\right)+10\left(\frac{1}{12}\right)+11\left(\frac{1}{18}\right)+12\left(\frac{1}{36}\right)=7.$$

5. **Explain why it is not possible for a random variable whose only possible values are 3, 4, and 5 to have an expected value greater than 6.**

Expected value is an average. The average of a large number of observations of this variable cannot be greater than 6 if the variable can only have values that are 6 or less.

6. **Consider a discrete random variable with possible values 1, 2, 3, and 4. Create a probability distribution for this variable so that its expected value would be greater than 3 by entering probabilities into the table below. Then calculate the expected value to verify that it is greater than 3.**

Value of variable	1	2	3	4
Probability				

Answers will vary. One possible answer is as follows:

Value of variable	1	2	3	4
Probability	0.1	0.1	0.1	0.7

with expected value $= 1(0.1) + 2(0.1) + 3(0.1) + 4(0.7) = 3.4$

Lesson 8: Interpreting Expected Value

Student Outcomes

- Students interpret expected value in context.

Lesson Notes

This lesson develops the interpretation of the expected value as a long-run average of the value of a discrete random variable. As previously observed, the more times an event occurs, the closer the distribution of outcomes gets to the probability distribution and the closer the average value of all the outcomes will get to the expected value. In this lesson, students calculate the expected value of the sum of two dice, roll the dice 10 times to calculate the average value, and then observe that rolling 40 times produces an average value closer to the expected value. Students interpret the expected value of a discrete distribution in the context of the problem. Each student will need two dice.

Classwork

Exploratory Challenge 1/Exercise 1 (2 minutes)

Allow students to read and answer Exercise 1.

Scaffolding:

- For students who struggle, consider using a modified version of this exercise that might include pulling different colored chips or marbles out of a bag or using four-sided dice.
- An extension for advanced students may be given as follows: Construct your own hypothetical discrete random variable with a probability distribution that also has an expected value of 7.

Exploratory Challenge 1/Exercises 1-8

Recall the following problem from the Problem Set in Lesson 7.

Suppose two dice are rolled. The sum of the two numbers showing is a discrete random variable. The following table displays the probability distribution of this random variable.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

1. **If you rolled two dice and added the numbers showing a large number of times, what would you expect the average sum to be? Explain why.**

   The expected sum of the two rolled dice is as follows

$$2\left(\frac{1}{36}\right)+3\left(\frac{1}{18}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{1}{9}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{1}{6}\right)+8\left(\frac{5}{36}\right)+9\left(\frac{1}{9}\right)+10\left(\frac{1}{12}\right)+11\left(\frac{1}{18}\right)+12\left(\frac{1}{36}\right)=7$$

   The expected average sum would be 7 because, after a large number of rolls, the distribution of sums would resemble the probability distribution above.

Exploratory Challenge 1/Exercises 2–4 (5 minutes)

Allow students to roll two dice, recording the sum for each of 10 rolls. Students should then use their results to answer Exercises 3 and 4.

2. Roll two dice. Record the sum of the numbers on the two dice in the table below. Repeat this nine more times for a total of 10 rolls.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks											
Relative frequency											

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks		I	I	II	II	I		II	I		
Relative frequency	0	0.1	0.1	0.2	0.2	0.1	0	0.2	0.1	0	0

3. What is the average sum of these 10 rolls?

Answers will vary. For the example above, the average of the sum for these 10 rolls is as follows

$$3(0.1) + 4(0.1) + 5(0.2) + 6(0.2) + 7(0.1) + 9(0.2) + 10(0.1) = 6.4$$

4. How does this average compare to the expected value in Exercise 1? Are you surprised? Why or why not?

Answers will vary, but for most students, the average will not be the same as the expected sum of 7 in Exercise 1. Students may say they are not surprised as there were only 10 rolls of the dice.

Exploratory Challenge 1/Exercises 5–7 (5 minutes)

Allow students to roll two dice, recording the sum of each of the 10 rolls. Combine the results of these 10 rolls to the previous 10 rolls in Exercise 2. Students should then use their combined results of the 20 rolls to answer Exercises 6 and 7.

5. Roll the two dice 10 more times, recording the sums. Combine the sums of these 10 rolls with the sums of the previous 10 rolls for a total of 20 sums.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks											
Relative frequency											

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks		III	I	III	~~IIII~~	IIII		II	II		
Relative frequency	0	0.15	0.05	0.15	0.25	0.20	0	0.10	0.10	0	0

6. **What is the average sum for these 20 rolls?**

Answers will vary. For the example above, the average sum for these 20 rolls is as follows:

$3(0.15) + 4(0.05) + 5(0.15) + 6(0.25) + 7(0.20) + 9(0.10) + 10(0.10) = 6.2$

7. **How does the average sum for these 20 rolls compare to the expected value in Exercise 1?**

Answers will vary, but for most students, the average will still not be the same as the expected sum of 7 in Exercise 1.

Exploratory Challenge 1/Exercise 8 (5 minutes)

Divide the class into pairs. The partners should combine their rolls for a total of 40 rolls. If there are an odd number of students, create one group of three students. The group of three students will combine their rolls for a total of 60 rolls. Students should then find the average.

8. **Combine the sums of your 20 rolls with those of your partner. Find the average of the sum for these 40 rolls.**

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency											
Relative frequency											

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency	0	3	3	6	8	5	5	4	4	1	1
Relative frequency	0	0.075	0.075	0.15	0.20	0.125	0.125	0.10	0.10	0.025	0.025

The average sum for these 40 rolls is as follows

$$3(0.075) + 4(0.075) + 5(0.15) + 6(0.2) + 7(0.125)$$
$$+ 8(0.125) + 9(0.10) + 10(0.10) + 11(0.025) + 12(0.025)$$
$$= 6.825$$

Exploratory Challenge 2/Exercise 9 (5 minutes)

Now assign two pairs to work together (four students). The pairs should combine their rolls for a total of 80 rolls. Students should then find the average sum.

Exploratory Challenge 2/Exercises 9–12

9. **Combine the sums of your 40 rolls above with those of another pair for a total of 80 rolls. Find the average value of the sum for these 80 rolls.**

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency	0	6	4	13	14	11	9	6	12	3	2
Relative frequency	0	0.075	0.05	0.1625	0.175	0.1375	0.1125	0.075	0.15	0.0375	0.025

The average sum for these 80 rolls is as follow

$$3(0.075) + 4(0.05) + 5(0.1625) + 6(0.175) + 7(0.1375)$$
$$+ 8(0.1125) + 9(0.075) + 10(0.15) + 11(0.0375) + 12(0.025)$$
$$= 7.0375$$

Exploratory Challenge 2/Exercises 10–11 (10 minutes)

Allow time for one person from each group of four students to put the results in a class chart on the board. The use of tally marks may aid students in combining their results. After the class chart is complete, allow students time to calculate the relative frequency for each sum rolled and to calculate the expected sum of the class rolls.

10. **Combine the sums of your 80 rolls with those of the rest of the class. Find the average sum for all the rolls.**

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency											
Probability											

Student responses will vary. Here is one example for a class of 20 students.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency	7	27	33	48	60	62	47	43	39	23	11
Probability	0.0175	0.0675	0.0825	0.12	0.15	0.155	0.1175	0.1075	0.0975	0.0575	0.0275

The average sum of these 400 rolls is as follows

$$2(0.0175) + 3(0.0675) + 4(0.0825) + 5(0.12) + 6(0.15) + 7(0.155)$$
$$+ 8(0.1175) + 9(0.1075) + 10(0.0975) + 11(0.0575) + 12(0.0275)$$
$$= 6.9975$$

Allow time for each group of students to discuss Exercise 11. Then, ask students to share their ideas. Use this opportunity to check for understanding of the lesson.

MP.3

11. **Think about your answer to Exercise 1. What do you notice about the averages you have calculated as the number of rolls increase? Explain why this happens.**

    As the number of rolls increase, the average value approaches the expected value. This happens because as the number of observed values of the random variable increase, the relative frequencies become closer to the actual probabilities in the probability distribution. The expected value of a discrete random variable is a long-run average value for the variable.

Exploratory Challenge 2/Exercise 12 (5 minutes)

Before having students complete Exercise 12, be sure to discuss that the interpretation for the expected value should include the long-run aspect of probability. The interpretation should also be written in the context related to the discrete random variable. Then, allow time for students to work Exercise 12. When students are finished, discuss the answer.

The expected value of a discrete random variable is the long-run mean value of the discrete random variable. Refer back to Exercise 1 where two dice were rolled, and the sum of the two dice was recorded. The interpretation of the expected value of a sum of 7 would be

When two dice are rolled over and over for a long time, the mean sum of the two dice is 7.

Notice that the interpretation includes the context of the problem, which is the random variable sum of two dice, and also includes the concept of *long-run average*.

12. **Suppose a cancer charity in a large city wanted to obtain donations to send children with cancer to a circus appearing in the city. Volunteers were asked to call residents from the city's telephone book and to request a donation. Volunteers would try each phone number twice (at different times of day). If there was no answer, then a donation of $\$0$ was recorded. Residents who declined to donate were also recorded as $\$0$. The table below displays the results of the donation drive.**

Donation	$\$0$	$\$10$	$\$20$	$\$50$	$\$100$
Probability	0.11	0.35	0.25	0.20	0.09

Find the expected value for the amount donated, *and* write an interpretation of the expected value in context.

The expected amount donated is as follows

$$0(0.11) + 10(0.35) + 20(0.25) + 50(0.20) + 100(0.09) = \$27.50.$$

When a large number of residents are contacted, the mean amount donated is $\$27.50$.

Closing (3 minutes)

- The expected value is the long-run mean of a discrete random variable.
- The interpretation of an expected value must contain the context related to the discrete random variable.
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

> **Lesson Summary**
>
> **The expected value of a discrete random variable is interpreted as the long-run mean of that random variable.**
>
> **The interpretation of the expected value should include the context related to the discrete random variable.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 8: Interpreting Expected Value

Exit Ticket

At a large university, students are allowed to register for no more than 7 classes. The number of classes for which a student is registered is a discrete random variable. The expected value of this random variable for students at this university is 4.15.

Write an interpretation of this expected value.

Exit Ticket Sample Solutions

At a large university, students are allowed to register for no more than seven classes. The number of classes for which a student is registered is a discrete random variable. The expected value of this random variable for students at this university is 4.15.

Write an interpretation of this expected value.

If many students at this university are asked how many classes they are registered for, the average number would be 4.15 classes.

Problem Set Sample Solutions

1. **Suppose that a discrete random variable is the number of broken eggs in a randomly selected carton of one dozen eggs. The expected value for the number of broken eggs is 0.48 eggs. Which of the following statements is a correct interpretation of this expected value? Explain why the others are wrong.**
   a. **The probability that an egg will break in one dozen cartons is 0.48, on average.**
   b. **When a large number of one dozen cartons of eggs are examined, the average number of broken eggs in a one dozen carton is 0.48 eggs.**
   c. **The mean number of broken eggs in one dozen cartons is 0.48 eggs.**

   The correct answer is b.

   Answer choice a relates the expected value to a probability, which is incorrect. The expected value is the long-run mean of a random variable.

   Answer choice c does not refer to the long-run aspect of the expected value.

2. **Due to state funding, attendance is mandatory for students registered at a large community college. Students cannot miss more than eight days of class before being withdrawn from a course. The number of days a student is absent is a discrete random variable. The expected value of this random variable for students at this college is 3.5 days. Write an interpretation of this expected value.**

   If many students at this college are asked how many days they have been absent, the average number would be 3.5 days.

3. **The students at a large high school were asked to respond anonymously to the question:**

   How many speeding tickets have you received?

   The table below displays the distribution of the number of speeding tickets received by students at this high school.

Number of tickets	0	1	2	3	4	5
Probability	0.55	0.28	0.09	0.04	0.03	0.01

   Compute the expected number of speeding tickets received. Interpret this mean in context.

   The expected number of speeding tickets received by students at this high school is as follows:

   $0(0.55) + 1(0.28) + 2(0.09) + 3(0.04) + 4(0.03) + 5(0.01) = 0.75$

   In the long-run, students at this high school have received an average of 0.75 speeding tickets.

4. Employees at a large company were asked to respond to the question:

How many times do you bring your lunch to work each week?

The table below displays the distribution of the number of times lunch was brought to work each week by employees at this company.

Number of times lunch brought to work each week	0	1	2	3	4	5
Probability	0.30	0.12	0.12	0.10	0.06	0.30

Compute the expected number of times lunch was brought to work each week. Interpret this mean in context.

$0(0.30) + 1(0.12) + 2(0.12) + 3(0.10) + 4(0.06) + 5(0.30) = 2.4$

In the long-run, employees at this company brought lunch on average lunch 2.4 times each week.

5. Graduates from a large high school were asked the following:

How many total AP courses did you take from Grade 9 through Grade 12?

The table below displays the distribution of the total number of AP courses taken by graduates while attending this high school.

Number of AP courses	0	1	2	3	4	5	6	7	8
Probability	0.575	0.06	0.09	0.12	0.04	0.05	0.035	0.025	0.005

Compute the expected number of total AP courses taken per graduate. Interpret this mean in context.

The expected number of AP courses taken is as follows

$0(0.575) + 1(0.06) + 2(0.09) + 3(0.12) + 4(0.04) + 5(0.05) + 6(0.035) + 7(0.025) + 8(0.005) =$ 1.435 *AP courses*

If many high school graduates are asked how many AP courses were taken between Grades 9–12, the average number would be 1.435 AP courses.

6. **At an inspection center in a large city, the tires on the vehicles are checked for damage. The number of damaged tires is a discrete random variable. Create two different distributions for this random variable that have the same expected number of damaged tires. What is the expected number of damaged tires for the two distributions? Interpret the expected value.**

Distribution 1:

Number of damaged tires	0	1	2	3	4
Probability					

Distribution 2:

Number of damaged tires	0	1	2	3	4
Probability					

Answers will vary. Here is one example.

Distribution 1:

Number of damaged tires	0	1	2	3	4
Probability	0.60	0.20	0.10	0.05	0.05

The expected number of damaged tires is as follows

$0(0.60) + 1(0.20) + 2(0.10) + 3(0.05) + 4(0.05) = 0.75$ *tires*

Distribution 2:

Number of damaged tires	0	1	2	3	4
Probability	0.55	0.30	0.05	0.05	0.05

The expected number of damaged tires is as follows

$$0(0.55) + 1(0.30) + 2(0.05) + 3(0.05) + 4(0.05) = 0.75 \textit{ tires}$$

Because this inspection center examines the tires on the vehicles of a large number of customers, the inspectors find an average of 0.75 damaged tires per vehicle.

Lesson 9: Determining Discrete Probability Distributions

Student Outcomes

- Given a description of a discrete random variable, students determine the probability distribution of that variable.

Lesson Notes

In this lesson, students are given a description of a chance experiment that results in a discrete random variable. Students derive the discrete probability distribution for that random variable, and use the discrete probability distribution to answer probability questions. It is important to be very specific with language. The discrete probability distribution is a mathematical calculation based on possible outcomes of an event. It does not say much about what would actually happen if the following experiments were attempted. A coin flipped 10 times could land on heads every time. A coin flipped $1{,}000$ times could land on heads every time. However, there could be some introductory discussion of the idea that the larger the number of times the event occurs, the closer to the discrete probability distribution the outcomes will be.

Classwork

Exercises 1–3 (10 minutes)

Students should complete Exercises 1–3 either independently or with a partner. Discuss the answers as a class when the students are done. As students work, take a look at what they are producing. As students' work is informally assessed, choose students to share their answers as part of the discussion process.

Scaffolding:

If students are struggling, consider using the following questions to guide them:

- What are the possible values for this random variable?

  The values for this random variable are 0, 1, and 2. In other words, 0 heads, 1 head, or 2 heads can be observed.

- How many possible outcomes are there for the chance experiment of flipping a penny and a nickel? (Hint: Use the counting principle.)

  There are $2 \times 2 = 4$ possible outcomes.

Exercises

A chance experiment consists of flipping a penny and a nickel at the same time. Consider the random variable of the number of heads observed.

MP.4

1. Create a discrete probability distribution for the number of heads observed.

Penny	Nickel	Calculation	Probability
H	*H*	0.5×0.5	0.25
H	*T*	0.5×0.5	0.25
T	*H*	0.5×0.5	0.25
T	*T*	0.5×0.5	0.25

Number of Heads	0	1	2
Probability	0.25	0.50	0.25

2. **Explain how the discrete probability distribution is useful.**

   It can be used to help make predictions. For example, if the scenario presents a game where I could win a prize for guessing the correct number of heads, I would choose 1 head since the probability of only one head appearing is 0.5.

3. **What is the probability of observing at least one head when you flip a penny and a nickel?**

   The probability of tossing at least one head is as follows

$$P(1\text{ head}) + P(2\text{ heads}) = 0.50 + 0.25 = 0.75$$

Scaffolding:

Consider having students above-grade level attempt the following extension to the lesson:

- Invent a scenario that requires the calculation of a probability distribution. (Perhaps provide a broad topic, such as *food* or *shoes* to help focus their brainstorming.)
- Their scenarios and probabilities do not necessarily need to be research-based (they could be fictional), but their probability distributions should be accurate for their scenarios.

Exercises 4–6 (12 minutes)

Students can work independently or with a partner to complete Exercises 4–6. Be sure that students complete the tables correctly. Discuss the answers when students are done. As students progress from Exercises 1–3 to Exercises 4–6, they should notice that, if outcomes are composed of j events, then there are 2^j possible outcomes. (This may be a good introductory question.) Also, they should understand that probabilities asking for *at least* or *at most* are calculated by addition. Since these outcomes are discrete, multiplication does not make sense. (It is not possible for two coins to come up both heads and both tails.)

MP.4

Suppose that on a particular island, 60% of the eggs of a certain type of bird are female. You spot a nest of this bird and find three eggs. You are interested in the number of male eggs. Assume the gender of each egg is independent of the other eggs in the nest.

4. **Create a discrete probability distribution for the number of male eggs in the nest.**

Egg 1	*Egg 2*	*Egg 3*	*Calculation*	*Probability*
F	*F*	*F*	$0.6 \times 0.6 \times 0.6$	0.216
F	*F*	*M*	$0.6 \times 0.6 \times 0.4$	0.144
F	*M*	*F*	$0.6 \times 0.4 \times 0.6$	0.144
F	*M*	*M*	$0.6 \times 0.4 \times 0.4$	0.096
M	*F*	*F*	$0.4 \times 0.6 \times 0.6$	0.144
M	*F*	*M*	$0.4 \times 0.6 \times 0.4$	0.096
M	*M*	*F*	$0.4 \times 0.4 \times 0.6$	0.096
M	*M*	*M*	$0.4 \times 0.4 \times 0.4$	0.064

Number of Male Eggs	0	1	2	3
Probability	0.216	0.432	0.288	0.064

5. **What is the probability that no more than two eggs are male?**

   The probability that no more than two eggs are male is as follows

$$0.216 + 0.432 + 0.288 = 0.936$$

6. **Explain the similarities and differences between this probability distribution and the one in the first part of the lesson.**

   The distributions are similar because the events are independent in both cases, so the probability of each outcome can be determined by multiplying the probabilities of the events. The distributions are different because the number of values for each of the random variables is different.

Exercise 7 (15 minutes)

Begin work on this exercise as a whole class. Assign various students to perform the individual probability calculations. Then, combine class results and have students complete the remainder of the exercise independently. Consider asking students what they think it means to be a *satisfied customer*.

7. **The manufacturer of a certain type of tire claims that only 5% of the tires are defective. All four of your tires need to be replaced. What is the probability you would be a satisfied customer if you purchased all four tires from this manufacturer? Would you purchase from this manufacturer? Explain your answer using a probability distribution.**

Tire 1	*Tire 2*	*Tire 3*	*Tire 4*	*Calculation*	*Probability*
D	*D*	*D*	*D*	$0.05 \times 0.05 \times 0.05 \times 0.05$	0.00000625
D	*D*	*D*	*ND*	$0.05 \times 0.05 \times 0.05 \times 0.95$	0.00011875
D	*D*	*ND*	*D*	$0.05 \times 0.05 \times 0.95 \times 0.05$	0.00011875
D	*ND*	*D*	*D*	$0.05 \times 0.95 \times 0.05 \times 0.05$	0.00011875
D	*D*	*ND*	*ND*	$0.05 \times 0.05 \times 0.95 \times 0.95$	0.00225625
D	*ND*	*D*	*ND*	$0.05 \times 0.95 \times 0.05 \times 0.95$	0.00225625
D	*ND*	*ND*	*D*	$0.05 \times 0.95 \times 0.95 \times 0.05$	0.00225625
D	*ND*	*ND*	*ND*	$0.05 \times 0.95 \times 0.95 \times 0.95$	0.04286875
ND	*D*	*D*	*D*	$0.95 \times 0.05 \times 0.05 \times 0.05$	0.00011875
ND	*D*	*D*	*ND*	$0.95 \times 0.05 \times 0.05 \times 0.95$	0.00225625
ND	*D*	*ND*	*D*	$0.95 \times 0.05 \times 0.95 \times 0.05$	0.00225625
ND	*ND*	*D*	*D*	$0.95 \times 0.95 \times 0.05 \times 0.05$	0.00225625
ND	*D*	*ND*	*ND*	$0.95 \times 0.05 \times 0.95 \times 0.95$	0.04286875
ND	*ND*	*D*	*ND*	$0.95 \times 0.95 \times 0.05 \times 0.95$	0.04286875
ND	*ND*	*ND*	*D*	$0.95 \times 0.95 \times 0.95 \times 0.05$	0.04286875
ND	*ND*	*ND*	*ND*	$0.95 \times 0.95 \times 0.95 \times 0.95$	0.81450625

Note: D stands for "defective," and ND stands for "not defective."

Number of Defective Tires	0	1	2	3	4
Probability	0.81450625	0.171475	0.0135375	0.000475	0.00000625

To be a satisfied customer, 0 tires would be defective. The probability of this happening is 0.81450625.

Answers will vary about whether students would purchase tires from this manufacturer, but this could lead to a discussion about whether this probability is "good enough."

Some sample discussion points:

- *81% is fairly high, but that means that there is a 19% chance that at least one tire is defective.*
- *According to this model, about 1 in 5 cars is expected to have at least one defective tire.*
- *Going further: Why is the probability for a vehicle to have at least one defective tire so much higher than the 5% defect rate? Is the advertised defect rate of 5% misleading?*

Closing (3 minutes)

- Ask students to explain the concept in writing and share their answer with a neighbor: Explain what a discrete probability distribution is and how is it useful.
  - *Sample response: A discrete probability distribution is the set of calculated probabilities of every possible outcome of a series of events. It is useful because it allows us to make predictions about what might happen if we attempted to actually test these events. It is also useful when determining the best course of action to take, as in a game, for instance.*
- To calculate the corresponding probabilities for the values of a random variable, add the individual probabilities for all outcomes that correspond to the value.
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

- **To derive a probability distribution for a discrete random variable, you must consider all possible outcomes of the chance experiment.**
- **A discrete probability distribution displays all possible values of a random variable and the corresponding probabilities.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 9: Determining Discrete Probability Distributions

Exit Ticket

Suppose that an estimated 10% of the inhabitants of a large island have a certain gene. If pairs of islanders are selected at random and tested for the gene, what is the probability that one or both islanders are carriers? Explain your answer using a probability distribution.

Exit Ticket Sample Solutions

Suppose that an estimated 10% of the inhabitants of a large island have a certain gene. If pairs of islanders are selected at random and tested for the gene, what is the probability that one or both islanders are carriers? Explain your answer using a probability distribution.

Person 1	*Person 2*	*Calculation*	*Probability*
Y	*Y*	0.10×0.10	0.01
Y	*N*	0.10×0.90	0.09
N	*Y*	0.90×0.10	0.09
N	*N*	0.90×0.90	0.81

Number of Islanders with the Gene	0	1	2
Probability	0.81	0.18	0.01

The probability that one or both islanders are carriers is as follows

$$0.18 + 0.01 = 0.19$$

Problem Set Sample Solutions

1. **About 11% of adult Americans are left-handed. Suppose that two people are randomly selected from this population.**
   a. **Create a discrete probability distribution for the number of left-handed people in a sample of two randomly selected adult Americans.**

Person 1	*Person 2*	*Calculation*	*Probability*
L	*L*	0.11×0.11	0.0121
L	*R*	0.11×0.89	0.0979
R	*L*	0.89×0.11	0.0979
R	*R*	0.89×0.89	0.7921

Note: L stands for "left-handed," and R stands for "right-handed"

Number of Left-handers	0	1	2
Probability	0.7921	0.1958	0.0121

   b. **What is the probability that at least one person in the sample is left-handed?**

The probability that at least one person is left-handed is as follows

$$0.1958 + 0.0121 = 0.2079$$

2. **In a large batch of M&M candies, about 24% of the candies are blue. Suppose that three candies are randomly selected from the large batch.**

   a. **Create a discrete probability distribution for the number of blue candies out of the three randomly selected candies.**

Candy 1	Candy 2	Candy 3	Calculation	Probability
B	*B*	*B*	$0.24 \times 0.24 \times 0.24$	0.013824
B	*B*	*NB*	$0.24 \times 0.24 \times 0.76$	0.043776
B	*NB*	*B*	$0.24 \times 0.76 \times 0.24$	0.043776
B	*NB*	*NB*	$0.24 \times 0.76 \times 0.76$	0.138624
NB	*B*	*B*	$0.76 \times 0.24 \times 0.24$	0.043776
NB	*B*	*NB*	$0.76 \times 0.24 \times 0.76$	0.138624
NB	*NB*	*B*	$0.76 \times 0.76 \times 0.24$	0.138624
NB	*NB*	*NB*	$0.76 \times 0.76 \times 0.76$	0.438976

Note: B stands for "blue candy," and NB stands for "not blue candy."

Number of Blue Candies	0	1	2	3
Probability	0.438976	0.415872	0.131328	0.013824

   b. **What is probability that at most two candies are blue? Explain how you know.**

   The probability that at most two candies are blue is as follows:

$$0.438976 + 0.415872 + 0.131328 = 0.986176$$

3. **In the 21st century, about 3% of mothers give birth to twins. Suppose three mothers-to-be are chosen at random.**

   a. **Create a discrete probability distribution for the number of sets of twins born from the sample.**

Mother 1	Mother 2	Mother 3	Calculation	Probability
T	*T*	*T*	$0.03 \times 0.03 \times 0.03$	0.000027
T	*T*	*NT*	$0.03 \times 0.03 \times 0.97$	0.000873
T	*NT*	*T*	$0.03 \times 0.97 \times 0.03$	0.000873
T	*NT*	*NT*	$0.03 \times 0.97 \times 0.97$	0.028227
NT	*T*	*T*	$0.97 \times 0.03 \times 0.03$	0.000873
NT	*T*	*NT*	$0.97 \times 0.03 \times 0.97$	0.028227
NT	*NT*	*T*	$0.97 \times 0.97 \times 0.03$	0.028227
NT	*NT*	*NT*	$0.97 \times 0.97 \times 0.97$	0.912673

Note: T stands for "twins," and NT stands for "not twins."

Number of Mothers who Have Twins	0	1	2	3
Probability	0.912673	0.084681	0.002619	0.000027

   b. **What is the probability that at least one of the three mothers did not give birth to twins?**

$$0.912673 + 0.084681 + 0.002619 = 0.999973$$

4. About three in 500 people have type O-negative blood. Though it is one of the least frequently-occurring blood types, it is one of the most sought after because it can be donated to people who have any blood type.

a. Create a discrete probability distribution for the number of people who have type O-negative blood in a sample of two randomly selected adult Americans.

Person 1	Person 2	Calculation	Probability
O	*O*	0.006×0.006	0.000036
O	*NO*	0.006×0.994	0.005964
NO	*O*	0.994×0.006	0.005964
NO	*NO*	0.994×0.994	0.988036

Note: O stands for "O-negative," and NO stands for "not O-negative."

Number of People with Type O-Negative Blood	0	1	2
Probability	0.988036	0.011928	0.000036

b. Suppose two samples of two people are taken. What is the probability that at least one person in each sample has type O-negative blood?

$$(0.011928 + 0.000036)^2 = 0.000143$$

5. The probability of being struck by lightning in one's lifetime is approximately 1 in $3,000$.

a. What is the probability of being struck by lightning twice in one's lifetime?

$$(0.000333)^2 = 0.000000111$$

b. In a random sample of three adult Americans, how likely is it that at least one has been struck by lightning exactly twice?

Person 1	Person 2	Person 3	Calculation	Probability
T	*T*	*T*	$0.000000111 \times 0.000000111 \times 0.000000111$	1.3676×10^{-21}
T	*T*	*NT*	$0.000000111 \times 0.000000111 \times 0.999999889$	1.2321×10^{-14}
T	*NT*	*T*	$0.000000111 \times 0.999999889 \times 0.000000111$	1.2321×10^{-14}
T	*NT*	*NT*	$0.000000111 \times 0.999999889 \times 0.999999889$	1.1099×10^{-7}
NT	*T*	*T*	$0.999999889 \times 0.000000111 \times 0.000000111$	1.2321×10^{-14}
NT	*T*	*NT*	$0.999999889 \times 0.000000111 \times 0.999999889$	1.1099×10^{-7}
NT	*NT*	*T*	$0.999999889 \times 0.999999889 \times 0.000000111$	1.1099×10^{-7}
NT	*NT*	*NT*	$0.999999889 \times 0.999999889 \times 0.999999889$	0.999999667

Note: T stands for "twice," and NT stands for "not twice."

Number of People Struck Twice by Lightning	0	1	2	3
Probability	0.999999667	3.3297×10^{-7}	3.6963×10^{-14}	1.3676×10^{-21}

The probability that at least one person has been struck exactly twice by lightning is as follows:

$3.3297 \times 10^{-7} + 3.6963 \times 10^{-14} + 1.3676 \times 10^{-21} = 3.32970037 \times 10^{-7}$

Lesson 10: Determining Discrete Probability Distributions

Student Outcomes

- Given a description of a discrete random variable, students determine the probability distribution of that variable.
- Students interpret probabilities in context.

Lesson Notes

In this lesson, students are again given a description of a chance experiment that results in a discrete random variable. Students derive the discrete probability distribution for that random variable, and use it to answer probability questions and interpret those probabilities in context. As in the previous lesson, students mathematically determine probability distributions based on the possible outcomes of an event. In this lesson, they realize that only after many trials do distributions of event outcomes approach those in the calculated probability distribution. Each student will need two pennies for this lesson.

Classwork

Exercise 1 (2 minutes)

Have students read and answer Exercise 1 independently. This should serve as a refresher from the previous lesson, but it will also provide the necessary probability distribution for the forthcoming experiment. Discuss the answer when the class is finished. Make sure that students carefully explaining their reasoning.

Exercises

Recall this example from Lesson 9:

A chance experiment consists of flipping a penny and a nickel at the same time. Consider the random variable of the number of heads observed.

The probability distribution for the number of heads observed is as follows.

Number of Heads	0	1	2
Probability	0.25	0.50	0.25

1. What is the probability of observing exactly 1 head when flipping a penny and a nickel?

The probability of observing exactly 1 head when flipping a penny and a nickel is 0.50.

Scaffolding:

- The teacher could demonstrate the experiment by flipping a penny and a nickel at the same time and finding some probabilities empirically first in a chart similar to the one shown.
- More advanced students could answer a more challenging question such as a similar question involving three coins.

Exercises 2–7 (12 minutes)

In these exercises, students flip two coins a small number of times and create an actual probability distribution. Have students complete Exercises 2–7 independently. Discuss the answers once the class has finished. As students share their probability distributions from Exercise 6, it is expected that the distributions will vary.

2. **Suppose you will flip two pennies instead of flipping a penny and a nickel. How will the probability distribution for the number of heads observed change?**

   The probability distribution for the number of heads observed when flipping two pennies would be the same as the probability distribution for the number of heads observed when flipping a penny and a nickel. This is because both the penny and the nickel have a 50% chance of landing on heads.

3. **Flip two pennies and record the number of heads observed. Repeat this chance experiment three more times for a total of four flips.**

   Student answers will vary. One example is shown below.

Flip	Number of heads
1	0
2	0
3	1
4	1

4. **What proportion of the four flips resulted in exactly 1 head?**

   Student answers will vary. Based on the sample answer in Exercise 3, the proportion of the four flips in which I observed exactly 1 head is 0.50.

5. **Is the proportion of the time you observed exactly 1 head in Exercise 4 the same as the probability of observing exactly 1 head when two coins are flipped (given in Exercise 1)?**

   Student answers will vary. Based on the sample answer in Exercise 3, the proportion in Exercise 4 is the same as the probability in Exercise 1. However, this will not be the case for all students.

6. **Is the distribution of the number of heads observed in Exercise 3 the same as the actual probability distribution of the number of heads observed when two coins are flipped?**

   Student answers will vary. Based on the sample answer in Exercise 3, my distribution for the number of heads observed will not be the same. One answer is given below.

Number of Heads	0	1	2
Proportion	0.50	0.50	0

7. **In Exercise 6, some students may have answered, “Yes, they are the same.” But, many may have said, “No, they are different.” Why might the distributions be different?**

   Since the chance experiment of flipping two pennies is only repeated four times, the proportions for the number of heads observed will probably not be the same as the actual probabilities for the number of heads observed.

Exercises 8–9 (10 minutes)

In the previous exercises, students flipped two coins four times and recorded the distributions of the outcomes. Because the number of trials was so small, there were probably a wide range of results. In this exercise, when all the trials are combined as a class, the overall distribution should be closer to the calculated distribution. Put the following table up on the board and have students make tally marks in the appropriate cells to add their four observations from Exercise 3. Then, have students calculate the proportions needed for Exercise 8. After students have completed Exercise 9, remind them that the probabilities describe the *long-run behavior* of the random variable number of heads observed when two pennies are flipped, and introduce the law of large numbers.

- In the case of coins, the more times the two pennies are flipped, the observed probabilities of getting 0, 1, and 2 heads should be getting closer to what values?
  - *In the case of coins, the more times two pennies are flipped, the closer the observed probabilities of* 0, 1, *and* 2 *heads get to the calculated distribution of* 0.25, 0.50, *and* 0.25, *respectively.*
- Even though students get varying results flipping on their own, when combined as a class, the results should start to approach the calculated distribution.

Number of Heads	0	1	2
Tally			

8. Combine your four observations from Exercise 3 with those of the rest of the class on the chart on the board. Complete the table below.

Class answers will vary. One example is given.

Number of Heads	0	1	2
Proportion	0.28	0.47	0.25

9. How well does the distribution in Exercise 8 estimate the actual probability distribution for the random variable number of heads observed when flipping two coins?

The proportions for the number of heads observed are approximately equal to the actual probabilities for the number of heads observed.

The probability of a possible value is the *long-run proportion of the time* that that value will occur. In the above scenario, after flipping two coins MANY times, the proportion of the time each possible number of heads is observed will be close to the probabilities in the probability distribution. This is an application of the *law of large numbers*, one of the fundamental concepts of statistics. The law says that the more times an event occurs, the closer the experimental outcomes naturally get to the theoretical outcomes.

Exercises 10–12 (13 minutes)

Give students time to read the text and then have them complete Exercise 10 independently. Discuss the answer when class is finished. When there is agreement on the answer to Exercise 10, have students complete Exercises 11–12. Discuss the answers when the class is finished.

Scaffolding:

- Alternatively, read the text aloud and ask students to restate the problem in their own words to a partner.
- It might also help to explain a little about the word *Gallup*: *Gallup* is a world-famous polling organization known for its public opinion polls. It was founded by George Gallup in 1935.

A May 2000 Gallup Poll found that 38% of the people in a random sample of $1,012$ adult Americans said that they believe in ghosts. Suppose that three adults will be randomly selected with replacement from the group that responded to this poll, and the number of adults (out of the three) who believe in ghosts will be observed.

MP.4

10. **Develop a discrete probability distribution for the number of adults in the sample who believe in ghosts.**

Person 1	Person 2	Person 3	Calculation	Probability
NG	*NG*	*NG*	$0.62 \times 0.62 \times 0.62$	0.238328
NG	*NG*	*G*	$0.62 \times 0.62 \times 0.38$	0.146072
NG	*G*	*NG*	$0.62 \times 0.38 \times 0.62$	0.146072
NG	*G*	*G*	$0.62 \times 0.38 \times 0.38$	0.089528
G	*NG*	*NG*	$0.38 \times 0.62 \times 0.62$	0.146072
G	*G*	*NG*	$0.38 \times 0.38 \times 0.62$	0.089528
G	*NG*	*G*	$0.38 \times 0.62 \times 0.38$	0.089528
G	*G*	*G*	$0.38 \times 0.38 \times 0.38$	0.054872

Note: G stands for "believing in ghosts," and NG stands for "not believing in ghosts."

Number of Adults who Believe in Ghosts	0	1	2	3
Probability	0.238328	0.438216	0.268584	0.054872

11. **Calculate the probability that at least one adult, but at most two adults, in the sample believe in ghosts. Interpret this probability in context.**

The probability that at least one adult but at most two adults believe in ghosts is $0.438216 + 0.268584 = 0.7068$.

If three adults were randomly selected and the number of them believing in ghosts was recorded many times, the proportion that at least one but at most two adults who believe in ghosts would be 0.7068.

MP.2

12. **Out of the three randomly selected adults, how many would you expect to believe in ghosts? Interpret this expected value in context.**

Out of three randomly selected adults, the expected number who believe in ghosts is

$$0 \cdot 0.238328 + 1 \cdot 0.438216 + 2 \cdot 0.268584 + 3 \cdot 0.054872 = 1.14 \text{ adults.}$$

The long-run average number of adults in a sample of three who believe in ghosts is 1.14 *adults.*

Scaffolding:

- For English language learners who may struggle with interpreting probabilities in context, consider providing a sentence frame on the front board of the classroom.
- For example:
  The long-run average number of adults in a sample of _____who believe in ghosts is _____.

Closing (3 minutes)

- In the previous lesson, we discussed the purposes of the discrete probability distribution. They were
  a. To make decisions and
  b. To make predictions.

  With a partner, discuss whether discrete probabilities are more useful in the short- or in the long-run and why.
  - *Sample response: Discrete probability distributions are more useful in the long-run because they provide the expected distribution after many occurrences of an event, rather than just a few. In the short-run, there is too much variability in outcomes for the probability distribution to be of much use. The law of large numbers tells us that the more times an event occurs, the closer its outcome distribution will be to the calculated probability distribution.*
- Remind students that the interpretation of a probability of observing a particular value for a discrete random variable must include a reference to *long-run* behavior.
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

- **To derive a discrete probability distribution, you must consider all possible outcomes of the chance experiment.**
- **The interpretation of probabilities from a probability distribution should mention that it is the *long-run proportion of the time* that the corresponding value will be observed.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 10: Determining Discrete Probability Distributions

Exit Ticket

23% of the cars a certain automaker manufactures are silver. Below is the probability distribution for the number of silver cars sold by a car dealer in the next five car sales.

Number of Silver Cars	0	1	2	3	4	5
Probability	0.27068	0.40426	0.24151	0.07214	0.01077	0.00064

1. What is the probability of selling at most three silver cars? Interpret this probability in context.

2. What is the probability of selling between one and four silver cars? Interpret this probability in context.

3. How many silver cars is the dealer expected to sell, on average, out of five cars? Interpret this expected value in context.

Exit Ticket Sample Solutions

23% of the cars a certain automaker manufactures are silver. Below is the probability distribution for the number of silver cars sold by a car dealer in the next five car sales.

Number of Silver Cars	0	1	2	3	4	5
Probability	0.27068	0.40426	0.24151	0.07214	0.01077	0.00064

1. **What is the probability of selling at most three silver cars? Interpret this probability in context.**

   The probability that the dealer sells at most three silver cars is
   $\mathbf{0.27068 + 0.40426 + 0.24151 + 0.07214 = 0.98859}$.

   After many car sales, the long-run proportion of the time that at most three silver cars are sold out of every five is $\mathbf{0.98859}$*.*

2. **What is the probability of selling between one and four silver cars? Interpret this probability in context.**

   The probability that the dealer sells between one and four silver cars is
   $\mathbf{0.40426 + 0.24151 + 0.07214 + 0.01077 = 0.72868}$.

   After many car sales, the long-run proportion of the time that between one and four silver cars are sold out of every five is $\mathbf{0.72868}$*.*

3. **How many silver cars is the dealer expected to sell, on average, out of five cars? Interpret this expected value in context.**

   The average number of silver cars the dealer is expected to sell out of five cars is

   $$\mathbf{0(0.27068) + 1(0.40426) + 2(0.24151) + 3(0.07214) + 4(0.01077) + 5(0.00064) = 1.15}.$$

   After many car sales, the dealer is expected to sell a long-run average of $\mathbf{1.15}$ *silver cars out of every five cars.*

Problem Set Sample Solutions

1. A high school basketball player makes 70% of the free-throws she attempts. Suppose she attempts seven free-throws during a game. The probability distribution for the number of made free-throws out of seven attempts is displayed below.

Number of Completed Free-throws	0	1	2	3	4	5	6	7
Probability	0.00022	0.00357	0.02501	0.09725	0.22689	0.31765	0.24706	0.08235

a. What is the probability that she completes at least three free-throws? Interpret this probability in context.

The probability that she completes at least three free-throws is

$$0.09725 + 0.22689 + 0.31765 + 0.24706 + 0.08235 = 0.9712.$$

If this basketball player attempts seven free-throws many times, the long-run proportion of the time that she will complete at least three free-throws is 0.972.

b. What is the probability that she completes more than two but less than six free-throws? Interpret this probability in context.

The probability that she completes more than two but less than six free-throws is

$$0.09725 + 0.22689 + 0.31765 = 0.64179.$$

If this basketball player attempts seven free-throws during a game for many games, the long-run proportion of the time that she will complete more than two but less than six free-throws is 0.64179.

c. How many free-throws will she complete on average? Interpret this expected value in context.

The average number of free-throws that this basketball player will complete out of seven attempts is

$0(0.00022) + 1(0.00357) + 2(0.025) + 3(0.9725) + 4(0.227) + 5(0.318) + 6(0.247) + 7(0.082) = 4.90$ *free-throws.*

If this basketball player attempts seven free-throws many times, the long-run average number of free-throws completed is 4.90.

2. **In a certain county, 30% of the voters are Republicans. Suppose that four voters are randomly selected.**

   a. **Develop the probability distribution for the random variable number of Republicans out of the four randomly selected voters.**

Voter 1	*Voter 2*	*Voter 3*	*Voter 4*	*Calculation*	*Probability*
R	*R*	*R*	*R*	$0.3 \times 0.3 \times 0.3 \times 0.3$	0.0081
R	*R*	*R*	*NR*	$0.3 \times 0.3 \times 0.3 \times 0.7$	0.0189
R	*R*	*NR*	*R*	$0.3 \times 0.3 \times 0.7 \times 0.3$	0.0189
R	*NR*	*R*	*R*	$0.3 \times 0.7 \times 0.3 \times 0.3$	0.0189
R	*R*	*NR*	*NR*	$0.3 \times 0.3 \times 0.7 \times 0.7$	0.0441
R	*NR*	*R*	*NR*	$0.3 \times 0.7 \times 0.3 \times 0.7$	0.0441
R	*NR*	*NR*	*R*	$0.3 \times 0.7 \times 0.7 \times 0.3$	0.0441
R	*NR*	*NR*	*NR*	$0.3 \times 0.7 \times 0.7 \times 0.7$	0.1029
NR	*R*	*R*	*R*	$0.7 \times 0.3 \times 0.3 \times 0.3$	0.0189
NR	*R*	*R*	*NR*	$0.7 \times 0.3 \times 0.3 \times 0.7$	0.0441
NR	*R*	*NR*	*R*	$0.7 \times 0.3 \times 0.7 \times 0.3$	0.0441
NR	*NR*	*R*	*R*	$0.7 \times 0.7 \times 0.3 \times 0.3$	0.0441
NR	*R*	*NR*	*NR*	$0.7 \times 0.3 \times 0.7 \times 0.7$	0.1029
NR	*NR*	*R*	*NR*	$0.7 \times 0.7 \times 0.3 \times 0.7$	0.1029
NR	*NR*	*NR*	*R*	$0.7 \times 0.7 \times 0.7 \times 0.3$	0.1029
NR	*NR*	*NR*	*NR*	$0.7 \times 0.7 \times 0.7 \times 0.7$	0.2401

Note: R stands for "Republican," and NR stands for "not Republican."

Number of Republicans	0	1	2	3	4
Probability	0.2401	0.4116	0.2646	0.0756	0.0081

   b. **What is the probability that no more than two voters out of the four randomly selected voters will be Republicans? Interpret this probability in context.**

   The probability that no more than two out of four randomly selected voters are Republicans is

   $$0.2401 + 0.4116 + 0.2646 = 0.9163.$$

   If four voters are randomly selected and the number of Republicans is recorded many, many times, the long-run proportion of the time that no more than two voters will be Republicans is 0.9163*.*

3. **An archery target of diameter 122 cm has a bulls-eye with diameter 12.2 cm.**

   a. **What is the probability that an arrow hitting the target hits the bulls-eye?**

   The area of the bulls-eye is $\pi(6.1)^2 = 37.21\pi\ cm^2$.

   The area of the target is $\pi(61)^2 = 3721\pi\ cm^2$.

   The probability of the arrow landing in the bulls-eye is $\frac{37.21\pi}{3721\pi} = 0.01$.

b. **Develop the probability distribution for the random variable number of bulls-eyes out of three arrows shot.**

Arrow 1	Arrow 2	Arrow 3	Calculation	Probability
NB	*NB*	*NB*	$0.99 \times 0.99 \times 0.99$	0.970299
NB	*NB*	*B*	$0.99 \times 0.99 \times 0.01$	0.009801
NB	*B*	*NB*	$0.99 \times 0.01 \times 0.99$	0.009801
NB	*B*	*B*	$0.99 \times 0.01 \times 0.01$	0.000099
B	*NB*	*NB*	$0.01 \times 0.99 \times 0.99$	0.009801
B	*B*	*NB*	$0.01 \times 0.01 \times 0.99$	0.000099
B	*NB*	*B*	$0.01 \times 0.99 \times 0.01$	0.000099
B	*B*	*B*	$0.01 \times 0.01 \times 0.01$	0.000001

Note: B stands for "bulls-eye," and NB stands for "not bulls-eye."

Number of Bulls-eyes	0	1	2	3
Probability	0.970299	0.029403	0.000297	0.000001

c. **What is the probability of an archer getting at least one bulls-eye? Interpret this probability in context.**

The probability of an archer getting at least one bulls-eye is $0.029403 + 0.000297 + 0.000001 = 0.029701$. *After shooting many arrows, the long-run proportion of an archer getting at least one bulls-eye out of three is* 0.029701.

d. **On average, how many bulls-eyes should an archer expect out of three arrows? Interpret this expected value in context.**

On average, an archer should expect $0(0.970299) + 1(0.029403) + 2(0.000297) + 3(0.000001) = 0.03$ *arrows. After shooting many arrows, the long-run average should be close to* 0.03 *bulls-eyes for every three arrows shot.*

4. The probability that two people have the same birthday in a room of 20 people is about 41.1%. It turns out that your math, science, and English classes all have 20 people in them.

   a. Develop the probability distribution for the random variable number of pairs of people who share birthdays out of three classes.

Math	Science	English	Calculation	Probability
NP	*NP*	*NP*	$0.589 \times 0.589 \times 0.589$	0.204336
NP	*NP*	*P*	$0.589 \times 0.589 \times 0.411$	0.142585
NP	*P*	*NP*	$0.589 \times 0.411 \times 0.589$	0.142585
NP	*P*	*P*	$0.589 \times 0.411 \times 0.411$	0.099494
P	*NP*	*NP*	$0.411 \times 0.589 \times 0.589$	0.142585
P	*P*	*NP*	$0.411 \times 0.411 \times 0.589$	0.099494
P	*NP*	*P*	$0.411 \times 0.589 \times 0.411$	0.099494
P	*P*	*P*	$0.411 \times 0.411 \times 0.411$	0.069427

Note: P stands for "pair," and NP stands for "no pair."

Number of Pairs of Birthday-Sharers	0	1	2	3
Probability	0.204336	0.427755	0.298482	0.069427

   b. What is the probability that one or more pairs of people share a birthday in your three classes? Interpret the probability in context.

   The probability is $0.427755 + 0.298482 + 0.069427 = 0.795664$. If you went to many classes or other events containing 20 people, the long-run proportion of groups in which at least one pair of people share the same birthday is 0.795664.

5. You go to the warehouse of the computer company you work for because you need to send eight motherboards to a customer. You realize that someone has accidentally reshelved a pile of motherboards you had set aside as defective. Thirteen motherboards were set aside, and 172 are known to be good. You are in a hurry, so you pick eight at random. The probability distribution for the number of defective motherboards is below.

Number of Defective Motherboards	0	1	2	3	4	5	6	7	8
Probability	0.5596	0.3370	0.0888	0.0134	0.0013	0.3177	7.6×10^{-5}	6.1×10^{-8}	5.76×10^{-10}

   a. If more than one motherboard is defective, your company may lose the customer's business. What is the probability of that happening?

$$1 - (0.5596 + 0.3370) = 0.1034$$

   b. You are in a hurry and get nervous, so you pick eight motherboards, then second-guess yourself and put them back on the shelf. You then pick eight more. You do this a few times then decide it is time to make a decision and send eight motherboards to the customer. On average, how many defective motherboards are you choosing each time? Is it worth the risk of blindly picking motherboards?

$$0(0.5596) + 1(0.3370) + 2(0.0888) + 3(0.0134) + 4(0.0013) + 5(0.3177) + 6(7.6 \times 10^{-5}) + 7(6.1 \times 10^{-8}) + 8(5.76 \times 10^{-10}) = 2.15$$

   On average, 2.15 motherboards are defective, which would suggest that you run the risk of losing the customer's business.

Lesson 11: Estimating Probability Distributions Empirically

Student Outcomes

- Students use empirical data to estimate probabilities associated with a discrete random variable.
- Students interpret probabilities in context.

Lesson Notes

This lesson engages students in estimating probability distributions of discrete random variables using data they have collected in class. If you have access to polling software (Google Forms is a free option and SurveyMonkey has a basic, free account option. There are many choices available for various platforms–computer, phone/tablet, clickers, etc.), you can collect the data in real time at the start of class. If not, have students complete a paper survey and then provide them with summarized data. If possible, try to provide the results graphically for Question 1 and in tabular form for Question 2, as the exercises ask for the alternate representation. Students should have technology available to do the calculations so they can focus on understanding what the results represent. Note that in some cases, the sum of the probabilities in an estimated probability distribution might not exactly equal 1 because of rounding.

Depending on the time available, you may choose to have some students analyze the data from Question 1 and the rest the data from Question 2, or you may want everyone to analyze the data from both questions, in which case you might like to give two polls, one at the beginning of Exploratory Challenge 1 and the second when you start Exploratory Challenge 2. If you have a small class, or would like a larger set of responses, you might choose to give the poll to other classes. You could also have the class design a way to randomly sample students in a certain grade or set of classes to revisit ideas about taking random samples.

If you divide the class and have each student work on only one of the two examples, you might have time to begin Lesson 12, which might take more than one class period.

A note about rounding decimals: Typically in statistics, rounding is not as big of an issue as it is in, for instance, a calculus or a chemistry course. Most of the time, statisticians are not looking for an absolutely correct numerical figure; they are trying to use numbers to explain trends and make generalizations. In general, rounding to two or three decimal places is sufficient. A teacher may want to establish a rule with a class, but it is also acceptable to have a bit of subjectivity.

Classwork

Exploratory Challenge 1/Exercise 1 (5 minutes)

Have students collect the data for Questions 1 and 2. This can be done as a *speed date*, in which the whole class pairs off, one student in each pair provides data to the other, and then after twenty seconds, students switch partners. This is done until all students have met one another and everyone has a complete set of data. Check to make sure all data is in whole numbers and that everyone has collected data from everyone else.

Exploratory Challenge 1/Exercise 1

In this lesson, you will use empirical data to estimate probabilities associated with a discrete random variable and interpret probabilities in context.

1. Collect the responses to the following questions from your class.

Question 1: **Estimate to the nearest whole number the number of hours per week you spend playing games on computers or game consoles.**

Question 2: **If you rank each of the following subjects in terms of your favorite (number 1), where would you put mathematics: $1, 2, 3, 4, 5$, or 6?**
English, foreign languages, mathematics, music, science, and social studies

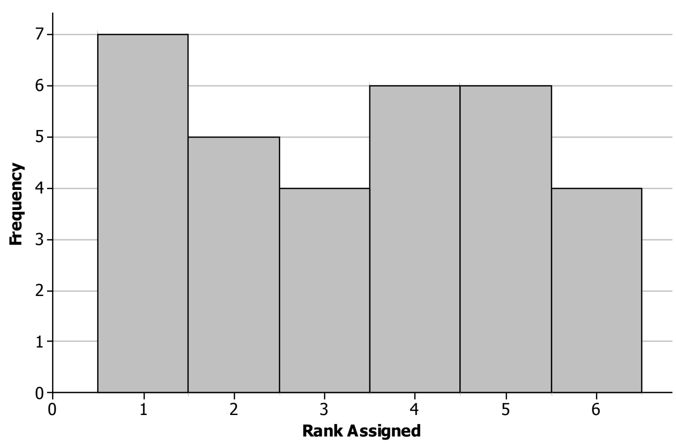
Responses may be displayed in raw form, tables, or graphs depending on the method you used to collect data from the class. The responses below are based on a class of 32 students.

Sample responses for Question 1 are given below.

Table: Number of hours playing games on computers or consoles

Hours	0	1	4	9	10	11	14	17	28	35
Frequency	3	2	5	4	6	5	3	2	1	1

Sample responses for Question 2 are given below.



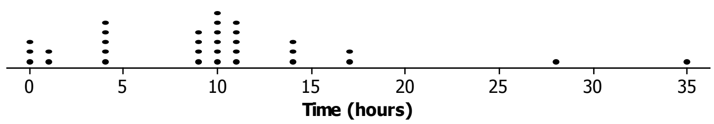
Exploratory Challenge 1/Exercises 2–5 (15 minutes): Computer Games

Students are asked to make a dot plot of the responses to Question 1. Be sure they have used a number line with the complete scale rather than only those numbers that were given as responses. (i.e., Even if no one responded 10 hours, but some did respond 11 hours, the horizontal axis should include both 10 and 11.)

Exploratory Challenge 1/Exercises 2–5: Computer Games

2. **Create a dot plot of the data from Question 1 in the poll: the number of hours per week students in class spend playing computer or video games.**

Responses will vary. The plot below is based on the sample data given in Exercise 1.



3. **Consider the chance experiment of selecting a student at random from the students at your school. You are interested in the number of hours per week a student spends playing games on computers or game consoles.**

   a. **Identify possible values for the random variable number of hours spent playing games on computers or game consoles.**

   Values include whole numbers from 1 to some number less than $24x7 = 168$. (Records show that some teens actually do play games for two or three days straight, but this is unusual.)

   b. **Which do you think will be more likely: a randomly chosen student at your school will play games for less than 9 hours per week or for more than 15 hours per week? Explain your thinking.**

   Responses will vary.

   Depending on the poll results students see displayed, if they think that their class is representative of students at the school, they may choose either option. Using the sample data above, students might suggest that chances seem more likely a randomly chosen student would play computer games less than 9 hours per week.

Scaffolding:

For advanced learners, consider replacing Exercises 2–4 with the following:

- How many hours do you think a typical student spends per week playing computer or video games?

Ask students to explain in writing how they arrived at their answer. Students should use an estimated probability distribution and expected value to support their answers.

For struggling students, consider reading the problems out loud, providing visual aids to help explain the problems, providing sample data for Exercise 3, or providing an exemplar response for the statements in Exercise 5.

MP.4

   c. **Assume that your class is representative of students at your school. Create an estimated probability distribution for the random variable number of hours per week a randomly selected student at your school spends playing games on computers or game consoles.**

   The table below is a sample answer based on data given in Exercise 1.

Hours	0	1	4	9	10	11	14	17	28	35
Probability	0.09	0.06	0.16	0.13	0.19	0.16	0.09	0.06	0.03	0.03

   d. **Use the estimated probability distribution to check your answer to part (b).**

   Responses will vary. A response based on sample data indicates that 0.25 of the class play less than 9 hour per week, while 0.12 play more than 15 hours of games on computers or game consoles.

MP.2

4. **Use the data your class collected to answer the following questions.**
   a. **What is the expected value for the number of hours students at your school play video games on a computer or game console?**

      Responses will vary. Based on the sample data, the expected value would be 9.7 hours.

   b. **Interpret the expected value you calculated in part (a).**

      Responses will vary. If you were to randomly select a student and then randomly select a student again and do this many times finding the number of hours spent playing games, the average number of hours would be about 9.7 hours per week.

5. **Again, assuming that the data from your class is representative of students at your school, comment on each of the following statements.**
   a. **It would not be surprising to have 20 students in a random sample of 200 students from the school who do not play computer or console games.**

      Responses will vary. Using the sample data, the estimated probability that students do not play any computer games is 0.09, which is about 18 of the 200 students. 20 is close to 18, so I would not be surprised.

   b. **It would be surprising to have 60 students in a random sample of 200 students from the school spend more than 10 hours per week playing computer or console games.**

      Responses will vary. According to the sample data, the probability of playing more than 10 hours a week is 0.37, which would be about 74 students. 60 students is not too close to 74, but I do not think it is far enough away to be that unlikely; so, I do not think it would be surprising.

   c. **It would be surprising if more than half of the students in a random sample of 200 students from the school played less than 9 hours of games per week.**

      Responses will vary. Using the sample data, the estimated probability that students play less than 9 hours of computer games per week is 0.31, which is about 62 students. Half of 200 is 100, so I would be surprised if 100 of the students played less than 9 hours of games per week.

Exploratory Challenge 2/Exercises 6–10 (15 minutes): Favorite Subject

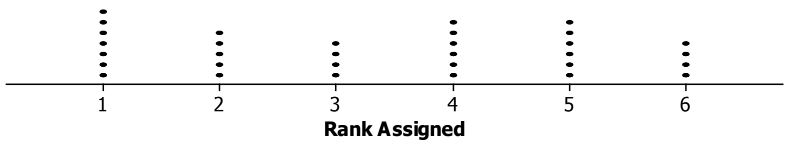
As mentioned before, you might begin this exercise set by giving students Question 2 as a second poll.

If students respond to Exercise 6, part (b), relying on their beliefs, stress the need to use data to support their answers; making decisions based on evidence is the point of studying statistics.

Exploratory Challenge 2/Exercises 6–10: Favorite Subject

6. Create a dot plot of your responses to Question 2 in the poll.

Responses will vary. Based on the sample data, the plot for Question 2 would be the following.



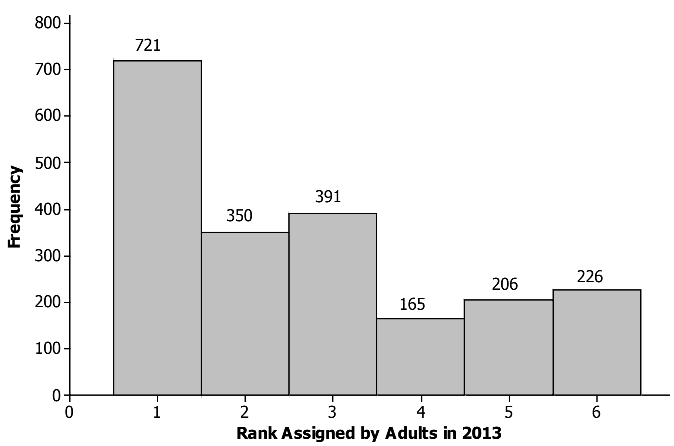
a. Describe the distribution of rank assigned.

Responses will vary. For the sample data, the distribution of the ranks is fairly uniform, with not that much difference in the frequency with which each of the ranks was chosen.

b. Do you think it is more likely that a randomly selected student in your class would rank mathematics high (1 or 2) or that he or she would rank it low (5 or 6)? Explain your reasoning.

Responses will vary. Using the counts in the sample data, students might observe that 12 people ranked math as 1 or 2, while only 10 people ranked math as a 5 or 6. So, two more people ranked it high than did low.

7. The graph displays the results of a 2013 poll taken by a polling company of a large random sample of $2,059$ adults 18 and older responding to Question 2 about ranking mathematics.



a. Describe the distribution of the rank assigned to mathematics for this poll.

Responses will vary.

The distribution is skewed right with more people ranking mathematics with a 1 or 2 than with a 5 or 6.

Scaffolding:

For advanced learners, consider replacing Exercises 1–5 with the following:

- How do you think a typical student would rank mathematics?
- Do you think a typical adult would rank mathematics higher or lower than a typical high school student?

Ask students to explain in writing how they arrived at their answer. Students should use an estimated probability distribution and expected value to support their answers.

For struggling students, consider providing sentence frames for English language learners, or altering Exercise 1, part (a), to be a multiple choice question:

- Would you describe the distribution as uniform, approximately normal, skewed left, or skewed right?

In addition, perhaps adding sentence starters or key points could be given for Exercise 2 to ensure thorough responses.

b. **Do you think the proportion of students who would rank mathematics 1 is greater than the proportion of adults who would rank mathematics 1? Explain your reasoning.**

Responses will vary. If students think that their class is representative of students in general, they may calculate the proportions or percentage and say more adults ranked it 1 than the students in their class. They may also ignore the data and respond with their belief.

8. **Consider the chance experiment of randomly selecting an adult and asking them what rank they would assign to mathematics. The variable of interest is the rank assigned to mathematics.**

a. **What are possible values of the random variable?**

The values are $1, 2, 3, 4, 5$, and 6.

MP.4

b. **Using the data from the large random sample of adults, create an estimated probability distribution for the rank assigned to mathematics by adults in 2013.**

Table: Rank assigned to mathematics (adults in 2013)

Rank	1	2	3	4	5	6
Probability	0.35	0.17	0.19	0.08	0.10	0.11

c. **Assuming that the students in your class are representative of students in general, use the data from your class to create an estimated probability distribution for the rank assigned to mathematics by students.**

Responses will vary. Sample response based on data from Question 2:

Table: Rank assigned to mathematics (based on sample class data for Question 2)

Rank	1	2	3	4	5	6
Probability	0.22	0.16	0.13	0.19	0.19	0.13

Note: Values may not add up to exactly 1 due to rounding.

9. **Use the two estimated probability distributions from Exercise 8 to answer the following questions.**

a. **Do the results support your answer to Exercise 7, part (b)? Why or why not?**

Responses will vary. The probability distributions indicate that adults rank mathematics as 1 with a probability of 0.35, while students do so with a probability of 0.22.

b. **Compare the probability distributions for the rank assigned to mathematics for adults and students.**

Responses will vary. Using the sample data, some students might note that the probabilities for ranking mathematics 6 and last for adults and for students were not that far apart. Others might note that for adults, the probabilities are quite large for a rank of 1, then decrease with a little fluctuation. The probabilities for students are all about the same for all of the ranks.

c. **Do adults or students have a greater probability of ranking mathematics in the middle (either a 3 or 4)?**

Responses will vary. Using the sample data, the estimated probability that a randomly chosen adult will rank mathematics as a 3 or 4 is 0.27 and that a randomly chosen student will rank it as a 3 or 4 is 0.32. These probabilities are not that far apart, so it does not seem that the two groups are that different.

10. Use the probability distributions from Exercise 2 to answer the following questions.

   a. Find the expected value for the estimated probability distribution of rank assigned by adults in 2013.

      The expected value is 2.74.

MP.2

   b. Interpret the expected value calculated in part (a).

      Responses will vary. Given that the sample of $2,059$ people was random, if you asked lots of adults how they would rank mathematics, in the long run, the average ranking would be 2.74.

   c. How does the expected value for the rank students assign to mathematics compare to the expected value for the rank assigned by adults?

      Responses will vary. Using the sample data, the expected value for students is 3.42, which indicates that students ranked mathematics lower than adults.

Closing (2 minutes)

- Explain how the average value of a probability distribution helps us to answer statistical questions.
   - *Responses will vary. The average value over the long run should approach the expected value and so should be somewhat close to the expected value. If it is not, I would probably be surprised.*
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

In this lesson you learned that

- **You can estimate probability distributions for discrete random variables using data collected from polls or other sources.**
- **Probabilities from a probability distribution for a discrete random variable can be interpreted in terms of long-run behavior of the random variable.**
- **An expected value can be calculated from a probability distribution and interpreted as a long-run average.**

Exit Ticket (8 minutes)

Name ____ Date ____________________

Lesson 11: Estimating Probability Distributions Empirically

Exit Ticket

The table shows the number of hours, to the nearest half-hour per day, that teens spend texting, according to a random sample of 870 teenagers aged 13–18 in a large urban city.

Table: Number of hours teenagers spend texting per day

Hours	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Frequency	170	82	220	153	92	58	40	15	12	18	10

1. What random variable is of interest here? What are the possible values for the random variable?

2. Create an estimated probability distribution for the time teens spend texting.

3. What is the estimated probability that teens spend less than an hour per day texting?

4. Would you be surprised if the average texting time for a smaller random sample of teens in the same city was three hours? Why or why not?

Exit Ticket Sample Solutions

The table shows the number of hours, to the nearest half-hour per day, that teens spend texting, according to a random sample of 870 teenagers aged 13–18 in a large urban city.

Table: Number of hours teenagers spend texting per day

Hours	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Frequency	170	82	220	153	92	58	40	15	12	18	10

1. What random variable is of interest here? What are the possible values for the random variable?

   The variable of interest is the number of hours to the nearest half-hour teens spend texting per day. Possible values are $0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5,$ *and* 5.

2. Create an estimated probability distribution for the time teens spend texting.

Hours	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
*Frequency**	0.20	0.09	0.25	0.18	0.11	0.07	0.05	0.02	0.01	0.02	0.01

   **Total does not equal* 1.00 *due to rounding.*

3. What is the estimated probability that teens spend less than an hour per day texting?

   0.29

4. Would you be surprised if the average texting time for a smaller random sample of teens in the same city was three hours? Why or why not?

   Responses will vary. Sample response:

   I would be surprised because the expected value is 1.36 *hours, and three hours is much larger than the expected value.*

Problem Set Sample Solutions

1. The results of a 1989 poll in which each person in a random sample of adults ranked mathematics as a favorite subject are in the table below. The poll was given in the same city as the poll in Exercise 6.

   Table: Rank assigned to mathematics by adults in 1989

Rank	1	2	3	4	5	6
Frequency	56	43	12	19	39	61

   a. Create an estimated probability distribution for the random variable that is the rank assigned to mathematics.

   Table: Rank assigned to mathematics by adults in 1989

Rank	1	2	3	4	5	6
Probability	0.24	0.19	0.05	0.08	0.17	0.27

b. **An article about the poll reported, "Americans have a bit of a love-hate relationship with mathematics." Do the results support this statement? Why or why not?**

Responses will vary.

Some students might note that the probability of ranks 1 and 2 is 0.43, and the probability of ranks 5 and 6 is 0.44. So, the probability that a randomly chosen adult from this city liked mathematics was almost the same as the probability that he or she did not like mathematics. Others might point out the probability that a randomly chosen person ranked math first was 0.24 and last was 0.27, which are only 0.03 apart, so the probability that they love math was about the same as the probability that they hate math.

c. **How is the estimated probability distribution of the rank assigned to mathematics by adults in 1989 different from the estimated probability distribution for adults in 2013?**

Table: Rank assigned to mathematics by adults in 2013

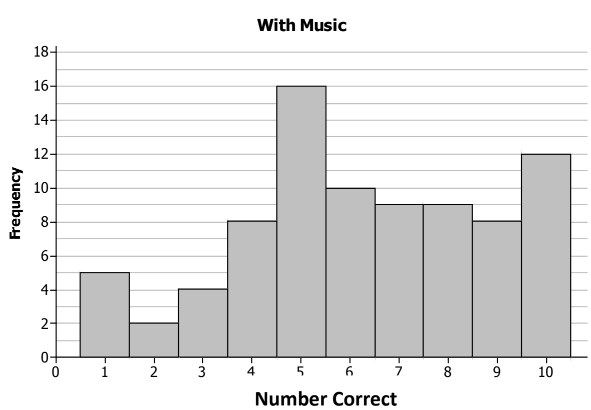
Rank	1	2	3	4	5	6
Frequency	0.35	0.17	0.19	0.08	0.1	0.11

Responses will vary.

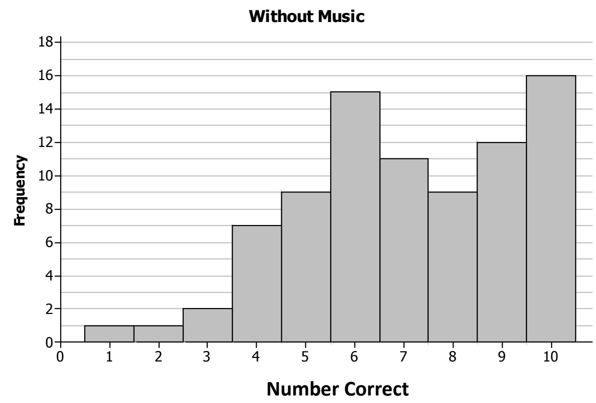
Students might say that the probability of ranking mathematics first was greater in 2013 or that the probabilities of ranking mathematics first and third in 2013 were greater than in 1989. The probability of ranking mathematics sixth in 2013 was less than half of the probability of ranking it sixth in 1989. The expected value in 2013 was a rank of 2.74, and in 1989 the expected value was lower with a rank of 3.56.

2. **A researcher investigated whether listening to music made a difference in people's ability to memorize the spelling of words. A random sample of 83 people memorized the spelling of 10 words with music playing, and then they were tested to see how many of the words they could spell. These people then memorized 10 different words without music playing and were tested again. The results are given in the two displays below.**

Number of words correctly spelled when memorized with music



Number of words correctly spelled when memorized without music



a. **What do you observe from comparing the two distributions?**

Responses will vary.

Some may observe that the people in the sample seemed to memorize more words without music than with music.

b. **Identify the variable of interest. What are possible values it could take on?**

The random variable is the number of words spelled correctly. Possible values are integers 0 to 10.

c. **Assume that the group of people that participated in this study are representative of adults in general. Create both a table and a graph of the estimated probability distributions for number of words spelled correctly when memorized with music and number of words spelled correctly when memorized without music. What are the advantages and disadvantages of using a table? A graph?**

Responses will vary.

With music:

Words	1	2	3	4	5	6	7	8	9	10
Frequency	5	2	4	8	16	10	9	9	8	12
Probability	0.06	0.02	0.05	0.10	0.19	0.12	0.11	0.11	0.10	0.14

Without music:

Words	1	2	3	4	5	6	7	8	9	10
Frequency	1	1	2	7	9	15	11	9	12	16
Probability	0.01	0.01	0.02	0.08	0.11	0.18	0.13	0.11	0.14	0.19

Students might suggest the advantages of using a table are that you can see the actual probabilities, and the advantages of using a graph are that you can see the pattern or trend. It is almost easier to see that more words were spelled correctly when they were memorized without music from the graph because you can see the distribution seems to be more skewed left than the one for the number of words memorized with music.

A disadvantage of using a table might be that the differences in the probabilities are not always easy to sort out quickly; a disadvantage of using the graph is that you almost have to estimate the probabilities because you cannot see the actual values. (Note that some interactive technology will show the values if the cursor is dragged over the bar or point.)

d. **Compare the probability that a randomly chosen person who memorized words with music will be able to correctly spell at least eight of the words to the probability for a randomly chosen person who memorized words without music.**

Responses will vary.

Students should compare the two probabilities: $P(W \geq 8 \text{ with music}) = 0.35$, while $P(W \geq 8 \text{ without music}) = 0.44$. There is a 0.9 difference in the probability they will be able to correctly spell at least eight words for the two conditions (music and no music).

e. **Make a conjecture about which of the two estimated probability distributions will have the largest expected value. Check your conjecture by finding the expected values. Explain what each expected value means in terms of memorizing with and without music.**

The expected value for the number of words spelled correctly when memorizing with music is 6.27 words; without music it is 6.99 words. Assuming that the group of people that participated in this study are representative of adults in general, if you gave people lots and lots of lists of 10 words to memorize with and without music, over the long run, with music they would be able to spell 6.27 words and without music 6.99 words. The difference does not seem to be very large.

3. **A random variable takes on the values $0, 2, 5$, and 10. The table below shows a frequency distribution based on observing values of the random variable and the estimated probability distribution for the random variable based on the observed values. Fill in the missing cells in the table.**

Table: Distribution of observed values of a random variable

Variable	0	2	5	10
Frequency	18	12	2	$?$
Probability	0.3	0.2	$??$	0.47

The missing cell in the probability row has to be 0.03 because the sum of the probabilities has to be 1. The missing cell in the frequency row has to be 28 because $\frac{18}{32+x} = 0.3$, $\frac{12}{32+x} = 0.2$, and $\frac{2}{32+x} = 0.03$, and solving for x in any of the equations yields $x = 28$.

Lesson 12: Estimating Probability Distributions Empirically

Student Outcomes

- Students use empirical data to estimate probabilities associated with a discrete random variable.
- Students interpret probabilities in context.

Lesson Notes

This lesson engages students in collecting empirical data associated with a discrete random variable by simulation. You may want to have them each do a few trials of the simulation and pool the data as a class before discussing the outcomes. Or, you may choose to have a small group of students collectively do as many as 50 trials of the simulation and then compare the results across groups. The use of technology for generating random samples will make the work easier for students, and some of the simulations can be completely done with simple steps to generate, collect, and display results.

In the first example, each student will need 2 dice if you plan to have them carry out a physical simulation. Or, you may choose to use technology to simulate tossing the dice. Students should use technology for the calculations so they can focus on the statistical concepts. Students make a dot plot of the results of the simulation and use that to estimate the probabilities in a probability distribution. In the second example, they go directly from the simulation to constructing a graph of the estimated probability distribution. Be sure they understand that the simulations produce frequencies, and these must be converted into relative frequencies to create the estimated probability distribution.

This lesson may take more than one class period. If you are short on time, consider choosing to do either Example 1 and Exercises 1–2 or Example 2 and Exercises 3–5.

Classwork

Exploratory Challenge 1/Exercises 1–2 (18 minutes): Moving Along

Students investigate tossing 2 dice and moving along a number line according to the absolute value of the difference of numbers showing on the faces. For example, if they toss a 6 and a 3 on the first toss, they would move from 0 to 3 on the number line. They may actually play the game, or they can simulate tossing 2 dice. They record the distance moved for each toss and the sum of the distances moved, targeting a total distance of at least 20. Using these data, students find the expected value and interpret it in context.

Let students work with a partner to toss the dice and play a few games. Students should keep track of the distance moved for each toss and the number of tosses it takes to move past 20 on the number line. After playing the game a few times, have students work on Exercises 1 and 2 with their partner to carry out the physical (or technological simulation) and estimate probabilities. (If using technology, students may choose to use either a graphing calculator or computer software to perform the simulation they describe in their answer to Exercise 1.)

MP.5

Exploratory Challenge 1/Exercises 1–2: Moving Along

In a certain game, you toss 2 dice and find the difference of the numbers showing on the faces. You move along a number line according to the absolute value of the difference. For example, if you toss a 6 and a 3 on the first toss, then you move 3 spaces from your current position on the number line. You begin on the number 0, and the game ends when you move past 20 on the number line.

1. **How many rolls would you expect it to take for you to get to 20? Explain how you would use simulation to answer this question.**

   Answers will vary. It appears that many rolls result in a difference of 1, 2, or 3. Considering that I need to move 20 spaces, I would say that I expect that it would take around 8, 9, or 10 rolls to get to 20.

   The random variable of interest is the distance moved on the number line for a toss of the dice. The possible values are 0, 1, 2, 3, 4, and 5. I would play the game with my partner at least 30 times and record the distance moved for each toss. Then, I would use the results to create an estimated probability distribution to determine expected value for the distance moved on 1 toss of the dice.

MP.2

2. **Perform the simulation with your partner.**

   a. **What is the expected value for the distance moved on 1 toss of 2 dice? Interpret your answer in terms of playing the game.**

   Responses will vary. One possible response based on 60 tosses is shown here:

   Distance moved on 1 toss of 2 dice

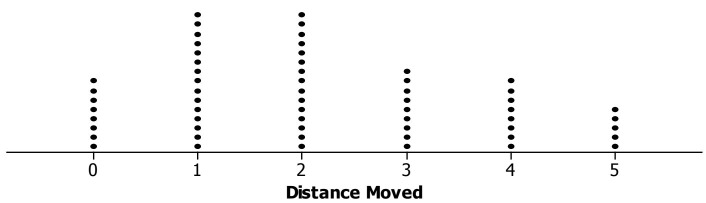


Table: Estimated probability distribution for the distance moved on 1 toss of 2 dice

Number of Moves	0	1	2	3	4	5
Estimated Probability	$\frac{8}{60} = 0.133$	$\frac{15}{60} = 0.25$	$\frac{15}{60} = 0.25$	$\frac{9}{60} = 0.15$	$\frac{8}{60} = 0.133$	$\frac{5}{60} = 0.083$

   The example results are

   $0(0.133) + 1(0.25) + 2(0.25) + 3(0.15) + 4(0.133) + 5(0.083) = 2.147.$

   In the long run, after tossing 2 dice many times, we expect the average distance moved to be close to 2.15.

   b. **Use your expected value from part (a) to find the expected number of tosses that would put you past 20 on the number line.**

   Responses will vary.

   For the sample data, in 10 tosses, you would expect to be at about 21 or 22.

Exploratory Challenge 2/Exercises 3–4 (15 minutes): Lemon Flavor

MP.5 Students can work alone or in small groups to collect their data. One strategy for carrying out the simulation is to generate about 200 random numbers from the set $\{0,1\}$ in a list with 1 representing lemon. Count down the list marking the number of cough drops before you have two 1s in a row. Note that the question is not whether there are 2 lemon-flavored cough drops in a row but rather how far into the package you have to go to get 2 lemon-flavored ones in a row.

Have students work alone or in a small group to complete the exercises.

Exploratory Challenge 2/ Exercises 3–4: Lemon Flavor

Cough drops come in a roll with 2 different flavors, lemon and cherry. The same number of lemon and cherry cough drops are produced. Assume the cough drops are randomly packed with 30 per roll and that the flavor of a cough drop in the roll is independent of the flavor of the others.

3. Suppose you really liked the lemon flavor. How many cough drops would you expect to go through before finding 2 lemon cough drops in a row? Explain how you would use simulation to answer this question.

Responses will vary. Students might answer anywhere from 3 to 20.

To simulate, I would generate random numbers from the set $\{0, 1\}$, with 1 representing lemon. The random variable is the number of cough drops before you have 2 lemon cough drops in a row. The possible values are $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, on up to 30. In a case where 2 lemon cough drops do not occur in the pack of 30, I would use 0 to represent the outcome.

I would simulate the experiment at least 50 times and record the number of cough drops I would go through before getting 2 lemon-flavored ones in a row. I would use the results to create a probability distribution that could be used to estimate the number of cough drops I expect to go through before finding 2 lemon-flavored ones in a row.

MP.2

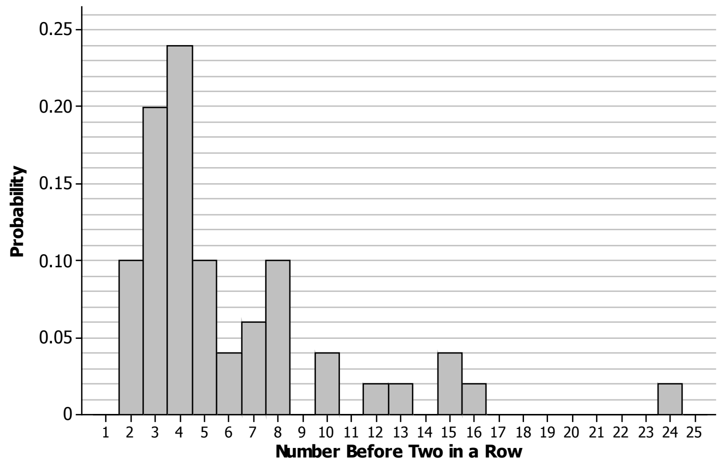
4. Carry out the simulation and use your data to estimate the average number of cough drops you would expect to go through before you found 2 lemon-flavored ones in a row. Explain what your answer indicates about 2 lemon-flavored cough drops in a row.

Responses will vary. Sample results are below. Students might be surprised that in one case, it took 24 cough drops before they had 2 lemon-flavored ones in a row.

Number before 2 in a row/frequency:

2	11111
3	1111111111
4	111111111111
5	11111
6	11
7	111
8	11111
9	

10	11
11	
12	1
13	1
14	
15	11
16	1
⋮	⋮
24	1



Using sample data above, the expected value would be 6.02. If you opened many packages of cough drops, on average you would go through about 6 cough drops before you got 2 lemon-flavored ones in a row.

Closing (2 minutes)

- How was simulation helpful in this lesson? What tools were used to help answer the questions posed in this lesson?
  - *Responses will vary. Sample response:*

    Simulation was used to help build estimated probability distributions. I used my graphing calculator to simulate the tossing of the 2 dice for my plan in Exploratory Challenge 1/Exercise 1.

- How did the estimated probability distributions for the distance moved in the game in Exploratory Challenge 1/Exercise 1 vary across the class?
  - *Responses will vary. Sample response:*

    They were fairly close but not exactly the same because of the variability in the outcomes of tossing 2 dice.
- Comment on the following statement: If the estimated expected value is 12 for the number of cough drops in a roll before 2 lemon-flavored ones, then you would expect to have 10 that were not lemon at the beginning.
  - *Responses will vary. Sample response:*

    You could have a mix of 10 lemon and cherry flavors but with no 2 lemons in a row. The eleventh and twelfth ones would be lemon.
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

In this lesson, you learned that

- **You can estimate probability distributions for discrete random variables using data from simulating experiments.**
- **Probabilities from an estimated probability distribution for a discrete random variable can be interpreted in terms of long-run behavior of the random variable.**
- **An expected value can be calculated from an estimated probability distribution and interpreted as a long-run average.**

Exit Ticket (10 minutes)

Name ____ Date ____________________

Lesson 12: Estimating Probability Distributions Empirically

Exit Ticket

A bus company has 9 seats on a shuttle between two cities, but about 10% of the time people do not show up for the bus even though they reserve a seat. The company compensates by reserving 11 seats instead of 9. Assume that whether or not a person with a reservation shows up is independent of what happens with the other reservation holders.

a. Consider the random variable number of people who are denied a seat because more than 9 people showed up for the shuttle. What are the possible values of this random variable?

b. The table displays the number of people who reserved tickets but did show up based on simulating 50 trips between the two cities. Use the information to estimate a probability distribution of the number of people denied a seat on the shuttle.

Table: Number of people who showed up for their reservation

10	10	11	9	10	11	9	9	11	11
7	10	10	8	9	10	11	9	10	10
8	11	11	10	11	10	10	11	9	11
11	10	8	11	9	11	9	10	11	9
10	9	9	10	10	9	9	10	10	9

c. In the long run, how many people should the company expect to be denied a seat per shuttle trip? Explain how you determined the answer.

Exit Ticket Sample Solutions

1. A bus company has 9 seats on a shuttle between two cities, but about 10% of the time people do not show up for the bus even though they reserve a seat. The company compensates by reserving 11 seats instead of 9. Assume that whether or not a person with a reservation shows up is independent of what happens with the other reservation holders.

   a. Consider the random variable number of people who are denied a seat because more than 9 people showed up for the shuttle. What are the possible values of this random variable?

      0, 1, and 2

   b. The table displays the number of people who reserved tickets but did show up based simulating 50 trips between the two cities. Use the information to estimate a probability distribution of the number of people denied a seat on the shuttle.

      Table: Number of people who showed up for their reservation

10	10	11	9	10	11	9	9	11	11
7	10	10	8	9	10	11	9	10	10
8	11	11	10	11	10	10	11	9	11
11	10	8	11	9	11	9	10	11	9
10	9	9	10	10	9	9	10	10	9

Table: Number of people denied a seat on shuttle

Number denied a seat	2	1	0
Frequency	14	18	18
Relative frequency	0.28	0.36	0.36

   c. In the long run, how many people should the company expect to be denied a seat per shuttle trip? Explain how you determined the answer.

      The company should expect about 0.92 passengers will be denied a seat over the long run, or just less than 1 per trip. I used the sample data to determine the expected value: $0 \cdot 0.36 + 1 \cdot 0.36 + 2 \cdot 0.28 = 0.92$.

Problem Set Sample Solutions

1. Suppose the rules of the game in Exploratory Challenge 1 changed.
   If you had an absolute difference of
   - 3 or more, you move forward a distance of 1;
   - 1 or 2, you move forward a distance of 2;
   - 0, you do not move forward.

   a. Use your results from Exploratory Challenge 1/Exercise 2 to estimate the probabilities for the distance moved on 1 toss of 2 dice in the new game.

   Responses will vary. Sample answer:

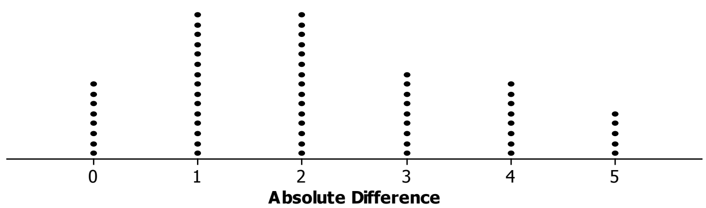


Table: Distance moved on 1 toss of the dice

Distance Moved	0 (*difference* = 0)	2 (*difference* = $1, 2$)	1 (*difference* = $3, 4, 5$)
Probability	$\frac{8}{60} = 0.133$	$\frac{15}{60} + \frac{15}{60} = 0.5$	$\frac{9}{60} + \frac{8}{60} + \frac{5}{60} = 0.366$

   b. Which distance moved is most likely?

   Responses will vary. Sample response:

   For the sample data, a distance of 2 is the most likely with a probability of 0.5.

   c. Find the expected value for distance moved if you tossed 2 dice 10 times.

   Responses will vary. Sample response:

   For the sample data, expected distance moved in 1 toss is $0(0.133) + 2(0.5) + 1(0.366) = 1.366$, so the expected distance moved in 10 tosses is 13.66.

   d. If you tossed the dice 20 times, where would you expect to be on the number line, on average?

   Responses will vary. Sample response:

   For the sample data, over the long run, you would expect to be at about 27 on the number line.

2. Suppose you were playing the game of Monopoly, and you got the Go to Jail card. You cannot get out of jail until you toss a double (the same number on both dice when 2 dice are tossed) or pay a fine.

   a. If the random variable is the number of tosses you make before you get a double, what are possible values for the random variable?

   $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$

   b. Create an estimated probability distribution for how many times you would have to toss a pair of dice to get out of jail by tossing a double. (You may toss actual dice or use technology to simulate tossing dice.)

   Responses will vary.

   For the sample data below, an estimated probability distribution would be as follows:

   Table: Number of times tossing a pair of dice before a double occurs

Number rolls	1	2	3	4	5	6	7	8
Probability	0.08	0.2	0.12	0.08	0.08	0.04	0.04	0.04
Number rolls	9	10	11	12	13	14	15	
Probability	0.08			0.08		0.04	0.04	

The table below displays the simulated tosses. The table is read down each column, from left to right. The first number represents the difference in the faces of simulated tosses of 2 dice, and the number 0 represents a double was tossed. All doubles have been highlighted. The number of times the dice were tossed between doubles appears in parentheses.

There were 8 tosses between doubles.

−1.	3.	0.(4)	3.	−4.	0.(2)	−2.	2.
0.(2)	1.	4.	−4.	3.	2.	−2.	−1.
2.	0.(3)	4.	−3.	−3.	−1.	5.	−1.
−3.	3.	−2.	1.	0.(14)	2.	0.(9)	−1.
−1.	−3.	−4.	3.	1.	0.(4)	1.	−2.
1.	0.(3)	−3.	5.	−2.	3.	1.	0.(12)
−3.	0.(1)	−1.	−4.	2.	−4.	4.	1.
−4.	−4.	−4.	5.	5.	−1.	4.	0.(2)
1.	−5.	1.	3.	0.(5)	−3.	−1.	−1.
0.(8)	−4.	0.(9)	2.	2.	−1.	−1.	1.

−3.	−1.	4.	0.(12)	−3.	−1.	−1.	−3.
3.	3.	1.	0.(1)	0.(3)	1.	−4.	0 (2)
1.	4.	−3.	−1.	3.	2.	−1.	
0.(6)	−4.	4.	4.	0.(2)	−2.	2.	
−2.	1.	2.	−1.	−2.	0.(6)	−5.	
−2.	−2.	−2.	4.	−3.	−2.	−2.	
2.	2.	−1.	2.	1.	3.	−1.	
−2.	−2.	3.	−4.	2.	2.	−2.	
−1.	0.(15)	−5.	0.(7)	0.(5)	1.	1.	
4.	−2.	−3.	−1.	−1.	4.	0.(15)	

Summary of number of tosses of dice from simulation

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
11	11111	111	11	11	11	1	1	11			11		1	11

c. **What is the expected number of tosses of the dice before you would get out of jail with a double?**

Responses will vary.

For the sample data, the expected number is 4.16. Over the long run, you would expect to toss the dice about 4 times before you got a double.

3. **The shuttle company described in the Exit Ticket found that when they make 11 reservations, the average number of people denied a seat per shuttle is about 1 passenger per trip, which leads to unhappy customers. The manager suggests they take reservations for only 10 seats. But his boss says that might leave too many empty seats.**

a. **Simulate 50 trips with 10 reservations, given that in the long run, 10% of those who make reservations do not show up. (You might let the number 1 represent a no-show and a 0 represent someone who does show up. Generate 10 random numbers from the set that contains one 1 and nine 0s to represent the 10 reservations and then count the number of $1'$s.)**

Responses will vary. Sample response:

Table: Simulated results of the number of no shows given that 10% are no shows

Number of no shows	0	1	2	3	4
Frequency	11	22	15	2	0

b. **If 3 people do not show up for their reservations, how many seats are empty? Explain your reasoning.**

2 seats are empty because there are only 9 seats on the shuttle.

c. **Use the number of empty seats as your random variable and create an estimated probability distribution for the number of empty seats.**

Responses will vary.

Note that there are no empty seats when 9 or 10 people with reservations show up because there are only 9 seats on the shuttle.

Table: Number of empty seats on shuttle

Number of empty seats	0	1	2	3
Frequency	33	15	2	0
Relative frequency	0.66	0.30	0.04	0

d. **What is the expected value for the estimated probability distribution? Interpret your answer from the perspective of the shuttle company.**

Responses will vary.

For the sample data, the expected value is 0.38, which means that over the long run, 0.38 seats are empty per shuttle.

e. **How many reservations do you think the shuttle company should accept and why?**

Responses will vary.

In the long run, about 1 person per trip would be denied a seat if they make 11 reservations, and they will have an empty seat over $\frac{1}{3}$ of the time if they make only 10 reservations. What they choose to do would depend on how much they have to compensate those who are denied seats.

Table of Random Numbers

00000110010001100011

10111001111010001100

01101001011011010101

11011011010100110010

01110001000100110011

00010000001100111011

11101001010010000110

10110001110001000100

11101100101101100110

11100010010011100011

Name ____ Date ______________________

1. In the game of tennis, one player serves to a second player to start a point. If the server misses landing the first serve (the ball is hit out of bounds and cannot be played), the server is allowed a second serve. Suppose a particular tennis player, Chris, has a 0.62 probability of a first serve landing in bounds and a 0.80 probability of a second serve landing in bounds. Once a serve has landed in bounds, the players take turns hitting the ball back and forth until one player hits the ball out of bounds or into the net. The player who hits the ball out or into the net loses the point, and the other player wins the point. For Chris, when the first serve lands in bounds, he has a 75% chance of winning the point; however, he has only a 22% chance of winning the point on his second serve.

   a. Calculate the probability that Chris lands his first serve in bounds and then goes on to win the point for a randomly selected point.

   b. Calculate the probability that Chris misses the first serve, lands the second serve, and then wins the point.

   c. Calculate the probability that Chris wins a randomly selected point.

2. In Australia, there are approximately 100 species of venomous snakes, but only about 12 have a deadly bite. Suppose a zoo randomly selects one snake from each of five different species to show visitors. What is the probability that exactly one of the five snakes shown is deadly?

3. In California (CA), standard license plates are currently of the form: 1 number–3 letters–3 numbers. Assume the numbers are $0–9$ and the letters are A–Z.

   a. In theory, how many different possible standard CA license plates are there, assuming we can repeat letters and numbers?

   b. How many different possible standard CA license plates are there if we are not allowed to repeat any letters or numbers?

   c. For part (b), did you use permutations or combinations to carry out the calculation? Explain how you know.

4. When there is a problem with a computer program, many people call a technical support center for help. Several people may call at once, so the center needs to be able to have several telephone lines available at the same time. Suppose we want to consider a random variable that is the number of telephone lines in use by the technical support center of a software manufacturer at a particular time of day. Suppose that the probability distribution of this random variable is given by the following table:

x	0	1	2	3	4	5
$p(x)$	0.35	0.20	0.15	0.15	0.10	0.05

a. Produce a graph of the probability distribution for this random variable, including all relevant labels.

b. Calculate the expected value of the random variable.

c. Explain how to interpret this expected value in context.

5. The following table lists the number of U.S. households (in thousands) with 0 vehicles, 1 vehicle, 2 vehicles, or 3 or more vehicles for all households responding to the 2009 National Household Travel Survey.

No vehicle	One vehicle	Two vehicles	Three or more vehicles
$9{,}828$	$36{,}509$	$41{,}077$	$25{,}668$

a. Use these data to create a table of relative frequencies that could be used as estimates of the probability distribution of number of vehicles for a randomly selected U.S. household (that responded to the survey).

No vehicle	One vehicle	Two vehicles	Three or more vehicles

b. Suppose you want to examine the distribution of the number of vehicles in all U.S. households. Define a random variable that corresponds to the probability distribution in part (a).

c. Assume for the moment that the last column corresponds to exactly three vehicles. Calculate the expected number of vehicles per household.

d. Now reconsider the last category. Suppose we were to find the information for the actual number of vehicles for these households. Would the expected number of vehicles per household with this new information be larger or smaller than the expected value you found in (c)? Explain your reasoning.

e. Suppose a town has about 4,500 households. What is a good estimate for the number of cars in town? Explain how you determined your answer.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	**STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	**STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	**STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a** S-CP.B.8	Student is not able to calculate a probability.	Student does not set up as a multiplication rule calculation.	Student recognizes the calculation method but substitutes the numbers incorrectly or does not explain solution approach sufficiently.	Student recognizes the conditional probabilities and uses the multiplication rule to combine.
	b S-CP.B.8	Student is not able to calculate a probability.	Student constructs a probability tree or table but is not able to finish it or does not have the correct outcomes.	Student recognizes the calculation method but does not utilize probability of first serve not landing in or does not explain the solution approach sufficiently.	Student recognizes the conditional probabilities and uses the multiplication rule to combine.
	c S-CP.B.8	Student is not able to calculate a probability or assumes 0.50.	Student does not weight probabilities of winning points by probabilities of serves landing in.	Student does not realize (a) and (b) are mutually exclusive events. <u>AND/OR</u> Student attempts a new, incorrect approach.	Student sums answers from (a) and (b).
2	S-CP.B.9	Student is not able to calculate a probability.	Student uses incorrect probability rules, such as $\frac{1}{5}$.	Student fails to consider the 88 nonvenomous species.	Student uses combinations to set up the problem.

3	**a** **S-CP.B.9**	Student is not able to approach problem.	Student calculates a probability.	Student employs multiplication rule but does not allow for repeats or does not take into account number in each category.	Student employs multiplication rule.
	b **S-CP.B.9**	Student is not able to approach problem.	Student calculates a probability.	Student employs multiplication rule but allows for repeats.	Student employs multiplication rule and does not allow repeats (could begin second set at 10 again; could fail to include zero in first set).
	c **S-CP.B.9**	Student does not address the question (e.g., "yes").	Student specifies an answer consistent with the approach but is not able to explain why in terms of order of outcomes.	Student specifies permutations but is not able to explain why. AND/OR Student incorrectly identifies a permutation approach as using combinations.	Student specifics permutations and relates to the distinctness of the outcomes based on order.
4	**a** **S-MD.A.1**	Student does not produce a graph.	Student produces a graph of x values that does not consider $p(x)$ values.	Student produces a graph but does not label both axes.	Student produces a graph of $p(x)$ vs. x with appropriate labels on both axes.
	b **S-MD.A.2**	Student does not use provided information.	Student calculates an average of the x values and does not use $p(x)$ information.	Student does not show work or has a substantial calculation error.	Student calculates expected value correctly and shows work.
	c **S-MD.A.2**	Student provides an incorrect statement about expected value (e.g., most probable value).	Student merely restates a definition/calculation of expected value without clarifying the interpretation.	Student indicates an average but does not sufficiently put statements in context (e.g., no "long-run").	Student interprets expected value as a long-run average in context.
5	**a** **S-MD.A.4**	Student provides values that do not sum to one.	Student does not use the counts provided to create the relative frequencies (e.g., assumes 0.25 for each).	Student creates a table of relative frequencies, but calculation details are unclear.	Student creates a table of relative frequencies, and calculation details are clear.
	b **S-MD.A.1**	Student creates a variable but is not considering probability distributions.	Student does not convey the mapping of outcomes to numbers, giving more of a definition of random outcomes.	Student discusses the 0, 1, 2, 3 outcomes but does not define a random variable.	Student creates a mapping of outcomes to numerical values.

	c S-MD.A.4	Student does not calculate an expected value.	Student calculates an expected value but not using 3 in the last category (e.g., ignores category or uses a number larger than 3).	Student makes a calculation error or does not show sufficient calculation details.	Student correctly calculates an expected value using 3 as the last outcome and shows sufficient calculation details.
	d S-MD.A.3	Student does not relate comments to calculation details of expected value.	Student claims the expected value will be smaller because the probabilities of different outcomes will be smaller.	Student indicates the expected value will be larger but does not have a clear explanation.	Student explains reasoning that larger numbers increase the weighted average.
	e S-MD.A.4	Student provides a response that is not in "number of cars."	Student does not utilize the information of expected value or the given probability distribution (e.g., assumes two cars per household).	Student does not explain solution approach but finds correct answer.	Student multiplies the expected value by the number of households, and calculation details are clear.

Name ______________________________ Date ________________

1. In the game of tennis, one player serves to a second player to start a point. If the server misses landing the first serve (the ball is hit out of bounds and cannot be played), the server is allowed a second serve. Suppose a particular tennis player, Chris, has a 0.62 probability of a first serve landing in bounds and a 0.80 probability of a second serve landing in bounds. Once a serve has landed in bounds, the players take turns hitting the ball back and forth until one player hits the ball out of bounds or into the net. The player who hits the ball out or into the net loses the point, and the other player wins the point. For Chris, when the first serve lands in bounds, he has a 75% chance of winning the point; however, he has only a 22% chance of winning the point on his second serve.

   a. Calculate the probability that Chris lands his first serve in bounds and then goes on to win the point for a randomly selected point.

      P(first serve in)P (wins point | first serve in)= 0.62 (0.75)=0.465

   b. Calculate the probability that Chris misses the first serve, lands the second serve, and then wins the point.

      P(first serve out and second serve in)P(wins point|second serve in)= 0.38(0.80)(0.22) =0.06688.

   c. Calculate the probability that Chris wins a randomly selected point.

      P(first serve in)P(wins point|first serve in)+P(second serve)P(wins point|second serve) = 0.465+0.06688= 0.53188.

2. In Australia, there are approximately 100 species of venomous snakes, but only about 12 have a deadly bite. Suppose a zoo randomly selects one snake from each of five different species to show visitors. What is the probability that exactly one of the five snakes shown is deadly?

$\frac{C(12,\ 1)C(88,4)}{C(100,5)} \approx 0.372.$

3. In California (CA), standard license plates are currently of the form: 1 number–3 letters–3 numbers. Assume the numbers are 0–9 and the letters are A–Z.

   a. In theory, how many different possible standard CA license plates are there, assuming we can repeat letters and numbers?

   $10 \times 26^3 \times 10^3 = 175{,}760{,}000$

   b. How many different possible standard CA license plates are there if we are not allowed to repeat any letters or numbers?

   $10 \times 26 \times 25 \times 24 \times 9 \times 8 \times 7 = 78{,}624{,}000$ (may start second sequence of numbers at 10 again)

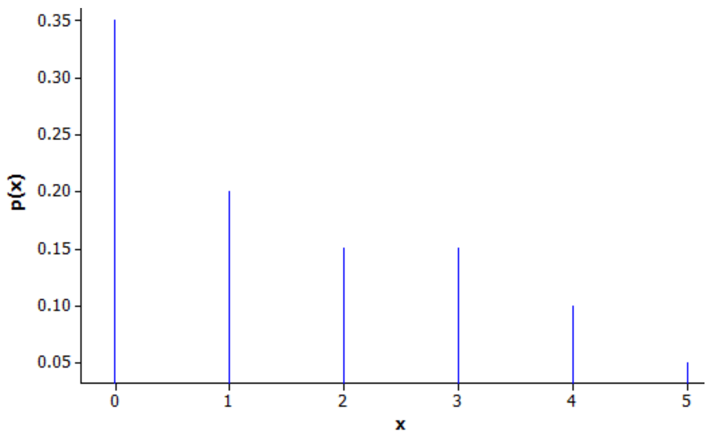
   c. For part (b), did you use permutations or combinations to carry out the calculation? Explain how you know.

   I used permutation because order matters.

4. When there is a problem with a computer program, many people call a technical support center for help. Several people may call at once, so the center needs to be able to have several telephone lines available at the same time. Suppose we want to consider a random variable that is the number of telephone lines in use by the technical support center of a software manufacturer at a particular time of day. Suppose that the probability distribution of this random variable is given by the following table:

x	0	1	2	3	4	5
$p(x)$	0.35	0.20	0.15	0.15	0.10	0.05

a. Produce a graph of the probability distribution for this random variable, including all relevant labels.



b. Calculate the expected value of the random variable.

Expected value = 0(0.35) + 1(0.20) + 2(0.15) + 3(0.15) + 4(0.10) + 5(0.05) = 1.6 telephone lines.

c. Explain how to interpret this expected value in context.

If we were to observe the number of phone lines in use over a very large number of days, the average number of lines in use will approach 1.6 in the long run.

5. The following table lists the number of U.S. households (in thousands) with 0 vehicles, 1 vehicle, 2 vehicles, or 3 or more vehicles for all households responding to the 2009 National Household Travel Survey.

No vehicle	One vehicle	Two vehicles	Three or more vehicles
$9,828$	$36,509$	$41,077$	$25,668$

a. Use these data to create a table of relative frequencies that could be used as estimates of the probability distribution of number of vehicles for a randomly selected U.S. household (that responded to the survey).

No vehicle	One vehicle	Two vehicles	Three or more vehicles	*Total*
0.0869	*0.323*	*0.363*	*0.227*	*113,082 (thousand)*

b. Suppose you want to examine the distribution of the number of vehicles in all U.S. households. Explain how you could represent this as a random variable.

We could let x represent the number of vehicles, so $x = 0, 1, 2, 3, \ldots$.

c. Assume for the moment that the last column corresponds to exactly three vehicles. Calculate the expected number of vehicles per household.

$0(0.0869) + (1)(0.323) + 2(0.363) + 3(0.227) = 1.73$ vehicles.

d. Now reconsider the last category. Suppose we were to find the information for the actual number of vehicles for these households. Would the expected number of vehicles per household with this new information be larger or smaller than the expected value you found in (c)? Explain your reasoning.

This would give us a larger expected value because we would decrease the probability of multiplying 3 vehicles and then add in larger numbers of vehicles.

e. Suppose a town has about 4,500 households. What is a good estimate for the number of cars in town? Explain how you determined your answer.

We expect 1.73 vehicles per household, so 4,500 households would correspond to $1.73 \times 4{,}500 = 7{,}785$ vehicles.

Topic C:

Using Probability to Make Decisions

S-MD.B.5, S-MD.B.6, S-MD.B.7

Focus Standards:	S-MD.B.5	(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*
	S-MD.B.6	(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).
	S-MD.B.7	(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).
Instructional Days:	7	
Lessons 13–14:	Games of Chance and Expected Value (P,P)[1]	
Lesson 15:	Using Expected Values to Compare Strategies (P)	
Lesson 16:	Making Fair Decisions (E)	
Lesson 17:	Fair Games (P)	
Lessons 18–19:	Analyzing Decisions and Strategies Using Probability (P,P)	

Topic C is a capstone topic for this module, where students use what they have learned about probability and expected value to analyze strategies and make decisions in a variety of contexts. They begin by analyzing simple games of chance and relate this to their work in previous lessons on expected value by calculating expected payoff (**S-MD.B.5**).

Students explore the notion of "fairness," both in the context of fair games and in the context of fair decisions (**S-MD.B.6**). In Lesson 16, students are given a scenario in which three basketball players must choose a fair way to determine which of the players keeps a new pair of shoes. Students are presented with three options

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S-**Socratic Lesson

for making the decision. They compare and contrast the methods and come to the conclusion that a decision is fair when one outcome is not favored over another. In Lesson 17, the notion of a fair game is developed as students explore the meaning in the context of a game in which a fee is paid to play. Students come to the conclusion that a game is fair if the fee paid to play is equal to the expected winnings.

In the final lessons of this topic, students are asked to use what they have learned about probability and expected value to make decisions in a variety of contexts (**S-MD.B.7**). Contexts include games of chance, medical testing, life insurance, and product testing. This provides students with a culminating experience that allows them to demonstrate their understanding of probability.

Lesson 13: Games of Chance and Expected Value

Student Outcomes

- Students analyze simple games of chance.
- Students calculate expected payoff for simple games of chance.
- Students interpret expected payoff in context.

Lesson Notes

When students are presented with a complete probability distribution for given random variable (X), they can calculate the *expected value*, $E(X)$, of the random variable. $E(X)$ is obtained from the following formula:

$$E(X) = \sum (\text{Value of } x) \cdot (\text{Probability of that value}) = \sum (X \cdot \text{Probability of } X)$$

This value, $E(X)$, can be interpreted as the long-run average of X over many repeated trials, such as over many rolls of the dice, plays at a lottery game, attempts at a carnival game, etc.

In addition to asking students to compute $E(X)$, this lesson has students examine the context of this value within the framework of payouts for simple games of chance. Specifically, students should recognize that a negative expected value (e.g., $-\$0.25$) states that on average, the player will lose this amount for every attempt at the game. While a player might not lose exactly $\$0.25$ on each game, if the player were to play the game over and over again, in the long run, we would expect the player's total losses to equal $\$0.25$ multiplied by the number of games played. Keep in mind that the actual total winnings (or losses) after a finite series of games will vary from player to player.

The material covered in Lessons 13 and 14 provides an opportunity for students to explore long-run behavior through simulation. Some students may benefit from developing or creating simulations with either manipulatives or technology. For example, on the ducks question below, students could use 10 colored disks (e.g., 6 red, 3 white, and 1 blue) in a bag and simulate the game by selecting a disk (and then returning it). Students could also make a spinning wheel as shown in Problem 2 of the Problem Set. Random number generators in popular and/or public software could also be used.

Classwork

Example 1 (2 minutes): Ducks at the Charity Carnival

MP.3

Have the class read the description of the game in Example 1. Ask students to write a brief response on paper around the following question to compare to Exercise 2 and be ready to share their response with a partner when they begin work on Example 3:

- You have to pay a fee to play a carnival game. What fee would you be willing to pay? Explain.
  - Sample Response: I would pay $\$1.00$ to play the game because I might not win, but the carnival needs the money to cover their costs. A dollar seems appropriate to pay for a couple of minutes of fun. More than $\$1.00$ might be too much, and not many people would play.

Make sure students understand the game before they begin to work on the exercises.

Example 1: Ducks at the Charity Carnival

One game that is popular at some carnivals and amusement parks involves selecting a floating plastic duck at random from a pond full of ducks. In most cases, the letters S, M, or L appear on the bottom of the duck signifying that the winner receives a small, medium, or large prize, respectively. The duck is then returned to the pond for the next game.

Although the prizes are typically toys, crafts, etc., suppose that the monetary values of the prizes are as follows: Small = $\$0.50$, Medium = $\$1.50$, and Large = $\$5.00$.

The probabilities of winning an item on 1 duck selection are as follows: Small 60%, Medium 30%, and Large 10%.

Suppose a person plays the game 4 times. What is the expected monetary value of the prizes won?

Scaffolding:

- For English language learners, consider displaying the following sentence frame on the board at the front of the classroom to aid in writing a response: I would be willing to pay ____ to play a game if the prize was worth ____.
- For advanced learners, provide an opportunity for further exploration around expected value. For instance:
- You want the carnival to make more money in the long run from these games. How would you change the expected value to help make that happen?

Exercises 1–4 (10 minutes)

Students should work in pairs to complete Exercises 1–4.

Exercises 1–4

1. **Let X = the monetary value of the prize that you win playing this game 1 time. Complete the table below and calculate $E(X)$.**

Event	X	Probability of X	$X \cdot$ Probability of X	
Small	$\$0.50$	0.6		
Medium	$\$1.50$	0.3		
Large	$\$5.00$	0.1		
		Sum:		$= E(X)$

Answer:

Event	X	Probability of X	$X \cdot$ Probability of X	
Small	$\$0.50$	0.6	$\$0.30$	
Medium	$\$1.50$	0.3	$\$0.45$	
Large	$\$5.00$	0.1	$\$0.50$	
		Sum:	$\$1.25$	$= E(X)$

2. **Regarding the $E(X)$ value you computed above, can you win exactly that amount on any 1 play of the game?**

   No, you cannot win exactly $\$1.25$ on any 1 play. On average, you are expected to win $\$1.25$ per play if you were to play many times.

3. **What is the least you could win in 4 attempts? What is the most you could win in 4 attempts?**

   The least amount that could be won in 4 attempts is $\$2.00$ ($4 \cdot \$0.50$), and the most is $\$20.00$ ($4 \cdot \$5.00$).

MP.2

4. **How would you explain the $E(X)$ value in context to someone who had never heard of this measurement? What would you *expect* for the total monetary value of your prizes from 4 attempts at this game? (To answer this question, check the introduction to this lesson; do not do anything complicated like developing a tree diagram for all of the outcomes from 4 attempts at the game.)**

   On average, a person is expected to win $\$5.00$ ($4 \cdot E(X)$) for every 4 attempts. The winnings could be as little as $\$2.00$ (4 small) or as large as $\$20.00$ (4 larges). If 4 attempts were made many times, the winnings would average out to be about $\$5.00$.

The question above gets at an important feature of expected value. If the probability distribution remains the same for each trial (also called *independent trials*), you can determine the expected value for N trials by computing the expected value of 1 trial and multiplying that $E(X)$ value by N. You do not have to develop a complicated probability distribution covering all the results of all N trials. This rule makes it far easier to compute expected value for situations where several hundred independent trials occur, such as a carnival, lottery, etc.

Example 2 (2 minutes): Expected Value for Repeated Trials

Read the details of the experiment as a class.

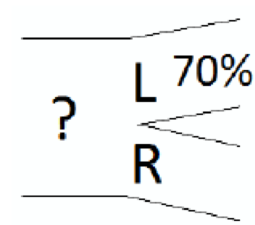
Example 2: Expected Value for Repeated Trials

In a laboratory experiment, 3 mice will be placed in a simple maze one at a time. In the maze, there is 1 decision point where the mouse can turn either left (L) or right (R). When the 1^{st} mouse arrives at the decision point, the direction the mouse chooses is recorded. The same is done for the 2^{nd} and the 3^{rd} mouse. The researchers conducting the experiment add food in the simple maze such that the long-run relative frequency of each mouse turning left is believed to be 0.7 or 70%.

Scaffolding:

For struggling students:

- Have them read the problem aloud and then turn to a partner and summarize, or
- Provide a visual such as the one below to assist in understanding the scenario.



Exercises 5–8 (8 minutes)

Have students work in pairs or small groups to answer Exercises 5–8. When students are finished, discuss the answers to make sure that all students were able to calculate the expected values correctly. Check in with students as they work to ensure quality responses. Each student's level of understanding can be gauged during this time, and misconceptions and errors can be addressed and corrected. Make sure all of the students are prepared to share their responses.

Exercises 5–8

5. Examining the outcomes for just 1 mouse, define the random variable and complete the following table:

Event	X	Probability of X	$X \cdot$ Probability of X	
Left	1	0.7		
Right	0	0.3		
		Sum:		$= E(X)$

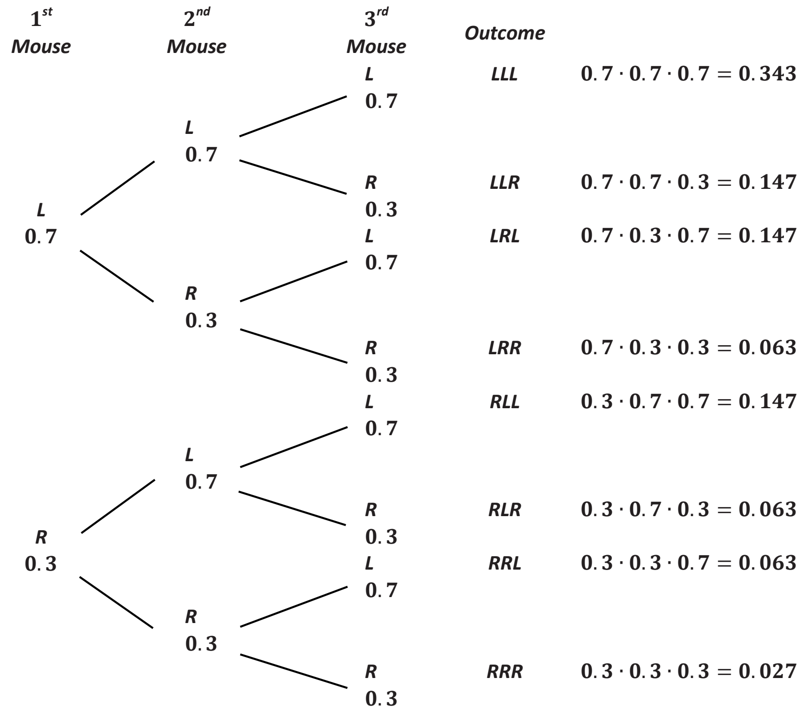
X = the number of left turns made by 1 mouse

Event	X	Probability of X	$X \cdot$ Probability of X	
Left	1	0.7	0.7	
Right	0	0.3	0	
		Sum:	0.7	$= E(X)$

6. Using this value and the rule mentioned above, determine the expected number of left turns for 3 mice. (Remember, the value you compute may not be an exact, attainable value for 1 set of 3 mice. Rather, it is the *average* number of left turns by 3 mice based on many sets of 3 mice.)

The expected value is 2.1 left turns. As stated in the student exercise, the value may not be an exact, attainable value for 1 set of 3 mice. Rather, it is the average number of left turns by 3 mice based on numerous attempts.

The tree diagram below demonstrates the 8 possible outcomes for 3 mice where the first stage of the tree represents the decision made by the 1^{st} mouse and the second stage represents the decision made by the 2^{nd} mouse, and so on.



Scaffolding:

- Tree diagrams were used in Grades 7 and 11. Students new to the curriculum may or may not be familiar with the construction or use of such diagrams. Briefly demonstrate how to construct the tree diagram presented in the text for students who are not familiar with the concept.
- For advanced learners, a more complex situation may be appropriate, such as a maze with three directional choices—right, left, and center.

7. Use the tree diagram to answer the following questions:

   a. Complete the following table and compute $E(Y)$, the expected number of left turns for 3 mice.

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
3 Lefts	3	0.343	
2 Lefts	2		
1 Left	1		
0 Lefts	0	0.027	
		Sum:	$= E(Y)$

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
3 Lefts	3	0.343	1.029
2 Lefts	2	$3 \cdot 0.147 = 0.441$	0.882
1 Left	1	$3 \cdot 0.063 = 0.189$	0.189
0 Lefts	0	0.027	0
		Sum:	2.1 $= E(Y)$

   b. Verify that the expected number of left turns for 3 mice, $E(Y)$, is the same as 3 times the expected number of left turns for 1 mouse, $3 \cdot E(X)$.

   Yes. $E(Y) = 2.1 = 3 \cdot E(X) = 3 \cdot 0.7$

8. Imagine that 200 mice are sent through the maze one at a time. The researchers believed that the probability of a mouse turning left due to the food is 0.7. How many left turns would they expect from 200 mice?

   $200 \cdot 0.7 = 140$ *left turns*

Example 3 (3 minutes): So How Does the Charity Make Money?

Revisit the charity game from Example 1. Students have expressed in writing what cost might justify them playing the game. Have students share their responses with their partners.

Now students have to consider why the charity is holding a carnival and how it will benefit from the game. Have students discuss the following questions with their partners:

- What is the purpose of having a charity carnival?
  - *To raise or earn money for a charitable cause.*
- Consider how much you would pay to play a game. How much should a prize be worth for the charity to earn money?
  - *Answers will vary. To earn money for the charity, the prizes cannot be worth more money than the game brings in. In addition, in the end, all the money brought in needs to be worth more than the time and expenses spent to hold the carnival. I would think in general that prizes should be worth* $1.00 *or less in order for the charity to raise a significant amount of money.*

Read the example as a class and remind them of the carnival game from Example 1.

Example 3: So How Does the Charity Make Money?

Revisiting the charity carnival of Example 1, recall that when selecting a duck, the average monetary value of the prizes you win per game is $\$1.25$. How can the charity running the carnival make any money if it is paying out $\$1.25$ to each player on average for each game?

To address this, in most cases a player must *pay* to play a game, and that is where the charity (or any other group running such a game) would earn its money.

Imagine that the cost to play the game is $\$2.00$. What are the expected net earnings for the charity? What are the expected net winnings for a player?

Scaffolding:

- English language learners may not understand the use of the word *net* in this context.
- Point out that *net earnings* refers to the amount of money the player won in the game, less the amount of money paid to play the game.
- For example, if the player won the $5.00 prize, he would actually have a net earnings of only $3.00 (equal to the $5.00 he wins, less the $2.00 he paid to play).

Exercises 9–13 (12 minutes)

Have students work on Exercises 9–13. Students should continue to work with partners. Informally assess student mastery by circulating and listening to students explain to one another and by looking at their work.

Exercises 9–13

9. Compute a player's net earnings for each of the 3 outcomes: small, medium, and large.

Small $= \$0.50 - \$2.00 = -\$1.50$

Medium $= \$1.50 - \$2.00 = -\$0.50$

Large $= \$5.00 - \$2.00 = +\$3.00$

10. For two of the outcomes, the net earnings result is negative. What does a negative value of net earnings mean in context as far as the player is concerned?

A negative expected value from the player's perspective means a loss.

Scaffolding:

For advanced learners, remove Exercises 9–11 and ask them to answer the following questions:

- What are the expected net winnings for a player?
- How does this relate to the earnings for the charity?

11. Let Y = the *net* amount that you win (or lose) playing the duck game 1 time. Complete the table below and calculate $E(Y)$.

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
Small		0.6	
Medium		0.3	
Large		0.1	
		Sum:	$= E(Y)$

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
Small	$-\$1.50$	0.6	$-\$0.90$
Medium	$-\$0.50$	0.3	$-\$0.15$
Large	$\$3.00$	0.1	$\$0.30$
		Sum:	$-\$0.75$ $= E(Y)$

MP.2

12. How would you explain the $E(Y)$ value in context to someone who had never heard of this measurement? Write a sentence explaining this value from the perspective of a player; then write a sentence explaining this value from the perspective of the charity running the game.

To a player: The game is designed such that if you play it over and over again, the people running the game expect that you will lose $\$0.75$ to them on average per attempt.

To the people running the game: The game is designed such that if people play this game over and over again, we can expect to make about $\$0.75$ on average per attempt.

13. How much money should the charity expect to earn from the game being played 100 times?

$100 \cdot \$0.75 = \$75.00$ *is expected from* 100 *attempts.*

Closing (3 minutes)

- How does a state lottery make money for its state? Thinking back to the duck game, imagine a simple lottery game that pays out $\$100.00$ in 1% of cases (like a scratch-off game). How much would the state expect to make on every 100 players if 99 of those players paid $\$2.00$ to play and won nothing while 1 of the 100 players paid $\$2.00$ and earned a net gain of $\$98.00$? Note: Make a table if you think it will help students.
    - *The state earned* $99 * \$2.00 = \$198.00$ *from the* 99 *losers and lost* $\$98.00$ *for the* 1 *winner. That means the state comes out ahead* $\$100.00$ *from these* 100 *players, or* $\$1.00$ *per ticket sold.*
    - You could also say that the state brought in $\$200.00$ ($\$2.00$ per player) and paid out $\$100.00$ to the 1 winner, thus coming out ahead $\$100.00$.
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

By computing the expected value, $E(X)$, for the earnings from a game of chance, one can determine the expected average payoff per game.

When this value is positive, the player can expect to come out ahead in the long run. However, in most games of chance, this value is negative and represents how much the group operating the game takes in on average per game. From a player's perspective, a negative expected value means that the player is expected to lose that $E(X)$ amount on average with each trial. Businesses and establishments that intend to make money from players, customers, etc., are counting on situations where the player's expected value is negative.

As long as the probabilities remain the same for each instance of a game or trial, you can compute the expected value of N games as N times the expected value of 1 game.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 13: Games of Chance and Expected Value

Exit Ticket

As posted on the Maryland Lottery's website for its Pick 3 game, the chance of winning with a Front Pair bet is 0.01.

A Front Pair bet is successful if the front pair of numbers you select match the Pick 3 number's first 2 digits. For example, a bet of $12X$ would be a winner if the Pick 3 number is $120, 121, 122$, etc. In other words, 10 of the $1{,}000$ possible Pick 3 numbers (1%) would be winners, and thus, the probability of winning is 0.01 or 1%.

A successful bet of $\$0.50$ pays out $\$25.00$ for a net gain to the player of $\$24.50$.

a. Define the random variable X and compute $E(X)$.

b. On average, how much does the Maryland Lottery make on each such bet?

c. Assume that for a given time period, $100{,}000$ bets like the one described above were placed. How much money should the Maryland Lottery Agency expect to earn on average from $100{,}000$ bets?

Note: According to the Maryland Lottery Gaming and Control Agency's Annual Report for Fiscal Year 2012, the Pick 3 game accounted for $\$254.60$ million in net sales. (http://mlgca.com/annual-report/ accessed November 17, 2013)

Exit Ticket Sample Solutions

As posted on the Maryland Lottery's website for its Pick 3 game, the chance of winning with a Front Pair bet is 0.01. (http://mdlottery.com/games/pick-3/payouts/ accessed on November 17, 2013)

A Front Pair bet is successful if the front pair of numbers you select match the Pick 3 number's first 2 digits. For example, a bet of $12X$ would be a winner if the Pick 3 number is $120, 121, 122$, etc. In other words, 10 of the $1,000$ possible Pick 3 numbers (1%) would be winners, and thus, the probability of winning is 0.01 or 1%.

A successful bet of $\$0.50$ pays out $\$25.00$ for a net gain to the player of $\$24.50$.

a. Define the random variable X and compute $E(X)$.

Let X = a player's NET gain or loss from playing 1 game in this manner.

$$E(X) = 0.99 \cdot -\$0.50 + 0.01 \cdot \$24.50 = -\$0.495 + \$0.245 = -\$0.25$$

b. On average, how much does the Maryland Lottery make on each such bet?

The Maryland Lottery makes on average $\$0.25$ for each such bet.

c. Assume that for a given time period, $100,000$ bets like the one described above were placed. How much money should the Maryland Lottery expect to earn on average from $100,000$ bets?

$$100,000 \cdot \$0.25 = \$25,000.00$$

Note: According to the Maryland Lottery Gaming and Control Agency's Annual Report for Fiscal Year 2012, the Pick 3 game accounted for $\$254.60$ million in net sales. (http://mlgca.com/annual-report/ accessed November 17, 2013)

Problem Set Sample Solutions

1. The Maryland Lottery Pick 3 game described in the Exit Ticket has a variety of ways in which a player can bet. Instead of the Front Pair bet of $\$0.50$ described above with a payout of $\$25.00$, a player could make a Front Pair bet of $\$1.00$ on a single ticket for a payout of $\$50.00$.

   Let Y = a player's NET gain or loss from playing 1 game in this manner.

   a. Compute $E(Y)$.

   $E(Y) = 0.99 \cdot -\$1.00 + 0.01 \cdot \$49.00 = -\$0.99 + \$0.49 = -\$0.50$

   b. On average, how much does the Maryland Lottery make on each such bet?

   The Maryland Lottery makes on average $\$0.50$ for each such bet.

   c. Assume that for a given time period, $100,000$ bets like the one described above were placed. How much money should the Maryland Lottery expect to earn on average from $100,000$ bets?

   $100,000 \cdot \$0.50 = \$50,000.00$

   d. Compare your answers to the three questions above with your Exit Ticket answers. How are the answers to these questions and the answers to the Exit Ticket questions related?

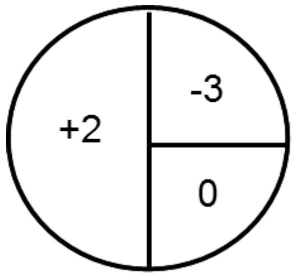
   Each of these answers is double the corresponding answers above.

2. Another type of carnival or arcade game is a spinning wheel game. Imagine that someone playing a spinning wheel game earns points (payoff) as follows for each spin:

- You gain 2 points 50% of the time.
- You lose 3 points 25% of the time.
- You neither gain nor lose any points 25% of the time.

The results of each spin are added to one another, and the object is for a player to accumulate 5 or more points. Negative total point values are possible.

a. Develop a model of a spinning wheel that would reflect the probabilities and point values.



b. Compute $E(X)$ where X = the number of points earned in a given spin.

$E(X) = 2 \cdot 0.50 + (-3) \cdot 0.25 + 0 \cdot 0.25 = 0.25$ points

c. Based on your computation, how many spins on average do you think it might take to reach 5 points?

$\frac{5}{0.25} = 20$. It would take on average 20 spins.

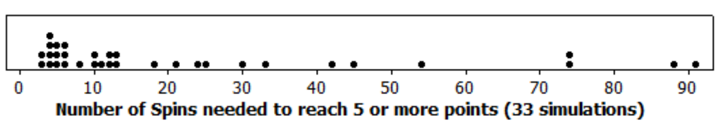
d. Use the spinning wheel you developed in part (a) (or some other randomization device) to take a few spins. See how many spins it takes to reach 5 or more points. Comment on whether this was fewer spins, more spins, or the same number of spins you expected in part (c) above.

Answers will vary. In most cases, a player can reach 5 or more points with very few spins, often far fewer than the average of 20.

e. Let Y = the number of spins needed to reach 5 or more points (like the number of spins it took you to reach 5 points in part (d) above), and repeat the simulation process from part (d) many times. Record on a dot plot the various values of Y you obtain. After several simulations, comment on the distribution of Y.

Answers will vary with students' work.

For parts (d) and (e), 1 simulation distribution (computer generated) of Y = the number of spins needed to reach 5 or more points was as follows:



The distribution is very skewed right, and sometimes it takes many spins to get to 5 or more points. The average number of spins needed based on these 33 simulations was 23.12 spins, close to the $E(Y) = 20$. In most cases, a player can reach 5 or more points with very few spins, often far fewer than the average of 20. (Note: This may be a good chance to remind students that it is important to see the center, shape, and spread of a distribution before drawing conclusions about a variable.)

Lesson 14: Games of Chance and Expected Value

Student Outcomes

- Students use expected payoff to compare strategies for a simple game of chance.

Lesson Notes

This lesson uses examples from the previous lesson as well as some new examples that expand on previous work. The lesson begins with an example in which students informally examine ticket payout distributions for three different games and form a conjecture around which game would be the best to play. Students then use expected value to support or revise their conjecture and answer questions about strategy. Throughout the lesson, students compare presented situations and choose the best course of action based on expected value.

Classwork

Example 1 (5 minutes): Which Game to Play?

MP.1

Explain the context of Example 1. Consider reading the example out loud as needed and using visuals to demonstrate the context of the problem. Make sure that students understand that they are to recommend which game should be played. Students informally examine payout distributions of three different games and conjecture about which game would be the best to play and evaluate their conjecture in the subsequent exercises.

- Now that you have investigated how expected value can be used to determine the average gain (or loss) associated with a random process such as a carnival game, lottery ticket, etc., you will investigate how to use expected value to determine the best strategy for reaching a certain goal. You will also further investigate some of the games listed in the previous lesson.

Example 1: Which Game to Play?

As mentioned in the previous lesson, games of chance are very popular. Some towns, amusement parks, themed restaurants, etc., have arcades that contain several games of chance. In many cases, tickets are awarded as a form of currency so that players can obtain tickets and eventually exchange them for a large prize at a prize center located within the arcade.

Suppose you are asked to give advice to a child who is interested in obtaining a prize that costs $1,000$ tickets. The child can choose from the following three games: a spinning wheel, a fishing game (very similar to the duck pond game described in the previous lesson), and a slot machine–style game with cartoon characters. Again, each of these is a game of chance; no skill is involved. Each game costs $\$0.50$ per play. The child will play only one of these three games but will play the game as many times as it takes to earn $1,000$ tickets.

Below are the ticket payout distributions for the three games:

Spinning Wheel		Fishing Game		Slot Machine	
Tickets	Probability	Tickets	Probability	Tickets	Probability
1	0.51	1	0.50	1	0.850
2	0.35	5	0.20	2	0.070
5	0.07	10	0.15	10	0.060
10	0.04	30	0.14	100	0.019
100	0.03	150	0.01	500	0.001

Which game would you recommend to the child?

Scaffolding:

- For English language learners, consider providing a sentence frame for Exercise 1:

  I think ____ might be the best choice because ____.
- Ask advanced learners to imagine designing their own game and answer the following:
- Describe your game.
- What is the expected payout of your game? Why?

Exercises 1–3 (15 minutes)

In this set of exercises, students use expected value to explore their conjecture about the best game to play. Consider having students work in groups of three on the exercises. That way, they can share the work of calculating the three expected values in Exercise 1.

MP.3

Exercises 1–3

1. **At first glance of the probability distributions of the three games, without performing any calculations, which do you think might be the best choice and why?**

   Answers will vary. Sample response: I think the spinning wheel might be the best choice because the probabilities are relatively high for winning 10 and 100 tickets.

2. **Perform necessary calculations to determine which game to recommend to the child. Explain your choice in terms of both tickets and price. Is this the result you anticipated?**

Spinning Wheel			Fishing Game			Slot Machine		
Tickets	Probability	$X \cdot P(X)$	Tickets	Probability	$X \cdot P(X)$	Tickets	Probability	$X \cdot P(X)$
1	0.51	0.51	1	0.50	0.50	1	0.850	0.85
2	0.35	0.70	5	0.20	1.00	2	0.070	0.14
5	0.07	0.35	10	0.15	1.50	10	0.060	0.60
10	0.04	0.40	30	0.14	4.20	100	0.019	1.90
100	0.03	3.00	150	0.01	1.50	500	0.001	0.50
	Sum = $E(X)$	$= 4.96$ *tickets*		*Sum = $E(X)$*	$= 8.7$ *tickets*		*Sum = $E(X)$*	$= 3.99$ *tickets*

The fishing game would be the best game to play because, on average, the child will earn about 9 tickets for each play—but for the spinning wheel and slot machine games, she will only win about 5 or 4 tickets (respectively, on average) for each play.

The cost of the prize, if won by playing the fishing game, will average $\frac{1000}{8.7} \cdot \$0.50 \approx \$57.47$. If the spinning wheel or slot machine game is played, the average cost of the prize becomes $\frac{1000}{4.96} \cdot \$0.50 \approx \$100.81$ or $\frac{1000}{3.99} \cdot \$0.50 \approx \$125.31$, respectively.

Answers about prediction will vary.

MP.3

3. The child states that she would like to play the slot machine game because it offers a chance of winning 500 tickets per game, and that means she might only have to play twice to reach her goal, and none of the other games offer that possibility. Using both the information from the distributions above and your expected value calculations, explain to her why this might not be the best strategy.

While the child is correct in her statement, the chances of earning 500 tickets with 1 play at the slot machine are very small. The event would happen in the long run only 1 out of $1,000$ tries. The chances of earning $1,000$ tickets in only 2 tries is very small (probability $= 0.001 \cdot 0.001 = 0.000001 = \frac{1}{1{,}000{,}000}$ or 1 in 1 million). Given the rare chance of a high payout and the high chance of a low payout, the slot machine is not the way to go.

With the fishing game, she will earn about 9 tickets on average for each play—but for the spinning wheel and slot machine games, she will earn about half that many tickets on average with each play. That means that she may have to spend nearly twice the amount of money for the same expected number of tickets.

Example 2 (5 minutes): Insurance

Explain the context of Example 2. Make sure that students understand the difference between the two plans. As in the first example of the lesson, students should informally examine the given information and form a conjecture around which plan might be the best option for the company to offer.

Example 2

Insurance companies consider expected value when developing insurance products and determining the pricing structure of these products. From the perspective of the insurance company, the company "gains" each time it earns more money from a customer than it needs to pay out to the customer.

An example of this would be a customer paying a one-time premium (that's the cost of insurance) of $\$500.00$ to purchase a one-year, $\$10,000.00$ casualty policy on an expensive household item that ends up never being damaged, stolen, etc., in that one-year period. In that case, the insurance company gained $\$500.00$ from that transaction.

However, if something catastrophic did happen to the household item during that one-year period (such that it was stolen or damaged so badly that it could not be repaired, etc.), the customer could then ask the insurance company for the $\$10,000.00$ of insurance money per the agreement, and the insurance company would lose $\$9,500.00$ from the transaction.

Imagine that an insurance company is considering offering two coverage plans for two major household items that owners would typically want to insure (or are required by law to insure). Based on market analysis, the company believes that it could sell the policies as follows:

- Plan A: Customer pays a one-year premium of $\$600.00$ and gets $\$10,000.00$ of insurance money if Item A is ever stolen or damaged so badly that it could not be repaired, etc., that year.
- Plan B: Customer pays a one-year premium of $\$900.00$ and gets $\$8,000.00$ of insurance money if Item B is ever stolen or damaged so badly that it could not be repaired, etc., that year.

It is estimated that the chance of the company needing to pay out on a Plan A policy is 0.09%, and the chance of the company needing to pay out on a Plan B policy is 3.71%.

Which plan should the company offer?

Exercise 4 (12 minutes)

In this exercise, students must make sense of the problem and construct a viable plan for supporting their conjecture from Example 2. Students will first need to determine the probability distributions for each plan and then use the distributions to calculate expected value. Let students continue to work in groups of three on Exercise 4.

MP.3

Exercise 4

4. **The company can market and maintain only one of the two policy types, and some people in the company feel it should market Plan B since it earns the higher premium from the customer and has the lower claim payout amount. Assuming that the cost of required resources for the two types of policies is the same (for the advertising, selling, maintaining, etc., of the policies) and that the same number of policies would be sold for either Plan A or Plan B. In terms of earning the most money for the insurance company, do you agree with the Plan B decision? Explain your decision.**

Plan A

Gain	*Probability*	$X \cdot P(X)$
$\$600.00$	0.9991	$\$599.46$
$-\$10,000.00$	0.0009	$-\$9.00$
	Sum = $E(X)$	$= \$590.46$

Plan B

Gain	*Probability*	$X \cdot P(X)$
$\$900.00$	0.9629	$\$866.61$
$-\$8,000.00$	0.0371	$-\$296.80$
	Sum = $E(X)$	$= \$569.81$

The Plan A policy would be better for the company as it has the higher expected value (and it is assumed that the same number of policies would be sold regardless of plan). This means the company will earn more on average for every Plan A policy sold than it would for every Plan B policy sold—specifically, $\$20.65$ ***more per policy sold on average. This difference will really add up as more and more policies are sold. So, even though Plan B has a higher premium and a lower payout, a payout is much more likely to occur with Plan B, making it the less attractive option.***

Closing (3 minutes)

- The "duck pond" game and Maryland Lottery game from the previous lesson both had a negative expected value from the player's perspective. However, these games of chance are very popular. Why do you suppose that games of chance, which have a negative expected value, are still widely played even when people are aware that they will most likely lose money in the long run? (Hint: Think about the child who wanted to play the slot machine game to earn her 1,000 tickets in this lesson.)
  - *The idea of a "big payout" is an appealing idea to most people who play games of chance. This is why lottery sales tend to increase dramatically when the jackpots become larger. Also, effective advertising about a "big payout" whether it is at a carnival or in media may influence people. Many people do not consider the long-run behavior of playing games of chance; rather, they are hopeful that the short-term immediate outcomes of a game will be favorable for them.*
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

The application of expected value is very important to many businesses, lotteries, and others. It helps to determine the average gain or loss that can be expected for a given iteration of a probability trial.

By comparing the expected value, $E(X)$, for different games of chance (or situations that closely mirror games of chance), one can determine the most effective strategy to reach one's goal.

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 14: Games of Chance and Expected Value

Exit Ticket

In the previous lesson, you examined the Maryland Lottery's Pick 3 game where the chance of winning with a Front Pair bet is 0.01. (http://mdlottery.com/games/pick-3/payouts)

In that game, a successful bet of $\$1.00$ pays out $\$50.00$ for a net gain to the player of $\$49.00$.

Imagine that the state also offers a $\$1.00$ scratch-off lottery game with the following net gain distribution:

Scratch-Off Lottery

Net Gain	Probability
$-\$1.00$	0.9600
$\$9.00$	0.0389
$\$99.00$	0.0010
$\$999.00$	0.0001

If you had a friend who wanted to spend $\$1.00$ each day for several days on only 1 of these 2 lottery games, which game would you recommend? Explain.

Exit Ticket Sample Solutions

In the previous lesson, you examined the Maryland Lottery's Pick 3 game where the chance of winning with a Front Pair bet is 0.01. (http://mdlottery.com/games/pick-3/payouts/)

In that game, a successful bet of $\$1.00$ pays out $\$50.00$ for a net gain to the player of $\$49.00$.

Imagine that the state also offers a $\$1.00$ scratch-off lottery game with the following net gain distribution:

Scratch-Off Lottery

Net Gain	Probability
$-\$1.00$	0.9600
$\$9.00$	0.0389
$\$99.00$	0.0010
$\$999.00$	0.0001

If you had a friend who wanted to spend $\$1.00$ each day for several days on only 1 of these 2 lottery games, which game would you recommend? Explain.

For the Maryland Lottery Pick 3, $E(X) = 0.99 \cdot -\$1.00 + 0.01 \cdot \$49.00 = -\$0.99 + \$0.49 = -\$0.50$ (determined in previous lesson).

Scratch-Off Lottery

Net Gain	Probability	$X \cdot P(X)$
$-\$1.00$	0.9600	-0.9600
$\$9.00$	0.0389	0.3501
$\$99.00$	0.0010	0.0990
$\$999.00$	0.0001	0.0999
	Sum $= E(X)$	$= -\$0.411$

So, if a person were going to play a $\$1.00$ game over and over again, he should play the scratch-off game as the player's average loss per game is lower by about $\$0.09$. (Note: Both games have negative expected values; placing the $\$1.00$ in a bank savings account each day is a much better idea in terms of earning money.)

Problem Set Sample Solutions

1. In the previous lesson, a duck pond game was described with the following payout distribution to its players:

Event	Y	Probability of Y
Small	$-\$1.50$	0.60
Medium	$-\$0.50$	0.30
Large	$\$3.00$	0.10

where $Y =$ the *net* amount that a player won (or lost) playing the duck game one time.

This led to a situation where the people running the game could expect to gain $\$0.75$ on average per attempt.

Someone is considering changing the probability distribution as follows:

Event	Y	Probability of Y
Small	$-\$1.50$	0.70
Medium	$-\$0.50$	0.18
Large	$\$3.00$	0.12

Will this adjustment favor the players, favor the game's organizers, or will it make no difference at all in terms of the amount the organization can expect to gain on average per attempt?

The expected value for the adjusted distribution:

Event	Y	Probability of Y	*$Y \cdot$ Probability of Y*
Small	$-\$1.50$	0.70	$-\$1.05$
Medium	$-\$0.50$	0.18	$-\$0.09$
Large	$\$3.00$	0.12	$\$0.36$
		Sum:	$-\$0.78 = E(Y)$

So, the new distribution will favor the game's organizers. On average, the organizers will earn an additional $\$0.03$ *for every game played.*

2. In the previous lesson's Problem Set, you were asked to make a model of a spinning wheel with a point distribution as follows:

- You gain 2 points 50% of the time.
- You lose 3 points 25% of the time.
- You neither gain nor lose any points 25% of the time.

When $X =$ the number of points earned in a given spin, $E(X) = 0.25$ points.

Suppose you change the probabilities by "moving" 10% of the distribution as follows:

- You gain 2 points 60% of the time.
- You lose 3 points 15% of the time.
- You neither gain nor lose any points 25% of the time.

a. **Without performing any calculations, make a guess as to whether or not this new distribution will lead to a player needing a fewer number of attempts than before on average to attain 5 or more points. Explain your reasoning.**

Since the chance of winning 2 points is higher, the chance of losing 3 points is lower, and the chance of earning 0 points is unchanged, that probably means that a person can reach 5 points more quickly than before.

b. **Determine the expected value of points earned from 1 game based on this new distribution. Based on your computation, how many spins on average do you think it might take to reach 5 points?**

$E(X) = 2 \cdot 0.60 + (-3) \cdot 0.15 + 0 \cdot 0.25 = 0.75$ *points. The expected value tripled. Now it is expected to only take* $\frac{5}{0.75} = 6.67$ *(about 7) spins on average to reach 5 points.*

c. **Does this value from part (b) support your guess in part (a)? (Remember that with the original distribution and its expected value of 0.25 points per play, it would have taken 20 spins on average to reach 5 points.)**

Yes. The new distribution will only require 6.67 spins on average to reach 5 points. That is far fewer than the previous expected number of spins (20) needed on average to reach 5 points. This is in line with our conjecture back in part (a) that a person would be able to reach 5 points more quickly than before.

3. **You decide to invest $\$1,000.00$ in the stock market. After researching, you estimate the following probabilities:**
   - **Stock A has a 73% chance of earning a 20% profit in 1 year, an 11% chance of earning no profit, and a 16% chance of being worthless.**
   - **Stock B has a 54% chance of earning a 75% profit in 1 year, a 23% chance of earning no profit, and a 23% chance of being worthless.**

a. **At first glance, which seems to be the most appealing?**

Answers vary. Sample response: Stock A seems more appealing because there are lower probabilities of making no profit and losing the entire $\$1,000.00$ investment.

b. **Which stock should you decide to invest in and why? Is this what you predicted?**

Let X = the value of the investment in Stock A after 1 year.

Event	X	$P(X)$	$X \cdot P(X)$
20% Profit	$\$1,200.00$	0.73	$\$876.00$
No Profit	$\$1,000.00$	0.11	$\$110.00$
Worthless	$-\$1,000.00$	0.16	$-\$160.00$
		Sum:	$\$826.00 = E(X)$

Let Y = the value of the investment in Stock B after 1 year.

Event	Y	$P(Y)$	$Y \cdot P(Y)$
75% Profit	$\$1,750.00$	0.54	$\$945.00$
No Profit	$\$1,000.00$	0.23	$\$230.00$
Worthless	$-\$1,000.00$	0.23	$-\$230.00$
		Sum:	$\$945.00 = E(Y)$

I should invest in Stock B because the total expected value of my investment is $\$945.00$, whereas Stock B is only expected to yield $\$826.00$. Stock B is riskier than Stock A, but the expected value is greater.

4. The clock is winding down in the fourth quarter of the basketball game. The scores are close. It is still anyone's game. As the team's coach, you need to quickly decide which player to put on the court to help ensure your team's success. Luckily, you have the historical data for Player A and Player B in front of you.

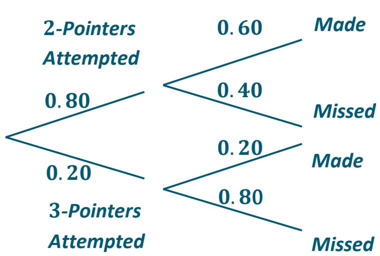
- 80% of Player A's shot attempts have been 2-point field goals, and 60% of them have hit their marks. The remaining shots have been 3-point field goals, and 20% of them have hit their marks.
- 85% of Player B's shot attempts have been 2-point field goals, and 62% of them have hit their marks. The remaining shots have been 3-point field goals, and 22% of them have hit their marks.

a. At first glance, whom would you put in and why?

Answers vary.

b. Based on these statistics, which player might be more likely to help lead your team to victory and why?

Player A Tree Diagram



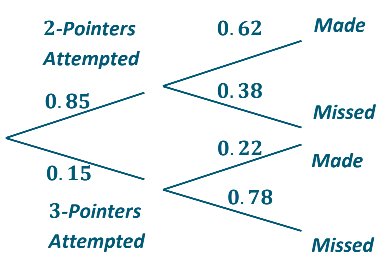
Percent of shots that were 2-pointers and successful $= 0.48$

Percent of shots that were 2-pointers and unsuccessful $= 0.32$

Percent of shots that were 3-pointers and successful $= 0.04$

Percent of shots that were 3-pointers and unsuccessful $= 0.16$

Player B Tree Diagram



Percent of shots that were 2-pointers and successful $= 0.53$

Percent of shots that were 2-pointers and unsuccessful $= 0.32$

Percent of shots that were 3-pointers and successful $= 0.03$

Percent of shots that were 3-pointers and unsuccessful $= 0.12$

Let X = Player A's points per shot.

Event	X	$P(X)$	$X \cdot P(X)$
2-Pointer	2	0.48	0.96
3-Pointer	3	0.04	0.12
Miss	0	0.48	0
		Sum:	$1.08 = E(X)$

Let Y = Player B's points per shot.

Event	Y	$P(Y)$	$Y \cdot P(Y)$
2-Pointer	2	0.53	1.06
3-Pointer	3	0.03	0.09
Miss	0	0.44	0
		Sum:	$1.15 = E(Y)$

From the analysis, it appears as though Player B might be able to score slightly more points than Player A with 1.15 points per shot versus 1.08. However, both players have very close expected points per shot, so either would be a viable candidate.

5. Prior versions of College Board examinations (SAT, AP) awarded the test taker with 1 point for each correct answer and deducted $\frac{1}{4}$ point for each incorrect answer. Current versions have eliminated the point deduction for incorrect responses (test takers are awarded 0 points).

The math section of the SAT contains 44 multiple-choice questions, with choices A–E. Suppose you answer all the questions but end up guessing on 8 questions. How might your math score look different on your score report using each point system? Explain your answer.

Let X = points per question with deduction.

Event	X	$P(X)$	$X \cdot P(X)$
Correct	1	0.2	0.2
Incorrect	-0.25	0.8	-0.2
		Sum:	$0 = E(X)$

Let Y = points per question without deduction.

Event	Y	$P(Y)$	$Y \cdot P(Y)$
Correct	1	0.2	0.2
Incorrect	0	0.8	0
		Sum:	$0.2 = E(Y)$

With the deduction system, a test taker would score an expected 0 points $(0 \cdot 8)$ on the 8 guessed questions, while on the nondeduction system, a test taker would score an expected 1.6 points $(0.2 \cdot 8)$ on the 8 guessed questions.

Lesson 15: Using Expected Values to Compare Strategies

Student Outcomes

- Students calculate expected values.
- Students make rational decisions based on calculated expected values.

Lesson Notes

In previous lessons, students analyzed and interpreted simple games of chance by calculating expected payoff. Further, they used expected payoff to compare strategies for such games. This lesson extends the use of expected value to comparing strategies in a variety of contexts.

Classwork

Example 1 (2 minutes)

This example gives three probability distributions. Ask your students to verify that they are probability distributions (i.e., in each case, the probabilities add to 1). Explain that the club is going to sell only one product this year. Then ask students to share with their neighbor which product they think the club should sell based on an initial examination of the probability distributions.

MP.3

Scaffolding:

For advanced learners, consider combining Exercises 1 and 2 and asking students to determine which product should be offered.

- The club advisors only want to offer one product this year. They have decided to let the club members choose which product to offer and shared the records from past years. The overhead costs are $\$80.00$ for candy, $\$20.00$ for magazine subscriptions, and $\$40.00$ for wrapping paper. Assuming that these probability distributions were to hold for the coming fundraiser, which product should the club members recommend they sell? Explain.

For struggling students, consider presenting only two products as possible options for the fundraiser.

- The math club must choose between selling candy and magazine subscriptions. Which product should the members recommend?

Example 1

A math club has been conducting an annual fundraiser for many years that involves selling products. The club advisors have kept records of revenue for the products that they have made and sold over the years and have constructed the following probability distributions for the three most popular products. (Revenue has been rounded to the nearest hundred dollars.)

Candy			Magazine Subscriptions			Wrapping Paper	
Revenue	Probability		Revenue	Probability		Revenue	Probability
$\$100.00$	0.10						
$\$200.00$	0.10		$\$200.00$	0.4			
$\$300.00$	0.25		$\$300.00$	0.4		$\$300.00$	1.0
$\$400.00$	0.45		$\$400.00$	0.2			
$\$500.00$	0.05						
$\$600.00$	0.05						

Exercises 1–2 (5 minutes)

Have students work with a partner or in a small group on Exercises 1 and 2. As students complete the exercises, rotate around the room to informally assess their work. Then discuss and confirm answers as a class.

Exercises 1–2

1. **The club advisors only want to offer one product this year. They have decided to let the club members choose which product to offer and have shared the records from past years. Assuming that these probability distributions were to hold for the coming fundraiser, which product should the club members recommend they sell? Explain.**

   The expected revenues for the three products are

   Candy
   $(100)(0.10) + (200)(0.10) + (300)(0.25) + (400)(0.45) + (500)(0.05) + (600)(0.05) = \$340.00.$

   (Be sure your students include the units and dollars, and not just the raw number 340*.)*

   Magazine subscriptions
   $(200)(0.4) + (300)(0.4) + (400)(0.2) = \$280.00.$

   Wrapping paper
   $(300)(1) = \$300.00.$

   Because the expected revenue is the highest for candy, it is recommended that the math club sell this product.

2. **The club advisors forgot to include overhead costs with the past revenue data. The overhead costs are $\$80.00$ for candy, $\$20.00$ for magazine subscriptions, and $\$40.00$ for wrapping paper. Will this additional information change the product that the math club members recommend they sell? Why?**

   The math club's decision should not be based solely on expected revenue. Expenses have been incurred in producing the product. Such expenses are referred to as overhead costs. The result of revenue minus overhead cost yields profit. Based on expected profit, the math club can randomly choose one of the products since they all yield the same expected profit.

   The expected profit for producing candy is $\$340.00 - \$80.00 = \$260.00.$

   The expected profit for producing magazine subscriptions is
   $\$280.00 - \$20.00 = \$260.00.$

   The expected profit for producing wrapping paper is $\$300.00 - \$40.00 = \$260.00.$

Scaffolding:

For students who are struggling or for English language learners, consider displaying the following on the front board of the classroom to help students answer Exercise 2.

- *Overhead costs* are expenses that have been incurred to produce a product.
- The profit from the fundraiser is the result of the revenue minus the overhead costs:
  Profit = revenue – overhead.

Example 2 (2 minutes)

The game described in this example is a very simple version of a real game on television called *Deal or No Deal*. You may want to demonstrate the game by putting four boxes on a front table in your classroom and have one of your students play the role of wanting to buy a $7,500.00 pre-owned car. This example and the following exercises provide students with an opportunity to practice making decisions and justifying their decisions using expected value.

Example 2

A game on television has the following rules. There are four identical boxes on a table. One box contains $\$1.00$, the second, $\$15.00$, the third, $\$15,000.00$; and the fourth, $\$40,000.00$. You are offered the choice between taking $\$5,000.00$ or taking the amount of money in one of the boxes you choose at random.

Exercises 3–4 (5 minutes)

MP.3

Have students work with a partner to complete the exercises. After students have had time to think about the questions in Exercises 3 and 4, discuss the answers as a class. In both Exercise 3 and 4, students should use expected value to support their strategy. Emphasize that there is not a right answer to these questions. Teachers should look for students to justify their answers using expected value.

Exercises 3–4

3. Suppose that you want to buy a $\$7,500.00$ pre-owned car. What should you do? Take the $\$5,000.00$ or choose a box? Why?

Answers will vary. Some students may say to take the $\$5,000.00$ since it is guarenteed money rather than potential money. But what if there is no way to earn the remaining $\$2,500.00$? Hopefully, a student will argue that a decision should be made based on the expected winning by playing the game. By choosing a box at random, the expected winning is $(1)\left(\frac{1}{4}\right)+(15)\left(\frac{1}{4}\right)+(15,000)\left(\frac{1}{4}\right)+(40,000)\left(\frac{1}{4}\right)=\$13,754.00$. That would be enough to buy the car, plus insurance and gas for an extended period of time.

4. What should you do if you want to buy a $\$20,000.00$ brand-new car? Take the $\$5,000.00$ or choose a box? Why?

Just as in Exercise 3, there really isn't a right or wrong answer to the question posed in Exercise 4. But it will be interesting to hear students' arguments on both sides.

Example 3 (3 minutes)

Read through the example as a class. Chess is an example of a game for which this example could apply. This example differs from the others in this lesson as students must determine several different probability distributions based upon the different combination of strategies for games 1 and 2.

Note that you may have to review the multiplication rule that says that the probability of the intersection of two independent events is the product of the event probabilities. Henry chooses a strategy for game 2 independently of his choice for game 1 and whether he won, tied, or lost game 1. So probabilities are being multiplied in the following calculations.

Example 3

In a certain two-player game, players accumulate points. One point is earned for a win, half a point is earned for a tie, and zero points are earned for a loss. A match consists of two games. There are two different approaches for how the game can be played—boldly (B) or conservatively (C). Before a match, Henry wants to determine whether to play both games boldly (BB), one game boldly and one game conservatively (BC or CB), or both games conservatively (CC). Based on years of experience, he has determined the following probabilities for a win, a tie, or a loss depending on whether he plays boldly or conservatively.

Approach	Win (W)	Tie (T)	Lose (L)
Bold (B)	0.45	0.0	0.55
Conservative (C)	0.0	0.8	0.2

How should Henry play to maximize the expected number of points earned in the match? The following exercises will help you answer this question.

Exercises 5–14 (20 minutes)

Have students work together in small groups to complete the following exercises. To be sure that all students understand how points are earned, work through Exercise 5 as a class.

MP.3

For Exercises 6–13, there are four possible strategies: play boldly in both games (BB), play conservatively in both games (CC), play boldly in the first game and conservatively in the second (BC), play conservatively in the first game and boldly in the second (CB). You may want to assign one strategy per group of students. As students work through the exercises, rotate around the room to be sure that their tables are correct. Once the expected number of points earned in the match are computed for each strategy, bring the class together and discuss answers to Exercise 14 where students are asked to make a recommendation to Henry on how he should play. Students may have initially thought that playing boldly in both games would yield a larger expected number of points, when in fact the number is fairly close between all of the strategies. Their answers to Exercise 14 should include expected value as rationale for choosing the best strategy.

Exercises 5–14

5. **What are the possible values for the total points Henry can earn in a match? For example, he can earn $1\frac{1}{2}$ points by WT or TW (he wins the first game and ties the second game or ties the first game and wins the second game). What are the other possible values?**

The possible numbers of points earned in a match are the entries in the following matrix. The possibilities are 0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2 as shown in the following table.

		Game 2		
		Win (1)	Tie $\left(\frac{1}{2}\right)$	Lose (0)
Game 1	Win (1)	2	$1\frac{1}{2}$	1
	Tie $\left(\frac{1}{2}\right)$	$1\frac{1}{2}$	1	$\frac{1}{2}$
	Lose (0)	1	$\frac{1}{2}$	0

6. If Henry plays both games boldly (BB), find the probability that Henry will earn
   a. 2 points 0.2025
   b. $1\frac{1}{2}$ points 0.0
   c. 1 point 0.495
   d. $\frac{1}{2}$ point 0.0
   e. 0 points 0.3025

If Henry plays both games boldly, the following table gives the possible points earned and the probabilities (printed below the points).

		Game 2 BOLD Win (1) 0.45	Tie $\left(\frac{1}{2}\right)$ 0.0	Lose (0) 0.55
Game 1 BOLD	*Win* (1) 0.45	2 $(0.45)(0.45) = 0.2025$	$1\frac{1}{2}$ 0.0	1 $(0.45)(0.55) = 0.2475$
	Tie $\left(\frac{1}{2}\right)$ 0.0	$1\frac{1}{2}$ 0.0	1 0.0	$\frac{1}{2}$ 0.0
	Lose (0) 0.55	1 $(0.55)(0.45) = 0.2475$	$\frac{1}{2}$ 0.0	0 $(0.55)(0.55) = 0.3025$

7. What is the expected number of points that Henry will earn if he plays using a BB strategy?

Henry's expected number of points won playing BB $= (2)(0.2025) + (1)(0.2475) + (1)(0.2475) + (0)(0.3025) = 0.9$ *points. (Be sure your students include the units.)*

8. If Henry plays both games conservatively (CC), find the probability that Henry will earn
   a. 2 points 0.0
   b. $1\frac{1}{2}$ points 0.0
   c. 1 point 0.64
   d. $\frac{1}{2}$ point 0.32
   e. 0 points 0.04

If Henry plays both games conservatively, the following table gives the possible points earned and the probabilities (printed below the points).

		Game 2 CONSERVATIVE Win (1) 0.0	Tie $\left(\frac{1}{2}\right)$ 0.8	Lose (0) 0.2
Game 1 CONS.	*Win* (1) 0.0	2 0.0	$1\frac{1}{2}$ 0.0	1 0.0
	Tie $\left(\frac{1}{2}\right)$ 0.8	$1\frac{1}{2}$ 0.0	1 $(0.8)(0.8) = 0.64$	$\frac{1}{2}$ $(0.8)(0.2) = 0.16$
	Lose (0) 0.2	1 0.0	$\frac{1}{2}$ $(0.2)(0.8) = 0.16$	0 $(0.2)(0.2) = 0.04$

9. **What is the expected number of points that Henry will earn if he plays using a CC strategy?**

   Henry's expected number of points won playing CC $= \mathbf{(1)(0.64) + \left(\frac{1}{2}\right)(0.16) + \left(\frac{1}{2}\right)(0.16) + (0)(0.04) = 0.8}$ ***points.***

10. **If Henry plays the first game boldly and the second game conservatively (BC), find the probability that Henry will earn**

    a. 2 points $\mathbf{0.0}$

    b. $1\frac{1}{2}$ points $\mathbf{0.36}$

    c. 1 point $\mathbf{0.09}$

    d. $\frac{1}{2}$ point $\mathbf{0.44}$

    e. 0 points $\mathbf{0.11}$

    If Henry plays the first game boldly and the second game conservatively, the following table gives the possible points earned and the probabilities (printed below the points).

		Game 2 *CONSERVATIVE*		
		Win (1) 0.0	*Tie* $\left(\frac{1}{2}\right)$ 0.8	*Lose* (0) 0.2
Game 1 *BOLD*	*Win* (1) 0.45	2 0.0	$1\frac{1}{2}$ $(0.45)(0.8) = 0.36$	1 $(0.45)(0.2) = 0.09$
	Tie $\left(\frac{1}{2}\right)$ 0.0	$1\frac{1}{2}$ 0.0	1 0.0	$\frac{1}{2}$ 0.0
	Lose (0) 0.55	1 0.0	$\frac{1}{2}$ $(0.55)(0.8) = 0.44$	0 $(0.55)(0.2) = 0.11$

11. **What is the expected number of points that Henry will earn if he plays using a BC strategy?**

    Henry's expected number of points won playing BC $= \mathbf{\left(1\frac{1}{2}\right)(0.36) + (1)(0.09) + \left(\frac{1}{2}\right)(0.44) + (0)(0.11) = 0.85}$ ***points.***

12. **If Henry plays the first game conservatively and the second game boldly (CB), find the probability that Henry will earn**

a. 2 points 0.0

b. $1\frac{1}{2}$ points 0.36

c. 1 point 0.09

d. $\frac{1}{2}$ point 0.44

e. 0 points 0.11

Note: Some students may see the symmetry between BC and CB and conclude without calculation that CB will have an expected number of points equal to 0.85 points.

If Henry plays the first game conservatively and the second game boldly, the following table gives the possible points earned and the probabilities (printed below the points).

		Game 2 BOLD		
		Win (1) 0.45	Tie $\left(\frac{1}{2}\right)$ 0.0	Lose (0) 0.55
Game 1 CONS	Win (1) 0.0	2 0.0	$1\frac{1}{2}$ 0.0	1 0.0
	Tie $\left(\frac{1}{2}\right)$ 0.8	$1\frac{1}{2}$ $(0.8)(0.45) = 0.36$	1 0.0	$\frac{1}{2}$ $(0.8)(0.55) = 0.44$
	Lose (0) 0.2	1 $(0.2)(0.45) = 0.09$	$\frac{1}{2}$ 0.0	0 $(0.2)(0.55) = 0.11$

13. **What is the expected number of points that Henry will earn if he plays using a CB strategy?**

Henry's expected number of points won playing CB $= \left(1\frac{1}{2}\right)(0.36) + (1)(0.09) + \left(\frac{1}{2}\right)(0.44) + (0)(0.11) =$ 0.85 *points.*

14. **Of the four possible strategies, which should Henry play in order to maximize his expected number of points earned in a match?**

Henry's best strategy is to play both games boldly (expected points = 0.9). Playing one game boldly and the other conservatively yields an expected 0.85 points, whereas playing both games conservatively is the least desirable strategy, 0.8 points.

Closing (2 minutes)

- Explain how to use expected value to make decisions.
  - *Sample response. Expected value can be used to determine the best option for maximizing or minimizing an objective.*
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

- **Making decisions in the face of uncertainty is one of the primary uses of statistics.**
- **Expected value can be used as one way to decide which of two or more alternatives is best for either maximizing or mininizing an objective.**

Exit Ticket (6 minutes)

Name ____ Date ____________________

Lesson 15: Using Expected Values to Compare Strategies

Exit Ticket

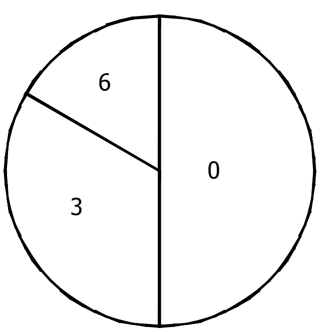
1. Your older sister asks you which of two summer job opportunities she should take. She likes them both. Job A is self-employed; Job B works with a friend. The probability distributions for the amount of money that can be earned per day, X, follow. In Job B, money earned will be split evenly between your sister and her friend.

Self-Employed Job A	
X	Probability
20	0.2
30	0.4
35	0.3
45	0.1

With Friend Job B	
X	Probability
50	0.3
75	0.6
100	0.1

Which opportunity would you recommend that your sister pursue? Explain why in terms of expected value.

2. A carnival game consists of choosing to spin the following spinner once or roll a pair of fair number cubes once. If the spinner lands on 0, you get no points; if it lands on 3, you get 3 points; if it lands on 6, you get 6 points. If the two number cubes sum to a prime number, you get 4 points. If the sum is not a prime number, you get 0 points. Should you spin the spinner or choose the number cubes if you want to maximize the expected number of points obtained? Explain why or why not. Note: The spinner is broken up into wedges representing $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$.



Exit Ticket Sample Solutions

1. **Your older sister asks you which of two summer job opportunities she should take. She likes them both. Job A is self-employed; Job B works with a friend. The probability distributions for the amount of money that can be earned per day, X, follow. In Job B, money earned will be split evenly between your sister and her friend.**

Self-Employed Job A	
X	Probability
20	0.2
30	0.4
35	0.3
45	0.1

With Friend Job B	
X	Probability
50	0.3
75	0.6
100	0.1

Which opportunity would you recommend that your sister pursue? Explain why in terms of expected value.

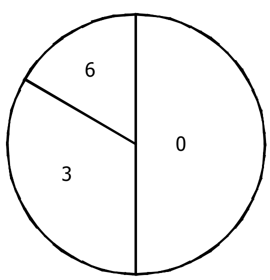
(Be sure your students first check that the probabilities add to 1 in each case.) Since your sister has no preference regarding the jobs, your recommendation should be based on expected earnings.

$E(\text{earnings for Job A}) = (20)(0.2) + (30)(0.4) + (35)(0.3) + (45)(0.1) = \$31.00$ *earnings per day*

$E(\text{earnings for Job B}) = (50)(0.3) + (75)(0.6) + (100)(0.1) = \$70.00$ *earnings per day for two people, so* $\$35.00$ *per day for each*

Recommend Job B because the expected earnings per day are $\$35.00$*, which is more than the expected earnings for Job A.*

2. **A carnival game consists of choosing to spin the following spinner once or roll a pair of fair number cubes once. If the spinner lands on 0, you get no points; if it lands on 3, you get 3 points; if it lands on 6, you get 6 points. If the two number cubes sum to a prime number, you get 4 points. If the sum is not a prime number, you get 0 points. Should you spin the spinner or choose the number cubes if you want to maximize the expected number of points obtained? Explain why or why not. Note: The spinner is broken up into wedges representing $\frac{1}{2}, \frac{1}{3}$, and $\frac{1}{6}$.**



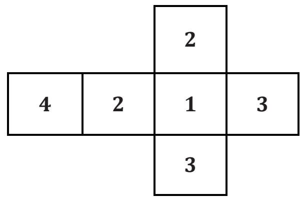
The expected number of points spinning the spinner is $(0)\left(\frac{1}{2}\right) + (3)\left(\frac{2}{6}\right) + (6)\left(\frac{1}{6}\right) = 2$ *points.*

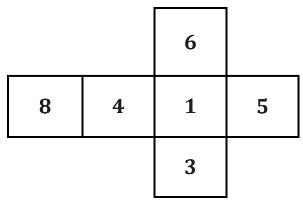
The expected number of points rolling the fair number cubes is $(4)\left(\frac{15}{36}\right) = 1\frac{2}{3}$ *points.*

Spin the spinner because the expected number of points from spinning the spinner is slightly higher than the expected number of points from rolling the fair number cubes.

Problem Set Sample Solutions

1. A game allows you to choose what number cubes you would like to use to play. One pair of number cubes is a regular pair in which the sides of each cube are numbered from 1 to 6. The other pair consists of two different cubes as shown below. For all of these number cubes, it is equally likely that the cube will land on any one of its six sides.





a. Suppose that you want to maximize the expected sum per roll in the long run. Which pair of number cubes should you use? Explain why.

For the regular fair number cubes, the expected sum per roll is $(2)\left(\frac{1}{36}\right)+(3)\left(\frac{2}{36}\right)+(4)\left(\frac{3}{36}\right)+(5)\left(\frac{4}{36}\right)+(6)\left(\frac{5}{36}\right)+(7)\left(\frac{6}{36}\right)+(8)\left(\frac{5}{36}\right)+(9)\left(\frac{4}{36}\right)+(10)\left(\frac{3}{36}\right)+(11)\left(\frac{2}{36}\right)+(12)\left(\frac{1}{36}\right)=7$.

For the Sicherman number cubes, first note that the distribution of the sum is (surprisingly) the same as that for the regular number cubes. The expected sum per roll is 7.

	1	3	4	5	6	8
1	2	4	5	6	7	9
2	3	5	6	7	8	10
2	3	5	6	7	8	10
3	4	6	7	8	9	11
3	4	6	7	8	9	11
4	5	7	8	9	10	12

Since the expected sum per roll is 7 *for each pair of cubes, you could choose either pair of number cubes.*

b. Imagine that you are playing a game in which you earn special privileges by rolling doubles (i.e., the same number on both cubes). Which number cubes would you prefer to use? Explain.

Students should then be able to suggest that the regular number cubes have 6 *possible doubles, but the Sicherman number cubes have only* 4*. So, for games that have special rules for rolling doubles, the regular number cubes are preferable despite the fact that the expected sum per roll is the same for each type of cube.*

Note on part (b): Whereas it doesn't matter which pair of number cubes is used if the game's rules were based strictly on the sum with no added conditions, ask your students if it would matter which pair were used to play Monopoly. You may have to tell students that in the rules of Monopoly there are special moves that can be made if the roll results in a pair (called rolling doubles). Students should then be able to suggest that the regular number cubes have 6 possible pairs but the Sicherman number cubes have only 4. So, for games that have special rules for pairs, the regular number cubes are preferable despite the fact that the expected sum per roll is the same.

2. Amy is a wedding planner. Some of her clients care about whether the wedding is held indoors or outdoors depending on weather conditions as well as respective costs. Over the years, Amy has compiled the following data for June weddings. (Costs are in thousands of dollars.)

Weather	Cost Indoors	Cost Outdoors	Probability
Cold and sunny	$\$29$	$\$33$	0.15
Cold and rainy	$\$30$	$\$40$	0.05
Warm and sunny	$\$22$	$\$27$	0.45
Warm and rainy	$\$24$	$\$30$	0.35

a. What is the expected cost of a June wedding held indoors?

Note: Be sure your students check that the probabilities add to 1.

The expected cost of a June wedding held indoors is $(29)(0.15) + (30)(0.05) + (22)(0.45) + (24)(0.35) = 24.15$ thousands of dollars, i.e., $\$24,150.00$.

b. What is the expected cost of a June wedding held outdoors?

The expected cost of a June wedding held outdoors is $(33)(0.15) + (40)(0.05) + (27)(0.45) + (30)(0.35) = 29.60$ thousands of dollars, i.e., $\$29,600.00$.

c. A new client has her heart set on an outdoor wedding. She has at most $\$25,000.00$ available. What do you think Amy told the client and why?

Amy would have told her new client that she has enough money to cover an indoor wedding, as the expected cost is $\$24,150.00$. However, no matter the weather conditions, the client will not be able to afford an outdoor wedding. The expected cost of an outdoor wedding (regardless of weather) is $\$29,600.00$, so the client's budget falls $\$4,600.00$ short.

3. **A venture capitalist is considering two investment proposals. One proposal involves investing $\$100,000.00$ in a green alternative energy source. The probability that it will succeed is only 0.05, but the gain on investment would be $\$2,500,000.00$. The other proposal involves investing $\$300,000.00$ in an existing textile company. The probability that it will succeed is 0.5, and the gain on investment would be $\$725,000.00$. In which proposal should the venture capitalist invest? Explain.**

The following are probability distributions for each proposal:

Green Energy	
Gain	*Probability*
$\$0.00$	0.95
$\$2,500,000.00$	0.05

Textile Company	
Gain	*Probability*
$\$0.00$	0.5
$\$725,000.00$	0.5

The expected gain for each proposal is as follows:

Green Energy: $0(0.95) + 2,500,000(0.05) = \$125,000.00$

Textile Company: $0(0.5) + 725,000(0.5) = \$362,500.00$

The venture capitalist's expected profit can be found by subtracting the amount invested from the expected gain.

The expected profit from the green energy proposal is $\$125,000.00 - \$100,000.00$*, or* $\$25,000.00$*.*

The expected profit from the textile proposal is $\$362,500.00 - \$300,000.00$*, or* $\$62,500.00$*.*

The venture capitalist should invest in the textile company because the expected profit is higher.

4. **A student is required to purchase injury insurance in order to participate on his high school football team. The insurance will cover all expenses incurred if the student is injured during a football practice or game, but the student must pay a deductible for submitting a claim. There is also an up-front cost to purchase the injury insurance.**
   - **Plan A costs $\$75.00$ up front. If the student is injured and files a claim, the deductible is $\$100.00$.**
   - **Plan B costs $\$100.00$ up front. If the student is injured and files a claim, the deductible is $\$50.00$.**

   Suppose there is a 1 in 5 chance of the student making a claim on the insurance policy. Which plan should the student choose? Explain.

The probability distributions for each plan are as follows:

Plan A	
Deductible	*Probability*
$\$0.00$	$\frac{4}{5}$
$\$100.00$	$\frac{1}{5}$

Plan B	
Deductible	*Probability*
$\$0.00$	$\frac{4}{5}$
$\$50.00$	$\frac{1}{5}$

The expected cost of submitting a claim for each plan is as follows:

Plan A: $0\left(\frac{4}{5}\right) + 100\left(\frac{1}{5}\right) = \$20.00$

Plan B: $0\left(\frac{4}{5}\right) + 50\left(\frac{1}{5}\right) = \$10.00$

Because there is also an up-front cost to purchase the insurance policy, the up-front cost should be combined with the expected cost of submitting a claim to determine the total out-of-pocket expense for the student.

The expected total cost of Plan A is $\$20.00 + \$75.00$ *or* $\$95.00$*. The expected total cost of Plan B is* $\$10.00 + \$100.00$ *or* $\$110.00$*.*

The student should choose Plan A in order to minimize his out-of-pocket expenses.

Lesson 16: Making Fair Decisions

Student Outcomes

- Students use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Materials List

In this lesson, students will work in groups of three. Each group will need the following:

- Bag, hat, or bowl
- A fair six-sided die
- Spinner
- Deck of cards (optional)

Random number generators are also used in this lesson. Students should have access to a graphing calculator, computer software, or a random number app on a smartphone, tablet, or computer. If needed, a table of random numbers can be used in the place of technology.

Lesson Notes

In this lesson, students determine if a decision is *fair*. Use the opening example as a guide to have students begin to articulate their own meaning of fair. Before students begin the exploratory component of the lesson, a class discussion should also revolve around the idea of how a seemingly fair decision can be manipulated to become unfair or biased. For example, a simple flip of a coin to determine a decision can be manipulated by a person if it is replaced by a coin with two heads.

The term *drawing by lots* is another name for selecting at random and may be the origin of the term *lottery*. Because the term can have some controversial interpretations, drawing by lots is not referred to in the student materials and should not be used in classroom discussions.

MP.5 Students use appropriate tools throughout the lesson as they discuss how to use probability, generally speaking, and various tools to aid in decision making. Students reflect on the use of probability to aid in decision making.

Students will work in groups of three throughout the lesson. This lesson can be shortened by not spending a great deal of time on the simulation activity if teachers need to get a start on Lesson 17 about fair games, which can take more than a full class period. However, spending time on the simulation activity provides a nice link to the use of simulation to estimate probabilities in earlier grades, and ample time should be allotted for a whole class discussion.

Classwork

Exploratory Challenge 1 (15 minutes): What Is a Fair Decision?

The opening example should be used to motivate a discussion around what a *fair decision* is and how a fair decision can be made. Break the class up into groups of three students. Give the groups three to five minutes to discuss the example and think about what is fair. Then come together for a whole class discussion using the following:

- Whose method is the most fair? Explain.
  - *Expect groups to choose Bobby or Andre. Using Bobby's method of drawing a name out of a hat, the probability that each player's name is drawn is* $\frac{10}{30}$ *or* $\frac{1}{3}$*. Using Andre's method of rolling a fair die, the probability that each player's number is rolled is* $\frac{2}{6}$ *or* $\frac{1}{3}$*. Students should not choose Chris.*
- Why is Chris's method not fair?
  - *If a coin is flipped two times, the outcomes are {HH, TT, HT, TH}. So, the probability that either Andre {HH} or Bobby {TT} gets the shoes is* $\frac{1}{4}$*, respectively. Since Chris could have either HT or TH, then the probability he gets the shoes is* $\frac{1}{2}$*.*
- Is it possible to modify Chris's plan in order for it to be considered fair?
  - *Yes. The order of the coin flip should matter. Chris could keep the shoes if a head appears on the first flip of the coin and a tail appears on the second. If a tail appears first and then a head, the coin has to be flipped two more times.*
- How can we define what is meant by fair here?
  - *Each player is equally likely to get the shoes.*

Explain that one goal of a fair decision is to eliminate bias.

- Is there anything that Bobby or Andre could have done to make their methods unfair (i.e., biased)?
  - *Expect multiple responses.*
  - *Bobby could have written his name on big pieces of paper and the names of the other players on small pieces of paper. If he was the one to draw the name from the hat, he could have felt the difference and chosen his name more easily.*
  - *If Chris rolled the die, he could replace it with a die that is unfair without the other players knowing. The sides could be numbered* 1, 2, 5, 5, 6, *and* 6*, which would give Chris an unfair advantage, and Bobby would have no chance at winning.*

Exploratory Challenge 1: What Is a Fair Decision?

Andre, Bobby, and Chris are competing in a 3-on-3 basketball tournament where a set number of teams compete to determine a winning team. Each basketball team plays with three players on the court at the same time.

The team of three wins the tournament. Part of the prize package is a pair of new basketball shoes. All three players want the shoes, but there is only one pair. The boys need to figure out a fair way to determine who gets to keep the new shoes.

- **Chris wants to flip a coin two times to decide who will get the shoes. If two heads appear, then Andre keeps the shoes. If two tails appear, then Bobby keeps the shoes. And if one head and one tail appear (in either order), then Chris will keep the shoes.**

- **Bobby wants to write each of their names 10 times on torn pieces of paper and put all 30 pieces in a hat. He will give the hat a good shake, and then Bobby will choose one piece of paper from the hat to determine who gets the shoes.**
- **Andre wants to roll a fair six-sided die to decide who will keep the shoes. If a 1 or 2 appears, Andre will keep the shoes. If a 3 or 4 appears, Bobby will keep the shoes. If a 5 or 6 appears, then Chris keeps the shoes.**

Which player's method is the most fair?

Exploratory Challenge 2 (20 minutes)

In this exercise, groups will experiment using dice, coins, lots, spinners, and random numbers to make the decision in Example 1. Before groups begin working on the exercise, discuss as a whole class how to use spinners and random number generators to make the decision.

- How could a spinner with three equal sections numbered 1, 2, and 3 be used to decide if Andre, Bobby, or Chris keeps the new shoes?
  - *Each player could be assigned a number: Andre $\{1\}$, Bobby $\{2\}$, and Chris $\{3\}$.*
- How could a random number generator be used to determine who keeps the shoes?
  - *Again, assign each player a number (Andre $\{1\}$, Bobby $\{2\}$, and Chris $\{3\}$), and then generate the random number using a calculator, computer software, or even a smartphone app. Depending on the range of numbers that are generated, the players could even be assigned multiple numbers as long as the probability that each is chosen is equally likely. For example, if a random number is generated from $[0, 9]$, then the following assignments could be used for each player: Andre $\{0, 1, 2\}$, Bobby $\{3, 4, 5\}$, Chris $\{6, 7, 8\}$, and $\{9\}$ is ignored.*

Depending on the size of the class and the number of groups, the teacher can either assign each group a different method to explore or have each group explore all methods. Each member of the group should assume the role of Andre, Bobby, or Chris. Each group should simulate the decision 30, 60, or 90 times (depending on teacher preference and time constraints). After all of the groups have finished the simulations, come back together for a whole class discussion and comparison of the outcomes. Although answers may vary, it is expected that each method produces a probability of getting the shoes of around $\frac{1}{3}$ for each player.

Exploratory Challenge 2

Work with your group to explore each of the decision-making methods. Your teacher will assign one or more methods to each group and specify the number of times each decision should be simulated. Record the outcomes in the table.

Method	Probability Andre keeps shoes	Probability Bobby keeps shoes	Probability Chris keeps shoes
Drawing a name out of a hat (Selecting at Random)			
Fair Coin			
Fair Six-Sided Die			
Random Number Generator (Technology Based)			
Spinner			

Which method do you think should be used to make a fair decision in this case? Explain.

Answers will vary.

Closing (5 minutes)

- Did Bobby have to write each player's name on 10 different slips of paper? Could something other than paper be used in the hat (or bag)? Explain.
  - *No. He could just use three pieces of paper and write each player's name. Or, he could use three different colored marbles, balls, etc.*
- If a decision had to be made involving a large number of people, which method might be the best to use? Explain. (For example, one person in a company of 450 employees will be chosen to win a vacation.)
  - *Random number generator would be the easiest. Each person could be assigned a number between* 1 *and* 450*, and a random number could be generated to choose a winner. A fair coin, spinner, or fair die could not easily be used to represent* 450 *people and make a decision. Drawing lots would take too much time (writing down* 450 *names and drawing, or finding* 450 *different colors to put into a bag).*
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

- **A decision can be considered fair if it does not favor one outcome over another.**
- **Fair decisions can be made by using several methods like selecting names from a hat or using a random number generator.**

Exit Ticket (5minutes)

Name ____ Date ____________________

Lesson 16: Making Fair Decisions

Exit Ticket

1. Both Carmen and her brother Michael want to borrow their father's car on a Friday night. To determine who gets to use the car, Carmen wants her father to roll a pair of fair dice. If the sum of the two dice is 2, 3, 11, or 12, Carmen gets to use the car. If the sum of the two dice is 4 or 10, then Michael can use the car. If the sum is any other number, then the dice will be rolled again. Michael thinks that this is not a fair way to decide. Is he correct? Explain.

2. Due to a technology glitch, an airline has overbooked the number of passengers in economy class on a flight from New York City to Los Angeles. Currently, there are 150 passengers that have economy class tickets, but there are only 141 seats on the plane. There are two seats available in first class and one seat available in business class.
   a. Explain how the ticket agent could use a random number generator to make a fair decision in moving some passengers to either the first or business class sections of the plane and to rebook the extra passengers to a later flight.

   b. Is there any other way for the ticket agent to make a fair decision? Explain.

Exit Ticket Sample Solutions

1. Both Carmen and her brother Michael want to borrow their father's car on a Friday night. To determine who gets to use the car, Carmen wants her father to roll a pair of fair dice. If the sum of the two dice is 2, 3, 11, or 12, Carmen gets to use the car. If the sum of the two dice is 4 or 10, then Michael can use the car. If the sum is any other number, then the dice will be rolled again. Michael thinks that this is not a fair way to decide. Is he correct? Explain.

   No. Michael is wrong. The probability of rolling a sum of a 2, 3, 11, or 12 is $\frac{1}{21}+\frac{1}{21}+\frac{1}{21}+\frac{1}{21}=\frac{4}{21}$. The probability of rolling a sum of a 4 or 10 is $\frac{2}{21}+\frac{2}{21}=\frac{4}{21}$. Because both Carmen and Michael are equally likely to get the car, this would yield a fair decision.

2. Due to a technology glitch, an airline has overbooked the number of passengers in economy class on a flight from New York City to Los Angeles. Currently, there are 150 passengers that have economy class tickets, but there are only 141 seats on the plane. There are two seats available in first class and one seat available in business class.

   a. Explain how the ticket agent could use a random number generator to make a fair decision in moving some passengers to either the first or business class sections of the plane and to rebook the extra passengers to a later flight.

      The ticket agent can assign each of the 150 passengers a number, 1 through 150. Since there are nine extra passengers, the ticket agent can generate nine random numbers between 1 and 150. The first two numbers will be moved to first class, the third number will be moved to business class, and the remaining six numbers will need to be moved to another flight. (If any random numbers repeat, then generate additional numbers as needed.)

   b. Is there any other way for the ticket agent to make a fair decision? Explain.

      The ticket agent could draw numbered balls from a bag. For example, use balls numbered 1 through 150 and choose one ball at a time.

Problem Set Sample Solutions

1. You and your sister each want to sit in the front seat of your mom's car. For each of the following, decide if the decision would be fair or unfair and explain your answer.

   a. You flip a two-sided coin.

      Answers may vary. Some students may say fair as it may be a two-sided coin with a 50/50 probability. Others may say unfair as the person flipping the coin could use a coin with two heads, which would favor one person over the other.

   b. Both you and your sister try to pick a number closest to one randomly generated on a smartphone.

      Fair. Neither person is favored over the other.

   c. You let your mom decide.

      Unfair. For example, you may not have cleaned your room that day and your mom may favor your sister instead.

2. Janice, Walter, and Brooke are siblings. Their parents need them to divide the chores around the house. The one task no one wants to volunteer for is cleaning the bathroom. Janice sees a deck of 52 playing cards sitting on the table and convinces her brother and sister to use the cards to decide who will clean the bathroom.

   - Janice thinks they should draw one card. If a heart is drawn, Janice cleans the bathroom. If a spade is drawn, then Walter cleans. If a diamond is drawn, then Brooke cleans. All of the club cards will be removed from the deck before they begin drawing a card.
   - Walter wants to draw two cards at a time. If both cards are red, then Janice cleans. If both cards are black, then Walter cleans. If one card is red and one card is black, then Brooke cleans the bathroom.
   - Brooke thinks they should draw cards until they get a heart. If the first card drawn is a heart, then Janice cleans the bathroom. If the second card drawn is a heart, then Walter cleans the bathroom. If it takes three or more times to draw a heart, then Brooke cleans the bathroom.

   Whose method is fair? Explain using probabilities.

   Janice's method is fair. They each have an equal probability of being chosen to clean the bathroom: $\frac{13}{39}$ *or* $\frac{1}{3}$.

   Walter's method is not fair. The probability of drawing two red cards is $\frac{26}{52} \cdot \frac{25}{51} \approx 0.245$. *The probability of drawing two black cards is the same,* $\frac{26}{52} \cdot \frac{25}{51} \approx 0.245$. *The probability of drawing a red card and a black card is* $1 - 0.245 - 0.245 \approx 0.51$. *So, Brooke is more likely to clean this bathroom using this method.*

   Brooke's method is not fair. The probability of the first card drawn being a heart is $\frac{13}{52} = 0.25$. *The probability of the second card drawn being a heart is* $\frac{39}{52} \cdot \frac{13}{51} \approx 0.191$. *The probability that it takes three or more cards to draw a heart is* $1 - 0.25 - 0.191 \approx 0.559$. *So, Brooke is more likely to clean this bathroom using this method.*

3. A large software company is moving into new headquarters. Although the workspace is larger, there is not enough space for each of the 239 employees to have his own office. It turns out that two of the employees will need to share an office. Explain how to use a random number generator to make a fair decision as to which employees will share an office.

   Answers will vary. Assign each of the employees a number, 1 *to* 239. *Then generate two random numbers between* 1 *and* 239. *The employees corresponding to these numbers will share an office.*

4. Leslie and three of her friends each want to eat dinner at different restaurants. Describe a fair way to decide to which restaurant the four friends should go to eat dinner.

   Answers will vary. Each of the four friends can write the name of the restaurant they prefer on a slip of paper (all slips the same) and draw out of a hat.

Lesson 17: Fair Games

Student Outcomes

- Students use probability to learn what it means for a game to be fair.
- Students determine whether or not a game is fair.
- Students determine what is needed to make an unfair game fair.

Lesson Notes

The previous lesson focused on making fair decisions. The concept of fairness in statistics requires that one outcome is not favored over the other. In this lesson, students use probability to determine if a game is fair. The lesson begins with a class discussion of the meaning of *fair* in the context of games. When a fee is incurred to play a game, fair implies that the expected winnings are equal in value to the cost incurred by playing the game. If the game is not fair, students use the expected winnings to determine the cost to play the game. Later in the lesson, this idea is extended to warranties and using expected value to determine a fair price for coverage.

Classwork

Example 1 (2 minutes): What Is a Fair Game?

In the previous lesson, students used probability to determine if a fair decision was made (i.e., if one outcome was not favored over another). Discuss as a class the meaning of *fair* as it relates to a fee to play a game. Encourage students to share their ideas about fair games. Consider using the following during the discussion:

- Is a pay-to-play game fair only if the chance of winning and losing are equally likely?
- Does the amount you pay to play the game have an effect on whether the game is fair? How about the amount you can potentially win?

An alternative setting for the instant lottery game card described is to have six similar paper bags labeled A through F. Five of the bags each contain a $\$1.00$ bill, and one contains a $\$10.00$ bill. Randomly scratching off two disks on the card is the same as choosing two bags at random without replacement.

Make sure that students understand the game, and then have them complete Exercises 1–5.

Scaffolding:
- The word *fair* has multiple meanings and may confuse English language learners.
- In some instances, fair means without unjust advantage or cheating.
- In statistics, fair requires that one outcome is not favored over the other.
- A game is *fair* if the expected winnings are equal in value to the cost incurred to play the game.

Before they begin, consider posing the following question. Ask students to write or share their answers with a neighbor.

- How much would you be willing to pay in order to play this game? Explain your answer.
  - *Answers will vary. Student responses should be from* $\$2.00$ *to* $\$11.00$*. For example, I would pay* $\$4.00$ *to play the game. I could potentially win either* $\$2.00$ *or* $\$11.00$*, and* $\$4.00$ *seems like a reasonable amount given the outcomes.*

Example 1: What Is a Fair Game?

An instant lottery game card consists of six disks labeled A, B, C, D, E, F. The game is played by purchasing a game card and scratching off two disks. Each of five of the disks hides $\$1.00$, and one of the disks hides $\$10.00$. The total of the amounts on the two disks that are scratched off is paid to the person who purchased the card.

Scaffolding:

Note that the word *fair* is used differently in this lesson compared to the last:

- Lesson 16: A random number generator can be used to make a fair decision for who gets to choose a song to play at a school dance.
- Lesson 17: Paying \$2.00 is a fair cost to play a carnival game where you have a $50/50$ chance of winning a stuffed animal.

Exercises 1–5 (7 minutes)

Have students work through the exercises with a partner, and then discuss answers as a class. The point of these exercises is for students to come to the conclusion that to justify the cost to play the game, the expected winnings should be equal to that cost, i.e., making the game fair.

As students are working, quickly check their work for Exercise 2 to be sure they are correctly identifying the number of ways for choosing the disks. When discussing the answers to Exercises 4 and 5 as a class, allow for multiple responses, but emphasize that the cost to play the game should be equal to the expected winnings for the game to be fair.

Exercises 1–5

1. **What are the possible total amounts of money you could win if you scratch off two disks?**

   If two $\$1.00$ disks are uncovered, the total is $\$2.00$. If one $\$1.00$ disk and the $\$10.00$ disk are uncovered, the total is $\$11.00$.

2. **If you pick two disks at random:**

   a. **How likely is it that you win $\$2.00$?**

   $$P(\text{win } \$2.00) = \frac{10}{15} = \frac{2}{3}$$

   b. **How likely is it that you win $\$11.00$?**

   $$P(\text{win } \$11.00) = \frac{5}{15} = \frac{1}{3}$$

   Following are two methods to determine the probabilities of getting $\$2.00$ and $\$11.00$.

   Method 1:

   List the possible pairs of scratched disks in a sample space, S, keeping in mind that two different disks need to be scratched and the order of choosing them does not matter. For example, you could use the notation AB that indicates disk A and disk B were chosen, in either order.

   $$S = \{AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF\}$$

   There are 15 different ways of choosing two disks without replacement and without regard to order from the six possible disks.

   Identify the winning amount for each choice under the outcome in S. Suppose that disks A–E hide $\$1.00$, and disk F hides $\$10.00$.

Scaffolding:

- For advanced learners, consider posing the following question, and allow students to conjecture and devise a way to support their claim with mathematics.
- To play the game, you must purchase a game card. The price of the card is set so that the game is fair. How much should you be willing to pay for a game card if the game is to be a fair one? Explain using a probability distribution to support your answer.

Outcomes	AB	AC	AD	AE	AF	BC	BD	BE	BF	CD	CE	CF	DE	DF	EF
Winnings	$\$2$	$\$2$	$\$2$	$\$2$	$\$11$	$\$2$	$\$2$	$\$2$	$\$11$	$\$2$	$\$2$	$\$11$	$\$2$	$\$11$	$\$11$

Since each of the outcomes in S is equally likely, the probability of winning $\$2.00$ is the number of ways of winning $\$2.00$, namely 10, out of the total number of possible outcomes, 15. $P(\text{win } \$2.00) = \frac{10}{15} = \frac{2}{3}$. Similarly, $P(\text{win } \$11.00) = \frac{5}{15} = \frac{1}{3}$.

Method 2:

Previous lessons studied permutations and combinations. Recall that counting when sampling was done without replacement and without regard to order involved combinations.

The number of ways of choosing two disks without replacement and without regard to order is ${}_6C_2 = \frac{6(5)}{2} = 15$. (${}_nC_k$ denotes the number of combinations of n items taken k at a time without replacement and without regard to order.)

To win $\$2.00$, two disks need to be chosen from the five $\$1.00$ disks. The number of ways of doing that is ${}_5C_2 = \frac{5(4)}{2} = 10$.

So the probability of winning $\$2.00$ is $\frac{10}{15} = \frac{2}{3}$.

To win $\$11.00$, one disk needs to be chosen from the five $\$1.00$ disks, and the $\$10.00$ disk needs to be chosen. The number of ways of doing that is $({}_5C_1)({}_1C_1) = 5(1) = 5$. So the probability of winning $\$11.00$ is $\frac{5}{15} = \frac{1}{3}$.

3. **Based on Exercise 3, how much should you expect to win on average per game if you played this game a large number of times?**

   The expected winning amount per play is $(2)\left(\frac{2}{3}\right) + (11)\left(\frac{1}{3}\right) = \$5.00$.

4. **To play the game, you must purchase a game card. The price of the card is set so that the game is fair. What do you think is meant by a fair game in the context of playing this instant lottery game?**

   Responses from students concerning what they think is meant by a fair game will no doubt vary. For example, the cost to play the game should be equal to the expected winnings.

5. **How much should you be willing to pay for a game card if the game is to be a fair one? Explain.**

   Responses will vary. In the context of this instant lottery game, the game is fair if the player is willing to pay $\$5.00$ (the expected winning per play) to purchase each game card.

Example 2 (2 minutes): Deciding Between Two Alternatives

Read through the example as a class, and answer any questions students may have about the game. Before having students complete the exercise, ask them if they were to encounter such a situation, would they actually play the game? Expect an interesting discussion that will involve risk taking. Those who would take the $\$10.00$ rather than play the game are risk-adverse—perhaps even if the payoffs amounted to a higher expected value than the given situation of $\$12.00$. (For example, three $\$1.00$ bills, one $\$5.00$ bill, and two $\$20.00$ bills yield an expected value of $\$16.00$.) Other students may be on the fence, perhaps tossing a fair coin to make their decision risk-neutral. And then there are the risk-seeking players who would play the game as long as the expected winnings exceeded the $\$10.00$ amount that Mom was paying.

Ask students to respond to the following in writing and share their response with a neighbor:

- Would you play your mom's game? Explain why or why not.
  - *Answers will vary. Some students may be conservative and want to stick with keeping the* $10.00 *for completing their chore and not risk playing the game only to walk away with* $4.00. *Others may argue that it is worth the risk to play Mom's game and get* $25.00.

Example 2: Deciding Between Two Alternatives

You have a chore to do around the house for which your mom plans to pay you $\$10.00$. When you are done, your mom, being a mathematics teacher, gives you the opportunity to change the amount that you are paid by playing a game. She puts three $\$2.00$ bills in a bag along with two $\$5.00$ bills and one $\$20.00$ bill. She says that you can take the $\$10.00$ she offered originally or you can play the game by reaching into the bag and selecting two bills without looking. You get to keep these two bills as your payment.

Exercise 6 (5 minutes)

This exercise presents two alternatives. Students should make a rational decision between choosing to take Mom's $10.00 offer or to play Mom's game based on expected value. Clearly, the expected value of choosing Mom's $10.00 offer is just $10.00. The expected value of playing the game involves finding probabilities of outcomes and then calculating the expected value.

Mom's game is like the instant lottery of six disks A, B, C, D, E, F in Example 1, but instead of having two different dollar amounts, there are three. In Mom's game, three disks hide $2.00 bills, two disks hide $5.00 bills, and one disk hides a $20.00 bill.

Have students work in pairs or small groups to complete Exercise 6. As students are working, check their probability distributions. Then discuss the answers as a class. Although answers will vary as to whether or not students will play the game, the expected winnings ($12.00) should be used to justify their responses.

Scaffolding:

For students who struggle to answer the question, use the following to help guide them through the problem:

- What are the possible amounts of money that you might be paid if you play the game?
- Determine a probability distribution for the winnings.
- What is the expected value of this random variable?

MP.3 & MP.4

Exercise 6–7

6. **Do you think you should take your mom's original payment of $\$10.00$ or play the "bag" game? In other words, is this game a fair alternative to getting paid $\$10.00$? Use a probability distribution to help answer this question.**

Suppose that disks (bags) A, B, C hide $\$2.00$ *each, disks D and E hide* $\$5.00$ *each, and disk F hides* $\$20.00$.

Outcomes	*AB*	*AC*	*AD*	*AE*	*AF*	*BC*	*BD*	*BE*	*BF*	*CD*	*CE*	*CF*	*DE*	*DF*	*EF*
Winnings	$\$4$	$\$4$	$\$7$	$\$7$	$\$22$	$\$4$	$\$7$	$\$7$	$\$22$	$\$7$	$\$7$	$\$22$	$\$10$	$\$25$	$\$25$

You could win $\$4.00$, $\$7.00$, $\$10.00$, $\$22.00$, *or* $\$25.00$.

Since each of the outcomes in S is equally likely, the probability of winning $\$4.00$ *is the number of ways of winning* $\$4.00$, *namely 3, out of the total number of possible outcomes, 15.* $P(win\ \$4.00) = \frac{3}{15}$. *Similarly,* $P(\text{win } \$7.00) =$ *the number of ways of winning $7.00 divided by the total number of possible outcomes, namely* $P(\text{win } \$7.00) = \frac{6}{15}$.

MP.3 & MP.4

The probability distribution for the winning amount per play is as follows:

Winning ($)	*Probability*
4	$\frac{3}{15}$
7	$\frac{6}{15}$
10	$\frac{1}{15}$
22	$\frac{3}{15}$
25	$\frac{2}{15}$

The expected winning amount per play is

$$(4)\left(\frac{3}{15}\right)+(7)\left(\frac{6}{15}\right)+(10)\left(\frac{1}{15}\right)+(22)\left(\frac{3}{15}\right)+(25)\left(\frac{2}{15}\right)=\$12.00.$$

The game is in your favor as its expected winning is $\$12.00$ compared to $\$10.00$, but answers will vary. The key is that students recognize that the expected payment for the game is greater than $\$10.00$ but that on any individual play, they could get less than $\$10.00$. In fact, the probability of getting less than $\$10.00$ is greater than the probability of getting $\$10.00$ or more.

Exercise 7 (5 minutes)

The game in Exercise 6 favors the player and not Mom. The purpose of this exercise it to have students explore how Mom's game can be changed so that it is fair on both sides, i.e., the expected winnings are equal to the cost to play ($\$10.00$). Have students work in a small group or with a partner to complete the exercise. There are multiple answers to this exercise and, if time allows, have groups share their answers with the class. Be sure that students support answers using expected value.

7. Alter the contents of the bag in Example 2 to create a game that would be a fair alternative to getting paid $\$10.00$. You must keep six bills in the bag, but you can choose to include bill-sized pieces of paper that are marked as $\$0.00$ to represent a $\$0.00$ bill.

Answers will vary. The easiest answer is to replace all six bills with $\$5.00$ bills. But other combinations are possible, such as three $\$0.00$ bills and three $\$10.00$ bills. If students come up with other possibilities, make sure they support their answer with an expected value calculation.

Example 3 (2 minutes): Is an Additional Year of Warranty Worth Purchasing?

Discuss the example with the class to make sure students understand the context. This example is an extension of the idea of a fair game: What cost would justify purchasing the warranty? That is, what is a fair price?

Example 3: Is an Additional Year of Warranty Worth Purchasing?

Suppose you are planning to buy a computer. The computer comes with a one-year warranty, but you can purchase a waranty for an additional year for $\$24.95$. Your research indicates that in the second year, there is a 1 in 20 chance of incurring a major repair that costs $\$180.00$ and a probability of 0.15 of a minor repair that costs $\$65.00$.

Exercises 8–9 (5 minutes)

Have students work through the exercises in a small group or with a partner. Students should use expected value to justify a fair price in Exercise 9.

Exercises 8–9

8. Is it worth purchasing the additional year warranty? Why or why not?

The expected cost of repairs in the second year is $(0.05)(180) + (0.15)(65) = \$18.75$. So, based on expected value, $\$24.95$ is too high.

9. If the cost of the additional year warranty is too high, what would be a fair price to charge?

A fair price for the waranty would be $\$18.75$.

Example 4 (5 minutes): Spinning a Pentagon

Lead a class discussion of Example 4, leading to the probabilities given below. Then have students complete Exercises 10 and 11. The probabilities of getting an odd sum or an even sum in spinning a regular pentagon can be found from the following matrix. The sums are the cell entries.

		SPIN 2				
		1	2	3	4	5
S	1	2	3	4	5	6
P	2	3	4	5	6	7
I	3	4	5	6	7	8
N	4	5	6	7	8	9
1	5	6	7	8	9	10

Scaffolding:

For advanced learners, consider providing the following extension to the lesson:

- Design your own fair game of chance. Use a probability distribution and expected value to explain why it is fair.

From the matrix, the probability of an odd sum is $\frac{12}{25}$ and the probability of an even sum is $\frac{13}{25}$.

Example 4: Spinning a Pentagon

Your math club is sponsoring a game tournament to raise money for the club. The game is to spin a fair pentagon spinner twice and add the two outcomes. The faces of the spinner are numbered $1, 2, 3, 4$, and 5. If the sum is odd, you win; if the sum is even, the club wins.

Exercises 10–11 (5 minutes)

The following exercises provide students with additional practice for determining how to make a game fair. Have students complete the exercises with a partner. If time is running short, work through the problems as a class.

Exercises 10–11

10. **The math club is trying to decide what to charge to play the game and what the winning payoff should be per play to make it a fair game. Give an example.**

    Answers will vary. If the game is to be fair, the expected amount of money taken in should equal the expected amount of payoff. Let x cents be the amount to play the game and y be the amount won by the player. The math club receives x cents whether or not the player wins. The math club loses y cents if the player wins. So, for the game to be fair, $x - y\left(\frac{12}{25}\right)$ must be 0. Any x and y that satisfy $25x - 12y = 0$ are viable. One example is for the math club to charge 12 cents to play each game with a payoff of 25 cents.

11. **What should the math club charge per play to make $\$0.25$ on average for each game played? Justify your answer.**

    Answers will vary. For the math club to clear 25 cents on average per game, the expected amount they receive minus the expected amount they pay out needs to equal 25, i.e., $x - y\left(\frac{12}{25}\right) = 25$, or $25x - 12y = 625$ where x and y are in cents. For example, if the math club charges 100 cents to play, then the player would receive 156.25 cents if she wins and the club would expect to clear 25 cents per game in the long run.

Closing (2 minutes)

- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

- **The concept of fairness in statistics requires that one outcome is not favored over another.**
- **In a game that involves a fee to play, a game is fair if the amount paid for one play of the game is the same as the expected winnings in one play.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 17: Fair Games

Exit Ticket

A game is played with only the four kings and four jacks from a regular deck of playing cards. There are three "one-eyed" cards: the king of diamonds, the jack of hearts, and the jack of spades. Two cards are chosen at random without replacement from the eight cards. Each one-eyed card is worth $\$2.00$, and non-one-eyed cards are worth $\$0.00$. In the following table, *JdKs* indicates that the two cards chosen were the jack of diamonds and the king of spades. Note that there are 28 pairings. The one-eyed cards are highlighted.

JcJd	Jc**Jh**	Jc**Js**	JcKc	Jc**Kd**	JcKh	JcKs
Jd**Jh**	Jd**Js**	JdKc	Jd**Kd**	JdKh	JdKs	
JhJs	**Jh**Kc	**JhKd**	**Jh**Kh	**Jh**Ks		
JsKc	**JsKd**	**Js**Kh	**Js**Ks			
Kc**Kd**	KcKh	KcKs				
KdKh	**Kd**Ks					
KhKs						

a. What are the possible amounts you could win in this game? Write them in the cells of the table next to the corresponding outcome.

b. Find the the expected winnings per play.

c. How much should you be willing to pay per play of this game if it is to be a fair game?

Exit Ticket Sample Solutions

A game is played with only the four kings and four jacks from a regular deck of playing cards. There are three "one-eyed" cards: the king of diamonds, the jack of hearts, and the jack of spades. Two cards are chosen at random without replacement from the eight cards. Each one-eyed card is worth $\$2.00$, and non-one-eyed cards are worth $\$0.00$. In the following table, *JdKs* indicates that the two cards chosen were the jack of diamonds and the king of spades. Note that there are 28 pairings. The one-eyed cards are highlighted.

JcJd	Jc**Jh**	Jc**Js**	JcKc	Jc**Kd**	JcKh	JcKs
Jd**Jh**	Jd**Js**	JdKc	Jd**Kd**	JdKh	JdKs	
JhJs	**Jh**Kc	**JhKd**	**Jh**Kh	**Jh**Ks		
JsKc	**JsKd**	**Js**Kh	**Js**Ks			
Kc**Kd**	KcKh	KcKs				
KdKh	**Kd**Ks					
KhKs						

a. **What are the possible amounts you could win in this game? Write them in the cells of the table next to the corresponding outcome.**

JcJd	$\mathbf{0}$	Jc**Jh**	$\mathbf{2}$	Jc**Js**	$\mathbf{2}$	JcKc	$\mathbf{0}$	Jc**Kd**	$\mathbf{2}$	JcKh	$\mathbf{0}$	JcKs	$\mathbf{0}$
Jd**Jh**	$\mathbf{2}$	Jd**Js**	$\mathbf{2}$	JdKc	$\mathbf{0}$	Jd**Kd**	$\mathbf{2}$	JdKh	$\mathbf{0}$	JdKs	$\mathbf{0}$		
JhJs	$\mathbf{4}$	**Jh**Kc	$\mathbf{2}$	**JhKd**	$\mathbf{4}$	**Jh**Kh	$\mathbf{2}$	**Jh**Ks	$\mathbf{2}$				
JsKc	$\mathbf{2}$	**JsKd**	$\mathbf{4}$	**Js**Kh	$\mathbf{2}$	**Js**Ks	$\mathbf{2}$						
Kc**Kd**	$\mathbf{2}$	KcKh	$\mathbf{0}$	KcKs	$\mathbf{0}$								
KdKh	$\mathbf{2}$	**Kd**Ks	$\mathbf{2}$										
KhKs	$\mathbf{0}$												

b. **Find the the expected winnings per play.**

Ask your students to verify their accumulated counts using combinations. There are $\mathbf{10}$ pairing where neither card is one-eyed (${}_{5}C_{2} = 10$); $\mathbf{15}$ pairings with one card one-eyed ($({}_{5}C_{1})({}_{3}C_{1}) = 15$); and $\mathbf{3}$ pairings in which both cards are one-eyed (${}_{3}C_{3} = 1$).

Expected winnings per play $= (0)\left(\frac{10}{28}\right) + (2)\left(\frac{15}{28}\right) + (4)\left(\frac{3}{28}\right) = \frac{42}{28}$, *or* $\$1.50$.

c. **How much should you be willing to pay per play of this game if it is to be a fair game?**

According to the expected value of part (b), a player should be willing to pay $\$1.50$ to play the game in order for it to be a fair game.

Problem Set Sample Solutions

1. **A game is played by drawing a single card from a regular deck of playing cards. If you get a black card, you win nothing. If you get a diamond, you win $\$5.00$. If you get a heart, you win $\$10.00$. How much would you be willing to pay if the game is to be fair? Explain.**

The following represents a probability distribution:

Outcomes	*Probability*
0	$\frac{1}{2}$
5	$\frac{1}{4}$
10	$\frac{1}{4}$

The expected winnings are $0\left(\frac{1}{2}\right)+5\left(\frac{1}{4}\right)+10\left(\frac{1}{4}\right)=3.75$. *For the game to be fair, the fee should be* $\$3.75$.

2. **Suppose that for the game described in Problem 1, you win a bonus for drawing the queen of hearts. How would that change the amount you are willing to pay for the game? Explain.**

The fee to play the game should increase, as the expected winnings will increase. For example, if a bonus of $\$10.00$ *is earned for drawing the queen of hearts, the probability distribution will be*

Outcomes	*Probability*
0	$\frac{1}{2}$
5	$\frac{1}{4}$
10	$\frac{12}{52}$
20	$\frac{1}{52}$

The expected winnings are $0\left(\frac{1}{2}\right)+5\left(\frac{1}{4}\right)+10\left(\frac{12}{52}\right)+20\left(\frac{1}{52}\right)\approx 3.94$. *For the game to be fair, the fee should be no more than* $\$3.94$, *which is an increase from* $\$3.75$.

3. You are trying to decide between playing two different carnival games and want to only play games that are fair.

One game involves throwing a dart at a balloon. It costs $\$10.00$ to play, and if you break the balloon with one throw, you win $\$75.00$. If you do not break the balloon, you win nothing. You estimate that you have about a 15% chance of breaking the balloon.

The other game is a ring toss. For $\$5.00$ you get to toss three rings and try to get them around the neck of a bottle. If you get one ring around a bottle, you win $\$3.00$. For two rings around the bottle, you win $\$15.00$. For three rings, you win $\$75.00$. If no rings land around the neck of the bottle, you win nothing. You estimate that you have about a 15% chance of tossing a ring and it landing around the neck of the bottle. Each toss of the ring is independent.

Which game will you play? Explain.

The following are probability distributions for each of the games:

Dart Outcomes	*Probability*
0 ($\$0.00$)	0.85
1 ($\$75.00$)	0.15

Ring Toss Outcomes	*Probability*
0 ($\$0.00$)	0.85^3
1 ($\$3.00$)	$3(0.15)(0.85)^2$
2 ($\$15.00$)	$3(0.15)^2(0.85)$
3 ($\$75.00$)	$(0.15)^3$

The expected winnings for the dart game are $0(0.85) + 75(0.15) = \$11.25$. The dart game is fair because the cost to play is less than the expected winnings.

The expected winnings for the ring toss game are $0 \cdot (0.85)^3 + 3 \cdot 3(0.15)(0.85)^2 + 15 \cdot 3(0.15)^2(0.85) + 75 \cdot (0.15)^3 \approx \$2.09$. For the ring toss outcomes of 1 and 2, there is a multiplier of 3 because there are three different ways to get 1 and 2 rings on a bottle in 3 tosses. The ring toss game is not fair because the cost to play is more than the expected winnings.

I would play the dart game.

4. Invent a fair game that involves three fair number cubes. State how the game is played and how the game is won. Explain how you know the game is fair.

Answers will vary. One example is given here:

The key is to determine probabilities correctly. Rolling three fair number cubes results in $(6)(6)(6) = 216$ possible ordered triples. Two events of interest could be "all same," i.e., 111, 222, 333, 444, 555, or 666. The probability of "all same" is $\frac{6}{216}$. Another event could be "all different." The number of ways of getting "all different" digits is the permutation, $(6)(5)(4) = 120$. Getting any other outcome would have probability $1 - \left(\frac{6}{216}\right) - \left(\frac{120}{216}\right) = \frac{90}{216}$.

Consider this game: Roll three fair number cubes. If the result is "all same digit," then you win $\$20.00$. If the result is "all different digits," you win $\$1.00$. Otherwise, you win $\$0.00$. It costs $\$1.11$ to play.

The game is approximately fair since the expected winnings per play $= (20)\left(\frac{6}{216}\right) + (1)\left(\frac{120}{216}\right) + (0)\left(\frac{90}{216}\right) \approx \$1.11$.

5. Invent a game that is not fair that involves three fair number cubes. State how the game is played and how the game is won. Explain how you know the game is not fair.

Answers will vary. One example is given here:

From Problem 4, any fee to play that is not $\$1.11$ results in a game that is not fair. If the fee to play is lower than $\$1.11$, then the game is in the player's favor. If the fee to play is higher than $\$1.11$, then the game is in the favor of whoever is sponsoring the game.

Lesson 18: Analyzing Decisions and Strategies Using Probability

Student Outcomes

- Students use probability concepts to make decisions in a variety of contexts.

Lesson Notes

In previous lessons, students have decided between strategies by comparing the expected payoff and have designed games to be fair by ensuring an equal probability of winning for those who play the game. In this lesson, students compare strategies by evaluating their associated probabilities of success. This lesson consists of a sequence of exercises where students have to draw on several probability concepts (e.g., conditional probability and the multiplication and addition rules). Therefore, it might be useful to have students who need help work with a classmate who can provide the necessary guidance.

Classwork

Exercise 1 (5 minutes)

MP.3

This first exercise is a relatively straightforward application of probability. Students compare strategies by evaluating their associated probabilities of success. In discussing the results of this exercise, students can support their answers by using exact probabilities or by generalizing cases as noted in the sample response below.

Exercise 1

Suppose that someone is offering to sell you raffle tickets. There are blue, green, yellow, and red tickets available. Each ticket costs the same to purchase regardless of color. The person selling the tickets tells you that 369 blue tickets, 488 green tickets, 523 yellow tickets, and 331 red tickets have been sold. At the drawing, one ticket of each color will be drawn, and four identical prizes will be awarded. Which color ticket would you buy? Explain your answer.

Suppose, for the sake of argument, that at the time of the drawing 500 blue tickets and 600 green tickets have been sold. If you hold one blue ticket, the probability that you will win is $\frac{1}{500}$. If you hold one green ticket, the probability that you will win is $\frac{1}{600}$. Since $\frac{1}{500}$ is greater than $\frac{1}{600}$, it would be to your advantage to have a blue ticket rather than a green one. Therefore, it is sensible to buy the color of ticket that has sold the fewest, which at the time of buying is red. You should buy a red ticket.

Scaffolding:

If students struggle with this concept, remind them how to compare fractions.

- If the numerators are the same, the larger denominator is the smaller fraction. $\frac{1}{500} > \frac{1}{600}$
- If the denominators are the same, the larger numerator is the larger fraction. $\frac{6}{3000} > \frac{5}{3000}$

For advanced learners, pose the following:

- Suppose you could purchase four raffle tickets. Would you purchase all tickets in the same color or one of each color? Explain.

Exercise 2 (7 minutes)

This exercise is another straightforward application of probability, but a certain amount of information has to be processed prior to evaluating the probabilities.

Exercise 2

Suppose that you are taking part in a TV game show. The presenter has a set of 60 cards, 10 of which are red and the rest are blue. The presenter randomly splits the cards into two piles and places one on your left and one on your right. The presenter tells you that there are 32 blue cards in the pile on your right. You look at the pile of cards on your left and estimate that it contains 24 cards. You will be given a chance to pick a card at random, and you know that if you pick a red card you will win $\$5,000$. If you pick a blue card, you will get nothing. The presenter gives you the choice of picking a card at random from the pile on the left, from the pile on the right, or from the entire set of cards. Which should you choose? Explain your answer.

Assuming that your estimate that there are 24 cards in the pile on the left is true, there are $60-24=36$ cards in the pile on the right. The number of blue cards on the right is 32, so the number of red cards on the right is $36-32=4$. There are 10 red cards altogether, so the number of red cards on the left is $10-4=6$. Therefore, if you pick a card at random from the pile on the left, the probability that you pick a red card is $\frac{6}{24}=0.25$. If you pick from the pile on the right, the probability that you get a red card is $\frac{4}{36}=0.111$. If you pick from the entire deck, the probability that you get a red card is $\frac{10}{60}=0.167$. Therefore, you should pick your card from the pile on the left.

Exercise 3 (8 minutes)

MP.3 In this exercise, which is about a test for Lyme disease, students use a hypothetical 100,000 table to evaluate a conditional probability. This probability is then used to assess the usefulness of the test in part (b).

Exercise 3

The American Lyme Disease Foundation states that a commonly used test called the ELISA test will be positive for virtually all patients who have the disease but that the test is also positive for around 6% of those who do not have the disease. (http://www.aldf.com/faq.shtml#Testing) For the purposes of this question, assume that the ELISA test is positive for all patients who have the disease and for 6% of those who do not have the disease. Suppose the test is performed on a randomly selected resident of Connecticut where, according to the Centers for Disease Control and Prevention, 46 out of every $100,000$ people have Lyme disease. (http://www.cdc.gov/lyme/stats/chartstables/reportedcases_statelocality.html)

a. **Complete the hypothetical $100,000$-person two-way frequency table below for $100,000$ Connecticut residents.**

	Test is Positive	Test is Negative	Total
Has the disease			
Does not have the disease			
Total			$100,000$

	Test is Positive	Test is Negative	Total
Has the disease	46	0	46
Does not have the disease	$5,997$	$93,957$	$99,954$
Total	$6,043$	$93,957$	$100,000$

b. **If a randomly selected person from Connecticut is tested for the disease using the ELISA test and the test is positive, what is the probability that the person has the disease? (Round your answer to the nearest thousandth.)**

$$P(\text{disease}\,|\,\text{positive}) = \frac{46}{6043} = 0.008$$

c. **Comment on your answer to part (b). What should the medical response be if a person is tested using the ELISA test and the test is positive?**

If a patient gets a positive test, the probability that the patient actually has the disease is very small (only around 8 in every $1,000$ people with positive tests actually have the disease). So, if a person has a positive test, then it is necessary to follow up with more accurate testing.

Example 1 (8 minutes)

Prior to tackling this example as a class, it would be worthwhile to remind students about two important probability rules—the multiplication rule and the addition rule. To review these rules, ask them the following questions:

- Suppose that a deck of cards consists of 10 green and 10 blue cards. Two cards will be selected at random from the deck without replacement. What is the probability that both cards are green? What is the probability that the two cards are the same color?
  - $P(\text{both cards green}) = \left(\frac{10}{20}\right)\left(\frac{9}{19}\right) = 0.237$
  - $P(\text{same color}) = P(BB) + P(GG) = \left(\frac{10}{20}\right)\left(\frac{9}{19}\right) + \left(\frac{10}{20}\right)\left(\frac{9}{19}\right) = \left(\frac{10}{20}\right)\left(\frac{9}{19}\right) \cdot 2 = 0.474$

Example 1

You are playing a game that uses a deck of cards consisting of 10 green, 10 blue, 10 purple, and 10 red cards. You will select four cards at random, and you want all four cards to be the same color. You are given two alternatives. You can randomly select the four cards one at a time, with each card being returned to the deck and the deck being shuffled before you pick the next card. Alternatively, you can randomly select four cards without the cards being returned to the deck. Which should you choose? Explain your answer.

With replacement:

$$P(\text{same color}) = P(GGGG) + P(BBBB) + P(PPPP) + P(RRRR) = \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^4 = 0.015625$$

Without replacement:

$$P(\text{same color}) = P(GGGG) + P(BBBB) + P(PPPP) + P(RRRR) = \left(\frac{10}{40}\right)\left(\frac{9}{39}\right)\left(\frac{8}{38}\right)\left(\frac{7}{37}\right) \cdot 4 = 0.009$$

Since 0.015625 is greater than 0.009, it is better to select the cards with replacement.

It is also possible to give an intuitive answer to this question. Think about the probability of selecting all green cards when you are selecting without replacement. The probability that the first card is green is $\frac{1}{4}$. However, having selected a green card first, the following cards are less likely to be green as there are only nine green cards remaining while there are 10 of each of the other three colors. Whereas, if you select the cards with replacement, then the probability of picking a green card remains at $\frac{1}{4}$ for all four selections. Thus, it is preferable to select the cards with replacement.

Exercise 4 (7 minutes)

This exercise concerns repeated trials and the probability of success on at least one of the trials. An important hint is provided, but some students might nonetheless need assistance understanding that the probability of success can be found by subtracting the probability that you fail on all three attempts from 1 (or $1 - \mathrm{P}(\text{fail, fail, fail})$. Students should recognize that because the outcomes of the throws are independent, the probability that you fail on all three attempts can be found by applying the multiplication rule. As you check their work, you may notice some students list all of the outcomes in the sample space and calculate probabilities accordingly. However, stress to students that it is more efficient to use the hint provided.

Exercise 4

You are at a stall at a fair where you have to throw a ball at a target. There are two versions of the game. In the first version, you are given three attempts, and you estimate that your probability of success on any given throw is 0.1. In the second version, you are given five attempts, but the target is smaller, and you estimate that your probability of success on any given throw is 0.05. The prizes for the two versions of the game are the same, and you are willing to assume that the outcomes of your throws are independent. Which version of the game should you choose? (Hint: In the first version of the game, the probability that you do not get the prize is the probability that you fail on all three attempts.)

First version of the game:
The probability that you do not get the prize $= P(\textit{fail, fail, fail}) = (0.9)^3 = 0.729$
So that the probability that you do get the prize $= 1 - 0.729 = 0.271$

Second version of the game:
The probability that you do not get the prize $= P(\textit{fail, fail, fail, fail, fail}) = (0.95)^5 = 0.774$
So that the probability that you do get the prize $= 1 - 0.774 = 0.226$

Since 0.271 is greater than 0.226, you should choose the first version of the game.

Closing (2 minutes)

Pose the following questions to the class. Have students express their answers in writing and share their responses with a partner:

- Explain how you used probability to make decisions during this lesson. Include at least one specific example in your answer. In what kind of a situation would a high probability determine the most desirable outcome? A low probability?
  - *Sample response: Probability was used to weigh options based on the likelihood of occurrences of random events. Decisions were made based on calculated probabilities. High probabilities are desirable in situations that involve making money or winning games. Low probabilities are desirable in situations such as losing money or getting sick.*
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

> **Lesson Summary**
>
> **If a number of strategies are available and the possible outcomes are success and failure, the best strategy is the one that has the highest probability of success.**

Exit Ticket (8 minutes)

Name ____ Date ____________________

Lesson 18: Analyzing Decisions and Strategies Using Probability

Exit Ticket

1. In a Home Décor store, 23% of the customers have reward cards. Of the customers who have reward cards, 68% use the self-checkout, and the remainder use the regular checkout. Of the customers who do not have reward cards, 60% use the self-checkout, and the remainder use the regular checkout.

   a. Construct a hypothetical $1,000$-customer two-way frequency table with columns corresponding to whether or not a customer uses the self-checkout and rows corresponding to whether or not a customer has a reward card.

	Self-Checkout	Regular Checkout	Total
Has Reward Card			
Does Not Have Reward Card			
Total			$1,000$

   b. What proportion of customers who use the self-checkout do not have reward cards?

   c. What proportion of customers who use the regular checkout do not have reward cards?

d. If a researcher wishes to maximize the proportion of nonreward cardholders in a study, would she be better off selecting customers from those who use the self-checkout or the regular checkout? Explain your reasoning.

2. At the end of a math contest, each team must select two students to take part in the countdown round. As a math team coach, you decide to randomly select two students from your team. You would prefer that the two students selected consist of one girl and one boy. Would you prefer to select your two students from a team of 6 girls and 6 boys or a team of 5 girls and 5 boys? Show your calculations and explain how you reached your conclusion.

Exit Ticket Sample Solutions

1. In a Home Décor store, 23% of the customers have reward cards. Of the customers who have reward cards, 68% use the self-checkout, and the remainder use the regular checkout. Of the customers who do not have reward cards, 60% use the self-checkout, and the remainder use the regular checkout.

   a. Construct a hypothetical $1,000$-customer two-way frequency table with columns corresponding to whether or not a customer uses the self-checkout and rows corresponding to whether or not a customer has a reward card.

	Self-Checkout	Regular Checkout	Total
Has Reward Card	156	74	230
Does Not Have Reward Card	462	308	770
Total	618	382	$1,000$

   b. What proportion of customers who use the self-checkout do not have reward cards?

$$\frac{462}{618} = 0.748$$

   c. What proportion of customers who use the regular checkout do not have reward cards?

$$\frac{308}{382} = 0.806$$

   d. If a researcher wishes to maximize the proportion of nonreward cardholders in a study, would the researcher be better off selecting customers from those who use the self-checkout or the regular checkout? Explain your reasoning.

   Since 0.806 is greater than 0.748, the researcher should select customers from those who use the regular checkout. The probability of customers not having reward cards in the regular checkout would be slightly higher than in the self-checkout, maximizing the researcher's chances of getting a higher proportion of non-cardholders for the study.

2. At the end of a math contest, each team must select two students to take part in the countdown round. As a math team coach, you decide that you will randomly select two students from your team. You would prefer that the two students selected consist of one girl and one boy. Would you prefer to select your two students from a team of 6 girls and 6 boys or a team of 5 girls and 5 boys? Show your calculations and explain how you reach your conclusion.

   6 girls, 6 boys: $P(1\text{ girl}, 1\text{ boy}) = P(GB) + P(BG) = \left(\frac{6}{12}\right)\left(\frac{6}{11}\right) + \left(\frac{6}{12}\right)\left(\frac{6}{11}\right) = 0.545$

   5 girls, 5 boys: $P(1\text{ girl}, 1\text{ boy}) = P(GB) + P(BG) = \left(\frac{5}{10}\right)\left(\frac{5}{9}\right) + \left(\frac{5}{10}\right)\left(\frac{5}{9}\right) = 0.555$

   The probabilities are nearly equal, but because 0.555 is greater than 0.545, it is preferable to select from 5 girls and 5 boys.

Problem Set Sample Solutions

1. Jonathan is getting dressed in the dark. He has three drawers of socks. The top drawer contains 5 blue and 5 red socks, the middle drawer contains 6 blue and 4 red socks, and the bottom drawer contains 3 blue and 2 red socks. Jonathan will open one drawer and will select two socks at random.

   a. Which drawer should he choose in order to make it most likely that he will select 2 red socks?

   The middle and bottom drawers both contain a minority of red socks. This is not the case for the top drawer. So Jonathan should choose the top drawer.

   b. Which drawer should he choose in order to make it most likely that he will select 2 blue socks?

   Blue socks are in the majority in the middle and bottom drawers but not in the top drawer.

   Middle drawer: $P(BB) = \left(\frac{6}{10}\right)\left(\frac{5}{9}\right) = 0.333$

   Bottom drawer: $P(BB) = \left(\frac{3}{5}\right)\left(\frac{2}{4}\right) = 0.3$

   Since 0.333 is greater than 0.3, he should choose the middle drawer.

   c. Which drawer should he choose in order to make it most likely that he will select a matching pair?

   Top drawer: $P(\text{matching pair}) = P(BB) + P(RR) = \left(\frac{5}{10}\right)\left(\frac{4}{9}\right) \cdot 2 = 0.444$

   Middle drawer: $P(\text{matching pair}) = P(BB) + P(RR) = \left(\frac{6}{10}\right)\left(\frac{5}{9}\right) + \left(\frac{4}{10}\right)\left(\frac{3}{9}\right) = 0.467$

   Bottom drawer: $P(\text{matching pair}) = P(BB) + P(RR) = \left(\frac{3}{5}\right)\left(\frac{2}{4}\right) + \left(\frac{2}{5}\right)\left(\frac{1}{4}\right) = 0.4$

   Since the probability of a matching pair is greatest for the middle drawer, Jonathan should choose the middle drawer.

2. Commuters in London have the problem that buses are often already full and, therefore, cannot take any further passengers. Sarah is heading home from work. She has the choice of going to Bus Stop A, where there are three buses per hour and 30% of the buses are full, or Bus Stop B, where there are four buses per hour and 40% of the buses are full. Which stop should she choose in order to maximize the probability that she will be able to get on a bus within the next hour? (Hint: Calculate the probability, for each bus stop, that she will fail to get on a bus within the next hour. You may assume that the buses are full, or not, independently of each other.)

   Bus Stop A: $P(\text{fails to get bus}) = P(\text{all three buses are full}) = (0.3)^3 = 0.027$
   So $P(\text{gets bus}) = 1 - 0.027 = 0.973$

   Bus Stop B: $P(\text{fails to get bus}) = P(\text{all four buses are full}) = (0.4)^4 = 0.0256$
   So $P(\text{gets bus}) = 1 - 0.0256 = 0.9744$

   Sarah is slightly more likely to get a bus if she chooses Bus Stop B.

3. An insurance salesman has been told by his company that about 20% of the people in a city are likely to buy life insurance. Of those who buy life insurance, around 30% own their homes, and of those who do not buy life insurance, around 10% own their homes. In the questions that follow, assume that these estimates are correct.

a. If a homeowner is selected at random, what is the probability that the person will buy life insurance? (Hint: Use a hypothetical $1,000$-person two-way frequency table.)

	Owns Home	*Does Not Own Home*	*Total*
Buys Life Insurance	60	140	200
Does Not Buy Life Insurance	80	720	800
Total	140	860	$1,000$

$$P(\text{buys life insurance}\,|\,\text{homeowner}) = \frac{60}{140} = 0.429$$

b. If a person is selected at random from those who do not own their homes, what is the probability that the person will buy life insurance?

$$P(\text{buys life insurance}\,|\,\text{does not own home}) = \frac{140}{860} = 0.163$$

c. Is the insurance salesman better off trying to sell life insurance to homeowners or to people who do not own their homes?

Since 0.429 is greater than 0.163, the salesman is better off trying to sell to homeowners.

4. You are playing a game. You are given the choice of rolling a fair six-sided number cube (with faces labeled 1–6) three times or selecting three cards at random from a deck that consists of

- 4 cards labeled 1
- 4 cards labeled 2
- 4 cards labeled 3
- 4 cards labeled 4
- 4 cards labeled 5
- 4 cards labeled 6

If you decide to select from the deck of cards, then you will not replace the cards in the deck between your selections. You will win the game if you get a triple (that is, rolling the same number three times or selecting three cards with the same number). Which of the two alternatives, the number cube or the cards, will make it more likely that you will get a triple? Explain your answer.

With number cube:

$$P(\text{triple}) = P(1,1,1) + P(2,2,2) + P(3,3,3) + P(4,4,4) + P(5,5,5) + P(6,6,6) = \left(\frac{1}{6}\right)^3 \cdot 6 = 0.028$$

With cards:

$$P(\text{triple}) = P(1,1,1) + P(2,2,2) + P(3,3,3) + P(4,4,4) + P(5,5,5) + P(6,6,6) = \left(\frac{4}{24}\right)\left(\frac{3}{23}\right)\left(\frac{2}{22}\right) \cdot 6 = 0.012$$

Since 0.028 is greater than 0.012, a triple is more likely with the number cube.

5. There are two routes Jasmine can take to work. Route A has five stoplights. The probability distribution of how many lights at which she will need to stop is below. The average amount of time spent at each stoplight on Route A is 30 seconds.

Number of Red Lights	0	1	2	3	4	5
Probability	0.04	0.28	0.37	0.14	0.11	0.06

Route B has three stoplights. The probability distribution of Route B is below. The average wait time for these lights is 45 seconds.

Number of Red Lights	0	1	2	3
Probability	0.09	0.31	0.40	0.20

a. In terms of stopping at the least number of stoplights, which route may be the best for Jasmine to take?

If she takes Route A, the expected number of stoplights that she will have to stop for is $(0 \cdot 0.04) + (1 \cdot 0.28) + (2 \cdot 0.37) + (3 \cdot 0.14) + (4 \cdot 0.11) + (5 \cdot 0.06) = 2.18$ *stoplights.*

If she takes Route B, the expected number of stoplights that she will have to stop for is $(0 \cdot 0.09) + (1 \cdot 0.31) + (2 \cdot 0.40) + (3 \cdot 0.20) = 1.71$ *stoplights.*

She might be stopped by fewer stoplights if she takes Route B.

b. In terms of least time spent at stoplights, which route may be the best for Jasmine to take?

The expected wait time on Route A is $2.18 \cdot 0.5 = 1.09$ *minutes.*

The expected wait time on Route B is $1.71 \cdot 0.75 = 1.28$ *minutes.*

Even though Jasmine is expected to hit fewer lights on Route B, her expected wait time is longer than that of Route A.

6. A manufacturing plant has been shorthanded lately, and one of its plant managers recently gathered some data about shift length and frequency of work-related accidents in the past month (accidents can range from forgetting safety equipment to breaking a nail to other, more serious injuries). Below is the table displaying his findings.

	Number of shifts with 0 accidents	Number of shifts with at least one accident	Total
$0 < x < 8$ hours	815	12	827
$8 \le x \le 10$ hours	53	27	80
$x > 10$ hours	52	41	93
Total	920	80	$1{,}000$

a. What is the probability that a person had an accident?

The probability is $\frac{80}{1000} = 0.08$.

b. What happens to the accident likelihood as the number of hours increases?

For an 8*-hour shift, the probability is* $\frac{12}{80} = 0.15$. *For an 8- to 10-hour shift, the probability is* $\frac{27}{80} = 0.34$. *For a* $10+$ *hour shift, the probability is* $\frac{41}{80} = 0.51$. *In general, as the length of the shift increases, so does the probability of having an accident.*

c. What are some options the plant could pursue in order to try to cut down or eliminate accidents?

Answers will vary. Assuming fatigue is a major factor for accident occurrence, the plant could hire more people. It could mandate shorter shifts. It could reduce its operating hours.

Lesson 19: Analyzing Decisions and Strategies Using Probability

Student Outcomes

- Students use probability concepts to make decisions in a variety of contexts.

Lesson Notes

In this lesson, students continue to compare strategies by evaluating the associated probabilities of success. Additionally, problems are included where students are asked whether the result of a probability experiment provides sufficient evidence to reject a hypothesis. This builds on the idea of statistical significance first introduced in Algebra II (Module 4, Lessons 25–27). Students should work in carefully selected pairs throughout the lesson.

Classwork

Example 1 (9 minutes): Using Probability to Make a Decision About a Hypothesis

This is an example of a question where students are asked whether the result of a probability experiment provides sufficient evidence to reject a hypothesis. It is important that students understand the approach involved as there are several exercises in this lesson that draw on this idea. A common convention used in statistics is that an outcome is deemed unlikely if its probability is less than 0.05. (This is often referred to as a 5% *significance level*.)

Draw students' attention to the fact that the answer to part (b) starts with an if clause. This is a vital part of the answer and should not be omitted.

Having completed the example, have students consider what the conclusion would have been had the answer to part (a) been greater than 0.05. The answer to part (b) would then be that there is not strong evidence that what the friend said was false. This does not mean that there is evidence that what the friend said is true.

Scaffolding:

For struggling students, consider providing a simpler version of the example:

- You put the MP3 player in a random shuffle mode, listen to three randomly selected songs, and discover they are all hip-hop. Do you believe $\frac{1}{3}$ of the songs are indie rock and the rest are hip-hop?
- Show a diagram with a series of question marks over Song 1 through Song 8 to describe the context. Show a diagram with hip-hop written over every song to indicate what you found when you listened to eight songs.

?	?	?	
Song 1	Song 2	Song 3	…
Hip Hop	*Hip Hop*	*Hip Hop*	
Song 1	Song 2	Song 3	…

Example 1: Using Probability to Make a Decision About a Hypothesis

Suppose that a friend lends you an MP3 player containing a large number of songs. Your friend tells you that $\frac{1}{3}$ of the songs are indie rock and the rest are hip-hop. You put the MP3 player in random shuffle mode, listen to eight randomly selected songs, and discover that they are all hip-hop. How could you determine whether your friend's statement is true or not?

a. **Do you believe your friend? Why or why not? Use probability to justify your answer.**

Sample response 1: No, because if $\frac{1}{3}$ of the songs were indie rock and the rest hip-hop, then 2 or 3 songs in the sample should have been indie rock and the rest hip-hop.

Sample response 2: $\frac{1}{3}$ is the probability of the occurrence of indie rock songs in any sample, but any sample could contain no indie rock songs.

b. **If it were true that $\frac{1}{3}$ of the songs on the MP3 player are indie rock and the rest are hip-hop, what would the probability be that all eight randomly selected songs are hip-hop? (Round your answer to the nearest thousandth.)**

$$\left(\frac{2}{3}\right)^8 = 0.039$$

c. **Does the result of your random selection lead you to suspect that what your friend told you is false? Explain your answer using the probability in part (a).**

If what the friend said is true, then it is very unlikely (probability $= 0.039$) that all eight randomly selected songs would be hip-hop. So, there is strong evidence that what the friend said is false.

Scaffolding:

For advanced learners, consider providing the following extension activity:

- Do you have a different estimate for the proportion of hip-hop songs? Explain using probabilities.

Exercise 1 (7 minutes)

The exercises in this lesson involve several different approaches to probability, so it would probably be helpful for students to work in carefully selected pairs throughout the lesson. The first exercise is a straightforward application of the multiplication rule.

Exercises

1. **A class consists of 12 boys and 12 girls. The teacher picks five students to present their work to the rest of the class and says that the five students are being selected at random. The students chosen are all girls.**

   a. **If the teacher were truly selecting the students at random, what would the probability be that all five students selected are girls? (Round your answer to the nearest thousandth.)**

   $$\left(\frac{12}{24}\right)\left(\frac{11}{23}\right)\left(\frac{10}{22}\right)\left(\frac{9}{21}\right)\left(\frac{8}{20}\right) = 0.019$$

   b. **Does the fact that all five students selected are girls lead you to suspect that the teacher was not truly selecting the students at random but that the teacher had a preference for choosing girls? Explain your answer using the probability in part (a).**

   MP.3

   If the teacher were selecting students at random, then it is very unlikely (probability = 0.019) that all the students selected would be girls. So, there is strong evidence that the teacher had a preference for choosing girls.

Exercise 2 (7 minutes)

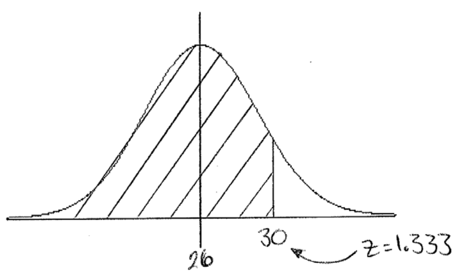
In this exercise, students choose a strategy using normal probabilities. Students learned how to calculate normal probabilities in Algebra II. A table of standard normal curve areas is located in the appendix of the teacher notes.

2. School starts at 8:00 a.m.; it is now 7:30 a.m., and you are still at home. You have to decide whether to leave now and ride your bicycle to school or to call a friend and ask the friend to pick you up. Your friend would take 10 minutes to get to your house. You know that when you ride your bicycle, the time it takes to get to school has a mean of 26 minutes and a standard deviation of 3 minutes. When your friend drives you, the time it takes to get from your home to school has a mean of 14 minutes and standard deviation of 5 minutes. Which of the two alternatives will make you more likely to get to school on time? Show the assumptions that must be made and the calculations that lead to your conclusion.

It should be assumed that the data is normally distributed. This is a valid assumption because:

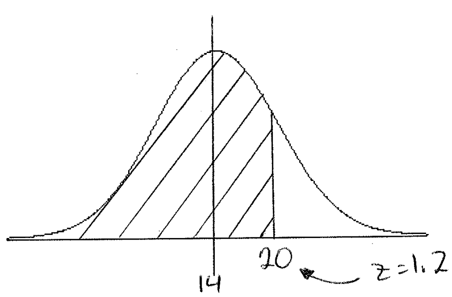
- *Each trip to school is an independent event.*
- *The dataset of all trips to school is continuous, not discrete.*
- *Barring anything drastic, almost all bicycle trips should take between 17 and 35 minutes (less than 20 minutes and more than 32 minutes being highly unlikely), while all car trips should take between 0 and 29 minutes (less than 4 minutes and more than 24 minutes being highly unlikely).*

If you choose to ride the bicycle, to get to school on time the journey time has to be less than 30 minutes.



$$P(X < 30) = P\left(Z < \frac{30-26}{3}\right) = P(Z < 1.333) = 0.909$$

If you choose to ride with the friend, in order to get to school on time the journey time has to be less than 20 minutes.



$$P(X < 20) = P\left(Z < \frac{20-14}{5}\right) = P(Z < 1.2) = 0.885$$

Since 0.909 is greater than 0.885, you are more likely to get to school on time if you ride the bicycle.

Scaffolding:

Students who are new to the curriculum or below grade level may not be familiar with normal probabilities. To calculate the area under a normal curve:

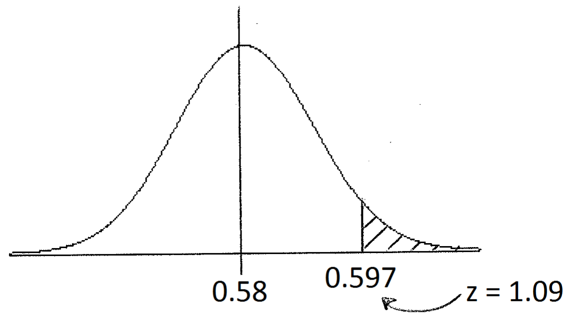
- Calculate the z-score using the formula $z = \frac{y-\mu}{\sigma}$ to two decimal places. The z-score is the number of standard deviations the data point is from the mean.
- Using a z-table, locate the percentage associated with that z-score. This value is the percentile associated with the data point, and the percentage of data under the curve is contained to the left of this point.
- Read the problem out loud and ask students to draw a picture to represent the information described.

Exercise 3 (7 minutes)

In this exercise, students examine a hypothesis about a proportion. Students should use probability to support their answer to part (b).

3. Recent polls have shown that 58% of voters in a city support a particular party. However, the party has just entered a new phase of its campaign, and in a new poll of $1,000$ randomly selected voters, the proportion of voters who support the party is found to be 0.597.

a. It is known that if 58% of all voters support the party, the proportion of people in a random sample of $1,000$ voters who support the party is approximately normally distributed with mean 0.58 and standard deviation 0.0156. If 58% of voters supported the party, what would be the probability of a sample proportion of 0.597 or more supporting the party? (Round your answer to the nearest thousandth.)



$$P(\text{sample proportion} \geq 0.597) = P\left(Z \geq \frac{0.597 - 0.58}{0.0156}\right) = P(Z \geq 1.090) = 0.138$$

b. Should the result of the new poll lead the party to think that support for the party has increased? Explain your answer using the probability in part (a).

If support for the party has remained the same, then it is not particularly unlikely (probability = 0.138) that you would get a sample proportion of 0.597 or more. So, there is no strong reason for the party to think that support has increased.

Exercise 4 (7 minutes)

MP.4 In this exercise, students use stated assumptions and compare strategies by calculating the probabilities of intersections of independent events in order to justify a decision. Students should recognize that because the outcomes of the sections are independent, the probability that Tim passes all three sections can be found by applying the multiplication rule. Some students may need assistance understanding that the probability of passing at least one section can be found by subtracting the probability that Tim fails on all sections from 1 (or $1 - P(\text{fail, fail, fail})$).

4. **In order to be admitted to a master's degree program, Tim must take a graduate record examination. A graduate record examination is in three sections: verbal reasoning, quantitative reasoning, and analytic writing. The exam is administered by two different companies. In Company A's version of the exam, Tim estimates that his probabilities of passing the three sections are 0.85, 0.95, and 0.90, respectively. In Company B's version, he estimates the probabilities of passing to be 0.92, 0.93, and 0.88 for the three sections. (You may assume that, with either of the companies, Tim's outcomes for the three sections of the exam are independent.)**

   a. **Which company should Tim choose if he must pass all three sections?**

   If Tim uses Company A, then the probability he will pass all three sections is $(0.85)(0.95)(0.90) = 0.727$.

   If Tim uses Company B, then the probability he will pass all three sections is $(0.92)(0.93)(0.88) = 0.753$.

   Since 0.753 *is greater than* 0.727*, Tim should use Company B.*

   b. **Which company should Tim choose if he must pass at least one of the sections?**

   If he uses Company A, then the probability that he will fail all three sections is $(0.15)(0.05)(0.10) = 0.000750$. *So, the probability that he passes at least one is* $1 - 0.000750 = 0.999250$.

   If he uses Company B, then the probability that he will fail all three sections is $(0.08)(0.07)(0.12) = 0.000672$. *So, the probability that he passes at least one is* $1 - 0.000672 = 0.999328$.

   Tim is almost certain to pass at least one section whichever company he uses, but his chances are slightly better with Company B.

Closing (2 minutes)

- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

Lesson Summary

If a number of strategies are available and the possible outcomes are success and failure, the best strategy is the one that has the highest probability of success.

Probability can be used to decide if there is evidence against a hypothesis. If, assuming that the hypothesis is true, the observed outcome is unlikely (probability < 0.05), then there is strong evidence that the hypothesis is false. If the observed outcome is not particularly unlikely, then there is *not* strong evidence that the hypothesis is false.

Exit Ticket (6 minutes)

Name ____ Date ____________________

Lesson 19: Analyzing Decisions and Strategies Using Probability

Exit Ticket

1. A coffee machine has two nozzles. It is known that the amount of coffee dispensed by the first nozzle is approximately normally distributed with mean 7.1 oz. and standard deviation 0.41 oz. and that the amount of coffee dispensed by the second nozzle is approximately normally distributed with mean 7.2 oz. and standard deviation 0.33 oz. If a person is using an 8 oz. cup, which nozzle should he use to minimize the probability that the cup will be overfilled and the coffee will spill?

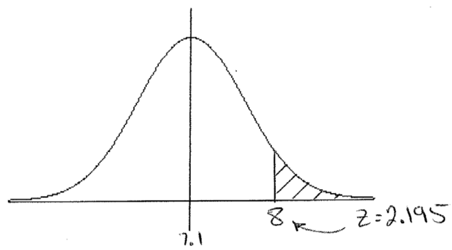
2. Ron's Joke Store offers a coin that is supposed to be weighted toward heads. Caitlin tries out one of these coins. She flips the coin three times and gets a head on all three flips.
   a. If the coin were fair, what would the probability be that Caitlin would get heads on all three flips? (Round your answer to the nearest thousandth.)

   b. Should Caitlin's result of heads on all three flips lead her to conclude that the coin is weighted toward heads? Explain.

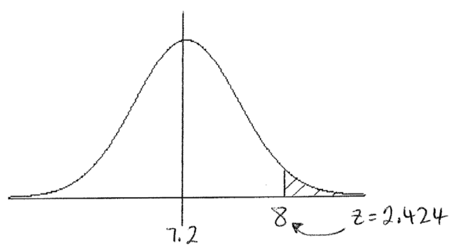
Exit Ticket Sample Solutions

1. A coffee machine has two nozzles. It is known that the amount of coffee dispensed by the first nozzle is approximately normally distributed with mean 7.1 oz. and standard deviation 0.41 oz. and that the amount of coffee dispensed by the second nozzle is approximately normally distributed with mean 7.2 oz. and standard deviation 0.33 oz. If a person is using an 8 oz. cup, which nozzle should he use to minimize the probability that the cup will be overfilled and the coffee will spill?

   Answer:



   Using the first nozzle, the probability that the coffee will spill is
   $$\boldsymbol{P(X > 8) = P\left(Z > \frac{8 - 7.1}{0.41}\right) = P(Z > 2.195) = 0.014.}$$



   Using the second nozzle, the probability that the coffee will spill is
   $$\boldsymbol{P(X > 8) = P\left(Z > \frac{8 - 7.2}{0.33}\right) = P(Z > 2.424) = 0.008.}$$

   The probability that the coffee will spill is less likely when using the second nozzle.

2. Ron's Joke Store offers a coin that is supposed to be weighted toward heads. Caitlin tries out one of these coins. She flips the coin three times and gets a head on all three flips.

   a. If the coin were fair, what would the probability be that Caitlin would get heads on all three flips? (Round your answer to the nearest thousandth.)

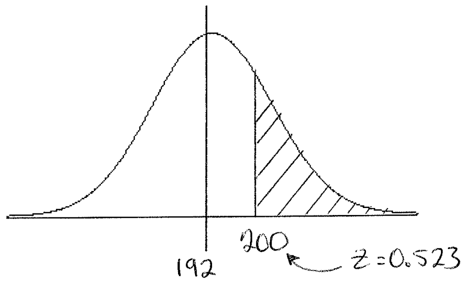
   $$\boldsymbol{\left(\frac{1}{2}\right)^3 = 0.125}$$

   b. Should Caitlin's result of heads on all three flips lead her to conclude that the coin is weighted toward heads? Explain.

   No. If the coin were fair, then it would not be particularly unlikely (probability $= 0.125$) that all three flips would result in heads. So, Caitlin's result does not give her strong evidence that the coin is weighted toward heads. If she had flipped the coin more times (say 5 or 6) and saw all heads, the evidence that the coin was weighted in favor of heads would be more compelling.

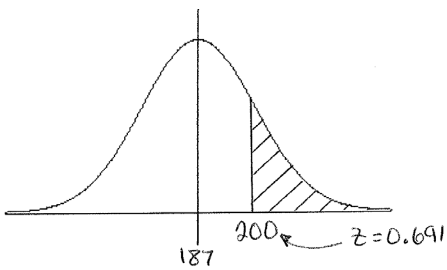
Problem Set Sample Solutions

1. To qualify for a vegetable-growing competition, onions must have a mass of at least 200 g. A grower has the choice of two different types of onion. The first type (when grown under certain conditions) grows to a mass that is approximately normally distributed with mean 192 g and standard deviation 15.3 g. The second (grown under the same conditions) grows to a mass that is approximately normally distributed with mean 187 g and standard deviation 18.8 g. Which of the two types is more likely to produce an onion that qualifies for the competition?



First type:

$$\begin{aligned} P(X \geq 200) &= P\left(Z \geq \frac{200-192}{15.3}\right) \\ &= P(Z \geq 0.523) \\ &= 0.300 \end{aligned}$$



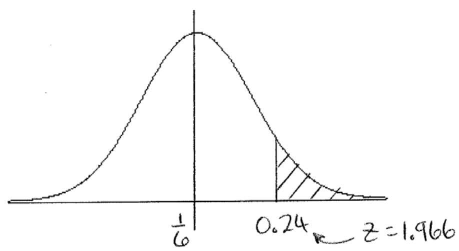
Second type:

$$\begin{aligned} P(X \geq 200) &= P\left(Z \geq \frac{200-187}{18.8}\right) \\ &= P(Z \geq 0.691) \\ &= 0.245 \end{aligned}$$

Since $0.300 > 0.245$, the first type is more likely to produce an onion that qualifies for the competition.

2. Ron's Joke Store sells both regular (fair) number cubes and weighted number cubes. Unfortunately, some of the number cubes have been mixed up. An employee rolls one of the number cubes 100 times, and the proportion of these rolls that result in sixes (the sample proportion) is 0.24.

   a. It is known that if a fair number cube is rolled 100 times, then the sample proportion of rolls that result in sixes is approximately normally distributed with mean $\frac{1}{6}$ and standard deviation 0.0373. If the number cube were fair, what would the probability be that the sample proportion in 100 rolls that result in sixes would be at least 0.24? (Round your answer to the nearest thousandth.)



$$P(\text{sample proportion} \geq 0.24) = P\left(Z \geq \frac{0.24 - \frac{1}{6}}{0.0373}\right) = P(Z \geq 1.966) = 0.025$$

   b. Does the employee's result provide strong evidence that the number cube is biased toward sixes? Explain your answer using the probability in part (a).

   Yes. If the number cube were fair, then it would be very unlikely (probability = 0.025) that you would get a sample proportion of sixes of at least 0.24. So, there is strong evidence that the number cube is biased toward sixes.

3. Alex and Max are twins, and they are both about to take exams in math, English, history, and science. Their parents have offered them special privileges (details to be announced) if they get A's in all their exams. For the four exams, Alex's probabilities of getting A's are 0.8, 0.9, 0.8, and 0.95, respectively. The equivalent probabilities for Max are 0.9, 0.9, 0.85, and 0.85. (You can assume that the results of the four tests are independent of one another.)

   a. Which of the twins is more likely to get *A*'s in all the exams?

$$P(\textit{Alex gets all A's}) = (0.8)(0.9)(0.8)(0.95) = 0.5472$$

$$P(\textit{Max gets all A's}) = (0.9)(0.9)(0.85)(0.85) = 0.5852$$

   Since 0.5852 is greater than 0.5472, Max is more likely to get all A's.

   b. Which of the twins is more likely to get at least one *A*?

   $P(\textit{Alex gets no A's}) = (0.2)(0.1)(0.2)(0.05) = 0.0002$
   So $P(\textit{Alex gets at least one A}) = 1 - 0.0002 = 0.9998$

   $P(\textit{Max gets no A's}) = (0.1)(0.1)(0.15)(0.15) = 0.000225$
   So $P(\textit{Max gets at least one A}) = 1 - 0.000225 = 0.999775$

   Alex is slightly more likely than Max to get at least one A.

4. **Roxy is a statistics teacher. She has a set of 52 cards, and she tells her class that there are 26 red and 26 black cards. Roxy shuffles the cards and offers cookies to the first student to select a red card. An eager volunteer starts to select cards at random; after each selection, the card is returned and the cards are shuffled. Having picked black cards on each of the first five selections, the volunteer exclaims that his teacher is up to one of her tricks. Does the student have strong evidence that the cards are not as Roxy described? Include a probability calculation in your answer.**

   If there were 26 red and 26 black cards in the deck, then the probability of randomly selecting five black cards would be $\left(\frac{1}{2}\right)^5 = 0.03125$; it would be very unlikely for all five of the student's cards to be black. So, yes, the student has strong evidence that the cards are not as Roxy described.

5. **You decide to lay tile on the floor in one room of your house. The room measures 120 sq. ft., and the tiles themselves each measure one square foot. You want to mix white, gray, and black tile in the ratio $3:2:1$, respectively. When the tile is delivered to your house, you open up the packages and realize that the tiles are completely assorted. You begin to sort them out. As you are sorting, you randomly pick six gray tiles in a row. You immediately look for the tile company's phone number and call them to have your order replaced because the ratio is not correct. Are you justified in doing this?**

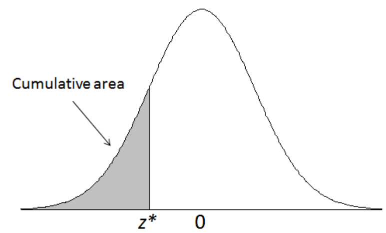
   A ratio of $3:2:1$ means that 20 tiles are black, 40 are gray, and 60 are white. The probability of choosing six gray tiles in a row is $\left(\frac{40}{120}\right)\left(\frac{39}{119}\right)\left(\frac{38}{118}\right)\left(\frac{37}{117}\right)\left(\frac{36}{116}\right)\left(\frac{35}{115}\right) \approx 0.00105$. Picking six gray tiles in this situation would occur about 1 in $1,000$ times, which is a very low probability. There is compelling evidence that the ratio of colors is not as it should be.

6. **The published universal distribution of M&Ms by color is 13% brown, 13% red, 14% yellow, 24% blue, 20% orange, and 16% green. You find a snack pack, which has about 20 candies; you open it, calculate the distribution, and then eat them all. In that pack, 15% are orange. Curious, you find a single-serve bag, containing about 50 candies. In this bag, 24% are orange. More intrigued, you go for a king-size bag, which ends up containing about 19% orange. You try some more bags, but none of the bags you try contain the actual published percentage of orange M&Ms. Does this make you think that their figures are wrong and that they should publish the correct percentages?**

   There is not enough evidence to claim that the candy company is wrong. The bags you chose are just a few of the millions of samples of M&Ms that exist, and the variation between bags is natural. In fact, it is probably normally distributed. The figure of 20% comes from two places: the average percentage of orange M&Ms across all bags and the possibility that the company actually counts the colors and sets the percentages as the candy is being made.

Appendix

Standard Normal Curve Areas



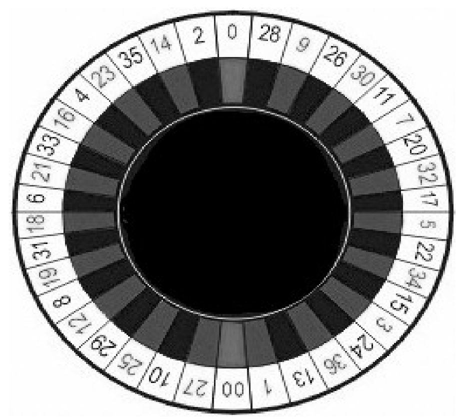
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0160	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0599
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Name ____ Date _____________________

1. An assembler of computer routers and modems uses parts from three sources. Company A supplies 60% of the parts, company B supplies 30% of the parts, and company C supplies the remaining 10% of the parts. From past experience, the assembler knows that 3% of the parts supplied by company A are defective, 5% of the parts supplied by company B are defective, and 8% of the parts supplied by company C are defective. If a part is selected at random, what is the probability it is defective?

2. In the game of roulette, a wheel with different slots is spun. A ball is placed in the wheel and bounces around until it settles in one of the 38 slots, all of equal size. There are 18 red slots, 18 black slots, and 2 green slots. Suppose it costs $\$1.00$ to play one round of the game. A “color bet” allows you to pick either red or black; if the ball lands on your color, you win $\$2.00$. If the ball lands on either of the other two colors, you do not win any money.



   a. In this game, for each spin you win either $\$2.00$ or $\$0.00$. Determine the probabilities of winning each amount.

   b. Calculate the expected winnings in one spin of the wheel.

   c. Are your expected winnings in one spin of the wheel larger or smaller than the $\$1.00$ bet?

d. A casino's profit is equal to the amount of bets minus the amount of winnings. For each spin of the Roulette wheel, the expected profit is $\$1.00$ minus your expected winnings. What is the casino's expected profit? Explain why the game of roulette is still attractive to a professional casino even though the expected profit to the casino is such a small amount on each spin.

e. The slots are also numbered 0, 00, 1–36. If you bet on a number and win, you win $\$36.00$. Which bet is better for you: a number bet or a color bet? Explain your decision.

f. Suppose you plan to construct a wheel with 12 slots: 5 red, 5 black, and 2 green. You plan to pay $\$5.00$ in winnings if someone picks a color (red or black) and the ball lands on that color. How much should you charge someone to play (the bet amount) so that you have created a fair game?

3. In New York, the Mega Ball jackpot lottery asks you to pick 5 numbers (integers) from 1 to 59 (the "upper section") and then pick a Mega Ball number from 1 to 35 (the "lower section"). You win $\$10{,}000$ if you match exactly 4 of the 5 numbers from the upper section and match the Mega Ball number from the lower section. The winning number(s) for each section are chosen at random without replacement. Determine the probability of correctly choosing exactly 4 of the winning numbers and the Mega Ball number.

4. A blood bank is screening a large population of donations for a particular virus. Suppose 5% of the blood donations contain the virus. Suppose you randomly select 10 bags of donated blood to screen. You decide to take a small amount from each of the bags and pool these all together into one sample. If that sample shows signs of the virus, you then test all of the original 10 bags individually. If the combined sample does not contain the virus, then you are done after just the one test.

   a. Determine the expected number of tests to screen 10 bags of donated blood using this strategy.

   b. Does this appear to be an effective strategy? Explain how you know.

5. Twenty-five sixth-grade students entered a math contest consisting of 20 questions. The student who answered the greatest number of questions correctly will receive a graphing calculator. The rules of the contest state that if two or more students tie for the greatest number of correct answers, one of these students will be chosen to receive the calculator.

   No student answered all 20 questions correctly, but four students (Allan, Beth, Carlos, and Denesha) each answered 19 questions correctly.

   What would be a fair way to use two coins (a dime and a nickel) to decide which student should get the calculator? Explain what makes your method fair.

6. A cell phone company offers cell phone insurance for $\$7.00$ a month. If your phone breaks and you submit a claim, you must first pay a $\$200.00$ deductible before the cell phone company pays anything. Suppose the replacement cost for a phone is $\$650.00$. This means if you break your phone and have insurance, you have to pay only $\$200.00$ toward the replacement cost. This plan has a limit of two replacements; if you break your phone more than twice in one year, you pay for the full replacement cost for the additional replacements.

   Suppose that within one year, there is a 48% chance that you do not break your phone, a 36% chance that you break it once, a 12% chance that you break it twice, a 3% chance that you break it three times, and a 1% chance that you break it four times.

   a. Calculate the expected one-year cost of this insurance plan based on the monthly cost and the expected repair costs.

   b. Determine your expected replacement costs if you do not purchase insurance.

   c. Does this insurance plan seem to be a good deal? Explain why or why not.

A Progression Toward Mastery					
Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	S-CP.B.8	Student does not recognize this as a probability calculation or combines the given numbers in a nonsensical way.	Student identifies components of the problem but is not able to combine the provided information in an appropriate way. AND/OR Student shows how the components are related to the law of total probability but ends up with a probability outside $(0,1)$. Student may also confuse the law of total probability with Bayes's theorem.	Student identifies the correct conditional and unconditional probabilities but makes a calculation error, such as weighting the companies equally $(0.03+0.05+0.08)/3$.	Student identifies the correct conditional and unconditional probabilities and correctly performs the probability calculation. Student supports answer with formulas and/or diagrams.
2	a S-MD.A.3 S-MD.B.5	Student assumes the probabilities are both 0.50 because there are only two outcomes (win, lose).	Student finds the probabilities for each slot but does not combine into the outcomes of winning and losing.	Student finds the probabilities of winning and losing but does not relate to dollar amounts.	Student finds the probabilities for $\$2.00$ and $\$0.00$ based on the roulette slots being equally likely.
	b S-MD.B.5	Student is not able to determine an expected value.	Student attempts to find an expected value but does not use the probability distribution from (a).	Student uses the probability distribution from (a) to calculate an expected amount but then rounds to $\$1.00$.	Student uses the probability distribution from (a) to calculate an expected amount.

	c S-MD.B.5	Student is not able to compare the values.	Student does not use the expected winnings calculation.	Student makes a comparison but not the correct values. For example, student tries to utilize "net winnings" instead.	Student compares the answer in (b) to $\$1.00$ and indicates which is larger.
	d S-MD.A.2	Student only focuses on the casino using unfair games.	Student brings in outside information but does not argue based on expected profit for large numbers of customers.	Student understands that the casino can be confident they will win $\$0.05$ per spin, on average, but does not relate this to large numbers of spins.	Student understands that the answer to (c) is not large but argues based on the law of large numbers.
	e S-MD.B.7	Student does not consider the probability distribution and the change in the amount won.	Student compares the bets but does not calculate expected winnings for the numbered bets.	Student argues based on expected values but does not compare bets or does not show calculation details to support answer.	Student calculates the expected winnings for the numbered bet and makes a comparative statement.
	f S-MD.B.6	Student does not use the given probability distribution.	Student calculates the probability of winning but does not relate to the amount bet.	Student calculates the probability of winning but assumes a $\$1.00$ bet.	Student calculates the expected winnings and indicates that the cost should be at least that amount.
3	S-CP.B.9	Student brings in outside information without using the provided probability distribution.	Student attempts to use the multiplication rule but is not able to correctly calculate the probability.	Student determines the probability as one out of the number of possible combinations but does not separate the upper section and lower section.	Student determines the probability as one out of the number of possible combinations.
4	a S-MD.B.7	Student is not able to calculate a probability or an expected value from the given information.	Student only calculates the probability of needing a retest. Student is not able to convert to an expected number of tests.	Student considers the initial test and retest but does not calculate the probability of needing a retest correctly or does not have the correct outcomes for $x =$ number of tests (e.g., 1 and 10).	Student calculates the expected number of tests considering the initial test and the retests.
	b S-MD.B.7	Student does not answer based on expected number of tests between the two strategies.	Student considers the expected number of tests from (a) but does not compare to 10.	Student finds the strategy to be effective but does not clearly explain why.	Student compares the expected number of tests to 10.

5	**S-MD.B.6**	Student is not able to describe a fair way to decide who should be awarded the calculator.	Student describes a method that is not fair or that does not use two coins.	Student describes a fair method using two coins, but the explanation does not indicate that each student is equally likely to be awarded the calculator.	Student describes a fair method using two coins, and the explanation indicates that each student is equally likely to be awarded the calculator.
6	**a** **S-MD.B.7**	Student does not calculate an expected cost.	Student considers the costs but does not combine them with the probabilities.	Student makes a calculation error (e.g., does not consider deductible) or does not show sufficient calculation detail.	Student combines the information to calculate an expected repair cost.
	b **S-MD.B.7**	Student is not able to calculate an expected cost.	Student states an expected cost but not based on the probabilities of number of phone breaks.	Student calculates an expected cost but makes a calculation error (e.g., considers information from the plan) or does not show sufficient calculation detail.	Student calculates the expected costs based on probabilities of breaking the phone 0–4 times.
	c **S-MD.B.7**	Student brings in outside information, and the argument is not based on previous calculations.	Student answers based on probability distributions rather than expected costs.	Student answers based on expected costs but does not clearly compare the answers to (a) and (b).	Student compares the two expected values.

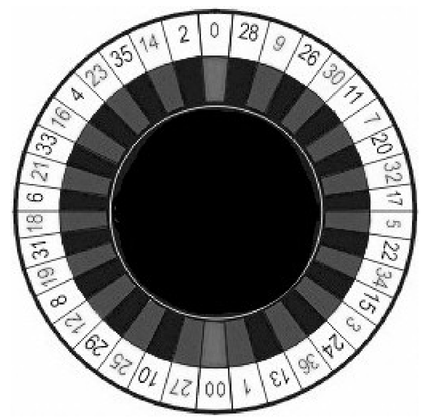
Name ____ Date ______________________

1. An assembler of computer routers and modems uses parts from three sources. Company A supplies 60% of the parts, company B supplies 30% of the parts, and company C supplies the remaining 10% of the parts. From past experience, the assembler knows that 3% of the parts supplied by company A are defective, 5% of the parts supplied by company B are defective, and 8% of the parts supplied by company C are defective. If a part is selected at random, what is the probability it is defective?

$P(\text{defective}) = P(\text{defective}|A)P(A) + P(\text{defective}|B)P(B) + P(\text{defective}|C)P(C)$

$= 0.6(0.03) + 0.3(0.05) + 0.1(0.08) = 0.041.$

2. In the game of roulette, a wheel with different slots is spun. A ball is placed in the wheel and bounces around until it settles in one of the 38 slots, all of equal size. There are 18 red slots, 18 black slots, and 2 green slots. Suppose it costs $\$1.00$ to play one round of the game. A "color bet" allows you to pick either red or black; if the ball lands on your color, you win $\$2.00$. If the ball lands on either of the other two colors, you do not win any money.



   a. In this game, for each spin you win either $\$2.00$ or $\$0.00$. Determine the probabilities of winning each amount.

   $P(\$2) = \frac{18}{38} = 0.4737.$

   $P(\$0) = \frac{20}{38} = 0.5263.$

   b. Calculate the expected winnings in one spin of the wheel.

   expected winnings $= 2(18/38) + 0(20/38) = \frac{36}{38} = 0.9474.$

   c. Are your expected winnings in one spin of the wheel larger or smaller than the $\$1.00$ bet?

   This is smaller than the $1.00 bet (about 5 cents).

d. A casino's profit is equal to the amount of bets minus the amount of winnings. For each spin of the Roulette wheel, the expected profit is $1.00 minus your expected winnings. What is the casino's expected profit? Explain why the game of roulette is still attractive to a professional casino even though the expected profit to the casino is such a small amount on each spin.

The casino's expected winnings are about 5 cents per spin. The law of large numbers states that if they can get enough people to play, the average winnings will be very close to the expected winnings. So, the casino will be pretty sure of being close to the expected winnings. Even a small amount of expected winnings will add up over a large number of spins.

e. The slots are also numbered 0, 00, 1–36. If you bet on a number and win, you win $36.00. Which bet is better for you: a number bet or a color bet? Explain how you are deciding.

expected winnings $= 36\left(\frac{1}{38}\right) = \frac{36}{38}$.
The expected winnings are the same for the two bets. (So, you could argue that there isn't a difference, or you could argue in terms of the thrill vs. security of the bets.)

f. Suppose you plan to construct a wheel with 12 slots: 5 red, 5 black, and 2 green. You plan to pay $5.00 in winnings if someone picks a color (red or black) and the ball lands on that color. How much should you charge someone to play (the bet amount) so that you have created a fair game?

expected winnings $= 5\left(\frac{5}{12}\right) = \frac{25}{12}$. *So, you should charge approximately $2.08 to break even. You should also accept $2.09 to cover the rounding from $2.083.*

3. In New York, the Mega Ball jackpot lottery asks you to pick 5 numbers (integers) from 1 to 59 (the "upper section") and then pick a Mega Ball number from 1 to 35 (the "lower section"). You win $10,000 if you match exactly 4 of the 5 numbers from the upper section and match the Mega Ball number from the lower section. The winning number(s) for each section are chosen at random without replacement. Determine the probability of correctly choosing exactly 4 of the winning numbers and the Mega Ball number.

P(4 winning numbers and 1 Mega Ball) $= \left(\frac{1}{C(56, 5)}\right) C(35, 1)$.
The probability of correctly choosing exactly 4 of the winning numbers and the Mega Ball number is 0.00000000748 (7.48 x10⁻⁹)

4. A blood bank is screening a large population of donations for a particular virus. Suppose 5% of the blood donations contain the virus. Suppose you randomly select 10 bags of donated blood to screen. You decide to take a small amount from each of the bags and pool these all together into one sample. If that sample shows signs of the virus, you then test all of the original 10 bags individually. If the combined sample does not contain the virus, then you are done after just the one test.

   a. Determine the expected number of tests to screen 10 bags of donated blood using this strategy.

   If none of the blood donations have the virus, then you only have to conduct one test. The probability that none of the (independent) people have the virus is $(1 - 0.05)^{10} = 0.5987$.
   So, the expected number of tests is $1(0.5987) + 11(1 - 0.5987) = 5.01$.

   b. Does this appear to be an effective strategy? Explain how you know.

   This expected value in part (a) is much less than 10, so this seems to be an effective strategy. In the long run, you will perform fewer tests.

5. Twenty-five sixth-grade students entered a math contest consisting of 20 questions. The student who answered the greatest number of questions correctly will receive a graphing calculator. The rules of the contest state that if two or more students tie for the greatest number of correct answers, one of these students will be chosen to receive the calculator.

   No student answered all 20 questions correctly, but four students (Allan, Beth, Carlos, and Denesha) each answered 19 questions correctly.

   What would be a fair way to use two coins (a dime and a nickel) to decide which student should get the calculator? Explain what makes your method fair.

   Answers will vary. One possible response is to toss the coins, and if both coins land heads up (HH), Allan gets the calculator. If the dime lands heads up and the nickel lands tails up (HT), Beth gets the calculator. If the dime lands tails up and the nickel lands heads up (TH), Carlos gets the calculator. If both coins land tails up (TT), Denesha gets the calculator. This method is fair because each of the four students has the same chance (a probability of $\frac{1}{4}$*) of getting the calculator.*

6. A cell phone company offers cell phone insurance for $7.00 a month. If your phone breaks and you submit a claim, you must first pay a $200.00 deductible before the cell phone company pays anything. Suppose the replacement cost for a phone is $650.00. This means if you break your phone and have insurance, you have to pay only $200.00 toward the replacement cost. This plan has a limit of two replacements; if you break your phone more than twice in one year, you pay for the full replacement cost for the additional replacements.

   Suppose that within one year, there is a 48% chance that you do not break your phone, a 36% chance that you break it once, a 12% chance that you break it twice, a 3% chance that you break it three times, and a 1% chance that you break it four times.

   a. Calculate the expected one-year cost of this insurance plan based on the monthly cost and the expected repair costs.

      If you do not break your phone, then you pay $7 × 12 = $84.
      If you break your phone once, then you pay $200 + $84 = $284.
      If you break your phone twice, then you pay $400 + $84 = $484.
      If you break your phone three times, then you pay $484 + $650 = $1134.
      If you break your phone four times, then you pay $1134 + $650 = $1784.
      The probability distribution of costs is as follows:

$84	$284	$484	$1134	$1784
0.48	0.36	0.12	0.03	0.01

      Therefore, your expected costs are
      $0.48(84) + 0.36(284) + 0.12(484) + 0.03(1134) + 0.01(1784) = \$252.50.$

   b. Determine your expected replacement costs if you do not purchase insurance.

      $0.48(0) + 0.36(650) + 0.12(1300) + 0.03(1950) + 0.01(2600) = \$474.50.$

   c. Does this insurance plan seem to be a good deal? Explain why or why not.

      Yes, the long-run average costs are considerably less with the insurance plan than without it.