Lesson 1: The General Multiplication Rule

Classwork

Example 1: Independent Events

Do you remember when breakfast cereal companies placed prizes in boxes of cereal? Possibly you recall that when a certain prize or toy was particularly special to children, it increased their interest in trying to get that toy. How many boxes of cereal would a customer have to buy to get that toy? Companies used this strategy to sell their cereal.

One of these companies put one of the following toys in its cereal boxes: a block (B), a toy watch (W), a toy ring (R), and a toy airplane (A). A machine that placed the toy in the box was programmed to select a toy by drawing a random number of 1 to 4. If a 1 was selected, the block (or B) was placed in the box; if a 2 was selected, a watch (or W) was placed in the box; if a 3 was selected, a ring (or R) was placed in the box; and if a 4 was selected, an airplane (or A) was placed in the box. When this promotion was launched, young children were especially interested in getting the toy airplane.

Exercises 1–8

1. If you bought one box of cereal, what is your estimate of the probability of getting the toy airplane? Explain how you got your answer.

2. If you bought a second box of cereal, what is your estimate of the probability of getting the toy airplane in the second box? Explain how you got your answer.

3. If you bought two boxes of cereal, does your chance of getting at least one airplane increase or decrease? Explain your answer.

4. Do you think the probability of getting at least one airplane from two boxes is greater than 0.5? Again, explain your answer.

5. List all of the possibilities of getting two toys from two boxes of cereals. (Hint: Think of the possible outcomes as ordered pairs. For example, BA would represent a block from the first box and an airplane from the second box.)

6. Based on the list you created, what do you think is the probability of each of the following outcomes if two cereal boxes are purchased?
   a. One (and only one) airplane

   b. At least one airplane

   c. No airplanes

7. Consider the purchase of two cereal boxes.
   a. What is the probability of getting an airplane in the first cereal box? Explain your answer.

   b. What is the probability of getting an airplane in the second cereal box?

   c. What is the probability of getting airplanes in both cereal boxes?

$P(A \text{ and } B)$ is the probability that Events A and B both occur and is the probability of the **intersection** of A and B. The probability of the intersection of Events A and B is sometimes also denoted by $P(A \cap B)$.

Multiplication Rule for Independent Events

If A and B are independent events, $P(A \text{ and } B) = P(A) \cdot P(B)$

This rule generalizes to more than two independent events, for example:

$P(A \text{ and } B \text{ and } C)$ or $P(A \text{ intersect } B \text{ intersect } C) = P(A) \cdot P(B) \cdot P(C)$

8. Based on the multiplication rule for independent events, what is the probability of getting an airplane in both boxes? Explain your answer.

Example 2: Dependent Events

Do you remember the famous line, "Life is like a box of chocolates," from the movie *Forrest Gump*? When you take a piece of chocolate from a box, you never quite know what the chocolate will be filled with. Suppose a box of chocolates contains 15 **identical-looking** pieces. The 15 are filled in this manner: 3 caramel, 2 cherry cream, 2 coconut, 4 chocolate whip, and 4 fudge.

Exercises 9–14

9. If you randomly select one of the pieces of chocolate from the box, what is the probability that the piece will be filled with fudge?

10. If you randomly select a second piece of chocolate (after you have eaten the first one, which was filled with fudge), what is the probability that the piece will be filled with caramel?

The events, *picking a fudge-filled piece on the first selection* and *picking a caramel-filled piece on the second selection*, are called **dependent** events.

Two events are dependent if knowing that one has occurred changes the probability that the other occurs.

Multiplication Rule for Dependent Events

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Recall from your previous work with probability in Algebra II that $P(B|A)$ is the conditional probability of event B given that event A occurred. If event A is *picking a fudge-filled piece on the first selection* and event B is *picking a caramel-filled piece on the second selection*, then $P(B|A)$ represents the probability of picking a caramel-filled piece second knowing that a fudge-filled piece was selected first.

11. If $A1$ is the event *picking a fudge-filled piece on the first selection* and $B2$ is the event *picking a caramel-filled piece on the second selection*, what does $P(A1 \text{ and } B2)$ represent? Find $P(A1 \text{ and } B2)$.

12. What does $P(B1 \text{ and } A2)$ represent? Calculate this probability.

13. If C represents selecting a coconut-filled piece of chocolate, what does $P(A1 \text{ and } C2)$ represent? Find this probability.

14. Find the probability that both the first and second pieces selected are filled with chocolate whip.

Exercises 15–17

15. For each of the following, write the probability as the intersection of two events. Then, indicate whether the two events are independent or dependent, and calculate the probability of the intersection of the two events occurring.
    a. The probability of selecting a 6 from the first draw and a 7 on the second draw when two balls are selected without replacement from a container with 10 balls numbered 1 to 10.

b. The probability of selecting a 6 on the first draw and a 7 on the second draw when two balls are selected with replacement from a container with 10 balls numbered 1 to 10.

c. The probability that two people selected at random in a shopping mall on a very busy Saturday both have a birthday in the month of June. Assume that all 365 birthdays are equally likely and ignore the possibility of a February 29 leap-year birthday.

d. The probability that two socks selected at random from a drawer containing 10 black socks and 6 white socks will both be black.

16. A gumball machine has gumballs of 4 different flavors: sour apple (A), grape (G), orange (O), and cherry (C). There are six gumballs of each flavor. When $50¢$ is put into the machine, two random gumballs come out. The event $C1$ means a cherry gumball came out first, the event $C2$ means a cherry gumball came out second, the event $A1$ means sour apple gumball came out first, and the event $G2$ means a grape gumball came out second.

a. What does $P(C2|C1)$ mean in this context?

b. Find $P(C1 \text{ and } C2)$.

c. Find $P(A1 \text{ and } G2)$.

17. Below are the approximate percentages of the different blood types for people in the United States.

    Type O 44%

    Type A 42%

    Type B 10%

    Type AB 4%

    Consider a group of 100 people with a distribution of blood types consistent with these percentages. If two people are randomly selected with replacement from this group, what is the probability that

    a. both people have type O blood?

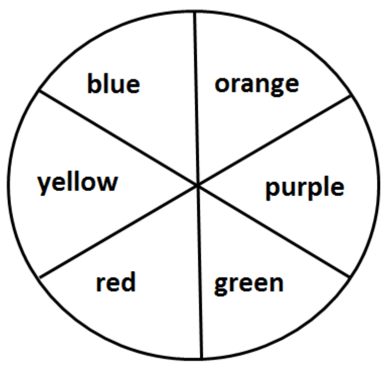
    b. the first person has type A blood and the second person has type AB blood?

Lesson Summary

- Two events are independent if knowing that one occurs does not change the probability that the other occurs.
- Two events are dependent if knowing that one occurs changes the probability that the other occurs.
- **GENERAL MULTIPLICATION RULE:**
  $P(A \text{ and } B) = P(A) \cdot P(B|A)$
  If A and B are independent events then $P(B|A) = P(B)$.

Problem Set

1. In a game using the spinner below, a participant spins the spinner twice. If the spinner lands on red both times, the participant is a winner.



   a. The event *participant is a winner* can be thought of as the intersection of two events. List the two events.
   b. Are the two events independent? Explain.
   c. Find the probability that a participant wins the game.

2. The overall probability of winning a prize in a weekly lottery is $\frac{1}{32}$. What is the probability of winning a prize in this lottery three weeks in a row?

3. A Gallup poll reported that 28% of adults (age 18 and older) eat at a fast food restaurant about once a week. Find the probability that two randomly selected adults would both say they eat at a fast food restaurant about once a week.

4. In the game *Scrabble*, there are a total of 100 tiles. Of the 100 tiles, 42 tiles have the vowels A, E, I, O, and U printed on them, 56 tiles have the consonants printed on them, and 2 tiles are left blank.
   a. If tiles are selected at random, what is the probability that the first tile drawn from the pile of 100 tiles is a vowel?
   b. If tiles drawn are not replaced, what is the probability that the first two tiles selected are both vowels?
   c. Event A is *drawing a vowel*, event B is *drawing a consonant*, and event C is *drawing a blank tile*. $A1$ means a vowel is drawn on the first selection, $B2$ means a consonant is drawn on the second selection, and $C2$ means a blank tile is drawn on the second selection. Tiles are selected at random and without replacement.
      i. Find $P(A1 \text{ and } B2)$
      ii. Find $P(A1 \text{ and } C2)$
      iii. Find $P(B1 \text{ and } C2)$

5. To prevent a flooded basement, a homeowner has installed two special pumps that work automatically and independently to pump water if the water level gets too high. One pump is rather old and does not work 28% of the time, and the second pump is newer and does not work 9% of the time. Find the probability that both pumps will fail to work at the same time.

6. According to a recent survey, approximately 77% of Americans get to work by driving alone. Other methods for getting to work are listed in the table below.

Method of getting to work	Percent of Americans using this method
Taxi	0.1%
Motorcycle	0.2%
Bicycle	0.4%
Walk	2.5%
Public Transportation	4.7%
Car Pool	10.7%
Drive Alone	77%
Work at Home	3.7%
Other	0.7%

   a. What is the probability that a randomly selected worker drives to work alone?
   b. What is the probability that two workers selected at random with replacement both drive to work alone?

7. A bag of M&Ms contains the following distribution of colors:

   9 blue

   6 orange

   5 brown

   5 green

   4 red

   3 yellow

   Three M&Ms are randomly selected without replacement. Find the probabilities of the following events.

   a. All three are blue.
   b. The first one selected is blue, the second one selected is orange, and the third one selected is red.
   c. The first two selected are red, and the third one selected is yellow.

8. Suppose in a certain breed of dog, the color of fur can either be tan or black. Eighty-five percent of the time, a puppy will be born with tan fur, while 15% of the time, the puppy will have black fur. Suppose in a future litter, six puppies will be born.

   a. Are the events *having tan fur* and *having black fur* independent? Explain.
   b. What is the probability that one puppy in the litter will have black fur and another puppy will have tan fur?
   c. What is the probability that all six puppies will have tan fur?
   d. Is it likely for three out of the six puppies to be born with black fur? Justify mathematically.

9. Suppose that in the litter of six puppies from Exercise 8, five puppies are born with tan fur, and one puppy is born with black fur.

   a. You randomly pick up one puppy. What is the probability that puppy will have black fur?
   b. You randomly pick up one puppy, put it down, and randomly pick up a puppy again. What is the probability that both puppies will have black fur?
   c. You randomly pick up two puppies, one in each hand. What is the probability that both puppies will have black fur?
   d. You randomly pick up two puppies, one in each hand. What is the probability that both puppies will have tan fur?

Lesson 2: Counting Rules—The Fundamental Counting Principle and Permutations

Classwork

Example 1: Fundamental Counting Principle

A restaurant offers a fixed-price dinner menu for $30. The dinner consists of three courses, and the diner chooses one item for each course.

The menu is shown below:

First Course	Second Course	Third Course
Salad	Burger	Cheesecake
Tomato Soup	Grilled Shrimp	Ice Cream Sundae
French Onion Soup	Mushroom Risotto	
	Ravioli	

Exercises 1–4

1. Make a list of all of the different dinner fixed-price meals that are possible. How many different meals are possible?

2. For many computer tablets, the owner can set a 4-digit pass code to lock the device.
   a. How many digits could you choose from for the first number of the pass code?

b. How many digits could you choose from for the second number of the pass code? Assume that the numbers can be repeated.

c. How many different 4-digit pass codes are possible? Explain how you got your answer.

d. How long (in hours) would it take someone to try every possible code if it takes three seconds to enter each possible code?

3. The store at your school wants to stock sweatshirts that come in four sizes (small, medium, large, xlarge) and in two colors (red and white). How many different types of sweatshirts will the store have to stock?

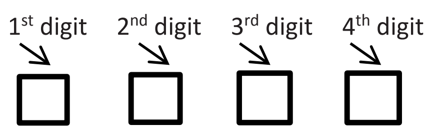
4. The call letters for all radio stations in the United States start with either a W (east of the Mississippi river) or a K (west of the Mississippi River) followed by three other letters that can be repeated. How many different call letters are possible?

Example 2: Permutations

Suppose that the 4-digit pass code a computer tablet owner uses to lock the device *cannot* have any digits that repeat. For example, 1234 is a valid pass code. However, 1123 is not a valid pass code since the digit "1" is repeated.

An arrangement of four digits with no repeats is an example of a permutation. A **permutation** is an arrangement in a certain order (a sequence).

How many different 4-digit pass codes are possible if digits cannot be repeated?



Exercises 5–9

5. Suppose a password requires three distinct letters. Find the number of permutations for the three letters in the code, if the letters may not be repeated.

6. The high school track has 8 lanes. In the 100 meter dash, there is a runner in each lane. Find the number of ways that 3 out of the 8 runners can finish first, second, and third.

7. There are 12 singers auditioning for the school musical. In how many ways can the director choose first a lead singer and then a stand-in for the lead singer?

8. A home security system has a pad with 9 digits (1 to 9). Find the number of possible 5-digit pass codes:
   a. if digits can be repeated.

b. if digits cannot be repeated,

9. Based on the patterns observed in Exercises 5–8, describe a general formula that can be used to find the number of permutations of n things taken r at a time or ${}_nP_r$.

Example 3: Factorials and Permutations

You have purchased a new album with 12 music tracks and loaded it onto your MP3 player. You set the MP3 player to play the 12 tracks in a random order (no repeats). How many different orders could the songs be played in?

This is the permutation of 12 things taken 12 at a time, or
${}_{12}P_{12} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 479{,}001{,}600$

The notation 12! is read 12 **factorial** and equals $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Factorials and Permutations

The factorial of a non-negative integer n is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 1$$

Note: $0!$ is defined to equal 1.

The number of permutations can also be found using factorials. The number of permutations of n things taken r at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!}.$$

Exercises 10–15

10. If 9! is $362{,}880$, find $10!$.

11. How many different ways can the 16 numbered pool balls be placed in a line on the pool table?

12. Ms. Smith keeps eight different cookbooks on a shelf in one of her kitchen cabinets. How many ways can the eight cookbooks be arranged on the shelf?

13. How many distinct 4-letter groupings can be made with the letters from the word *champion* if letters may not be repeated?

14. There are 12 different rides at an amusement park. You buy five tickets that allow you to ride on five different rides. In how many different orders can you ride the five rides in? How would your answer change if you could repeat a ride?

15. In the summer Olympics, 12 divers advance to the finals of the 3-meter springboard diving event. How many different ways can the divers finish 1st, 2nd, or 3rd?

Lesson Summary

- Let n_1 be the number of ways the first step or event can occur and n_2 be the number of ways the second step or event can occur. Continuing in this way, let n_k be the number of ways the kth stage or event can occur. Then based on fundamental counting principle, the total number of different ways the process can occur is $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.
- The factorial of a non-negative integer n is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1.$$

  Note: $0!$ is defined to equal 1.
- The number of permutations of n things taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

Problem Set

1. For each of the following, show the substitution in the permutation formula and find the answer.
   a. $_4P_4$
   b. $_{10}P_2$
   c. $_5P_1$

2. A serial number for a TV begins with three letters, is followed by six numbers, and ends in one letter. How many different serial numbers are possible? Assume the letters and numbers can be repeated.

3. In a particular area code, how many phone numbers (###-####) are possible? The first digit cannot be a zero and assume digits can be repeated.

4. There are four NFL teams in the AFC east: Bills, Jets, Dolphins, and Patriots. How many different ways can two of the teams finish first and second?

5. How many ways can 3 of 10 students come in first, second, and third place in a spelling contest, if there are no ties?

6. In how many ways can a president, a treasurer, and a secretary be chosen from among nine candidates if no person can hold more than one position?

7. How many different ways can a class of 22 second graders line up to go to lunch?

8. Describe a situation that could be modeled by using $_5P_2$.

9. To order books from an online site, the buyer must open an account. The buyer needs a username and a password.
   a. If the username needs to be eight letters, how many different usernames are possible:
      i. If letters can be repeated?
      ii. If the letters cannot be repeated?

   b. If the password must be eight characters, which can be any of the 26 letters, 10 digits, and 12 special keyboard characters, how many passwords are possible:
      i. If characters can be repeated?
      ii. If characters cannot be repeated?

   c. How would your answers to part (b) change if the password is case-sensitive? (In other words, *Password* and *password* are considered different because the letter *p* is in uppercase and lowercase.)

10. Create a scenario to explain why ${}_3P_3 = 3!$.

11. Explain why ${}_nP_n = n!$ for all positive integers n.

Lesson 3: Counting Rules—Combinations

Classwork

Example 1

Seven speed skaters are competing in an Olympic race. The first-place skater earns the gold medal, the second-place skater earns the silver medal, and the third-place skater earns the bronze medal. In how many different ways could the gold, silver, and bronze medals be awarded? The letters A, B, C, D, E, F, and G will be used to represent these seven skaters.

How can we determine the number of different possible outcomes? How many are there?

Now consider a slightly different situation. Seven speed skaters are competing in an Olympic race. The top three skaters move on to the next round of races. How many different "top three" groups can be selected?

How is this situation different from the first situation? Would you expect more of fewer possibilities in this situation? Why?

Would you consider the outcome where skaters B, C, and A advance to the final to be a different outcome than A, B, and C advancing?

A permutation is an ordered arrangement (a sequence) of k items from a set of n distinct items.

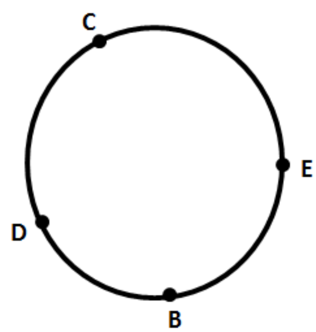
In contrast, a combination is an unordered collection (a set) of k items from a set of n distinct items.

When we wanted to know how many ways there are for seven skaters to finish first, second, and third, order was important. This is an example of a permutation of 3 selected from a set of 7. If we want to know how many possibilities there are for which three skaters will advance to the finals, order is not important. This is an example of a combination of 3 selected from a set of 7.

Exercises 1–4

1. Given four points on a circle, how many different line segments connecting these points do you think could be drawn? Explain your answer.

2. Draw a circle and place four points on it. Label the points as shown. Draw segments (chords) to connect all the pairs of points. How many segments did you draw? List each of the segments that you drew. How does the number of segments compare to your answer in Exercise 1?



You can think of each segment as being identified by a subset of two of the four points on the circle. Chord ED is the same as chord DE. The order of the segment labels is not important. When you count the number of segments (chords), you are counting combinations of two points chosen from a set of four points.

3. Find the number of permutations of two points from a set of four points. How does this answer compare to the number of segments you were able to draw?

4. If you add a fifth point to the circle, how many segments (chords) can you draw?

   If you add a sixth point, how many segments (chords) can you draw?

Example 2

Let's look closely at the four examples we have studied so far.

Choosing gold, silver, and bronze medal skaters	Choosing groups of the top three skaters
Finding the number of segments that can be drawn connecting two points out of four points on a circle	Finding the number of *unique* segments that can be drawn connecting two points out of four points on a circle

What do you notice about the way these are grouped?

The number of combinations of k items selected from a set of n distinct items is

$$ {}_nC_k = \frac{{}_nP_k}{k!} \text{ or } \quad {}_nC_k = \frac{n!}{k!\,(n-k)!}\,. $$

Exercises 5–11

5. Find the value of each of the following:

   a. ${}_9C_2$

   b. ${}_7C_7$

   c. ${}_8C_0$

   d. ${}_{15}C_1$

6. Find the number of segments (chords) that can be drawn for each of the following:

   a. 5 points on a circle

   b. 6 points on a circle

   c. 20 points on a circle

   d. n points on a circle

7. For each of the following questions, indicate whether the question posed involves permutations or combinations. Then provide an answer to the question with an explanation for your choice.

   a. A student club has 20 members. How many ways are there for the club to choose a president and a vice-president?

   b. A football team of 50 players will choose two co-captains. How many different ways are there to choose the two co-captains?

c. There are seven people who meet for the first time at a meeting. They shake hands with each other and introduce themselves. How many handshakes have been exchanged?

d. At a particular restaurant, you must choose two different side dishes to accompany your meal. If there are eight side dishes to choose from, how many different possibilities are there?

e. How many different four-letter sequences can be made using the letters A, B, C, D, E, and F if letters may not be repeated?

8. How many ways can a committee of 5 students be chosen from a student council of 30 students? Is the order in which the members of the committee are chosen important?

9. Brett has ten distinct t-shirts. He is planning on going on a short weekend trip to visit his brother in college. He has enough room in his bag to pack four t-shirts. How many different ways can he choose four t-shirts for his trip?

10. How many three-topping pizzas can be ordered from the list of toppings below? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?

Pizza Toppings

sausage	pepperoni	meatball	onions	olives	spinach
pineapple	ham	green peppers	mushrooms	bacon	hot peppers

11. Write a few sentences explaining how you can distinguish a question about permutations from a question about combinations.

Lesson Summary

A **combination** is a subset of k items selected from a set of n distinct items.

The number of combinations of k items selected from a set of n distinct items is

$$_nC_k = \frac{_nP_k}{k!} \quad \text{or} \quad _nC_k = \frac{n!}{k!\,(n-k)!}.$$

Problem Set

1. Find the value of each of the following:
   a. $_9C_8$
   b. $_9C_1$
   c. $_9C_9$

2. Explain why $_6C_4$ is the same value as $_6C_2$.

3. Pat has 12 books he plans to read during the school year. He decides to take 4 of these books with him while on winter break vacation. He decides to take *Harry Potter and the Sorcerer's Stone* as one of the books. In how many ways can he select the remaining 3 books?

4. In a basketball conference of 10 schools, how many conference basketball games are played during the season if the teams all play each other exactly once?

5. Which scenario(s) below is represented by $_9C_3$?
   a. Number of ways 3 of 9 people can sit in a row of 3 chairs.
   b. Number of ways to pick 3 students out of 9 students to attend an art workshop.
   c. Number of ways to pick 3 different entrees from a buffet line of 9 different entrees.

6. Explain why $_{10}C_3$ would not be used to solve the following problem:

   There are 10 runners in a race. How many different possibilities are there for the runners to finish first, second, and third?

7. In a lottery, players must match five numbers plus a bonus number. Five white balls are chosen from 59 white balls numbered from 1 to 59 and one red ball (the bonus number) is chosen from 35 red balls numbered 1 to 35. How many different results are possible?

8. In many courts, 12 jurors are chosen from a pool of 30 perspective jurors.
   a. In how many ways can 12 jurors be chosen from the pool of 30 perspective jurors?
   b. Once the 12 jurors are selected, 2 alternates are selected. The order of the alternates is specified. If a selected juror cannot complete the trial, the first alternate is called on to fill that jury spot. In how many ways can the 2 alternates be chosen after the 12 jury members have been chosen?

9. A band director wants to form a committee of 4 parents from a list of 45 band parents.
   a. How many different groups of 4 parents can the band director select?
   b. How many different ways can the band director select 4 parents to serve in the band parents' association as president, vice-president, treasurer, and secretary?
   c. Explain the difference between parts (a) and (b) in terms of how you decided to solve each part.

10. If you roll a cube with the numbers from 1 to 6 on the faces of the cube 4 times, how many different outcomes are possible?

11. Write a problem involving students that has an answer of ${}_6C_3$.

12. Suppose that a combination lock is opened by entering a three-digit code. Each digit can be any integer between 0 and 9, but digits may not be repeated in the code. How many different codes are possible? Is this question answered by considering permutations or combinations? Explain.

13. Six musicians will play in a recital. Three will perform before intermission, and three will perform after intermission. How many different ways are there to choose which three musicians will play before intermission? Is this question answered by considering permutations or combinations? Explain.

14. In a game show, contestants must guess the price of a product. A contestant is given nine cards with the numbers 1 to 9 written on them (each card has a different number). The contestant must then choose three cards and arrange them to produce a price in dollars. How many different prices can be formed using these cards? Is this question answered by considering permutations or combinations? Explain.

15.
   a. Using the formula for combinations, show that the number of ways of selecting 2 items from a group of 3 items is the same as the number of ways to select 1 item from a group of 3.
   b. Show that ${}_nC_k$ and ${}_nC_{n-k}$ are equal. Explain why this makes sense.

Lesson 4: Using Permutations and Combinations to Compute Probabilities

Classwork

Exercises 1–6

1. A high school is planning to put on the musical *West Side Story*. There are 20 singers auditioning for the musical. The director is looking for two singers who could sing a good duet. In how many ways can the director choose two singers from the 20 singers?

   Indicate if this question involves a permutation or a combination. Give a reason for your answer.

2. The director is also interested in the number of ways to choose a lead singer and a backup singer. In how many ways can the director choose a lead singer and then a backup singer?

   Indicate if this question involves a permutation or a combination. Give a reason for your answer.

3. For each of the following, indicate if it is a problem involving permutations, combinations, or neither, and then answer the question posed. Explain your reasoning.

   a. How many groups of five songs can be chosen from a list of 35 songs?

   b. How many ways can a person choose three different desserts from a dessert tray of eight desserts?

   c. How many ways can a manager of a baseball team choose the lead-off batter and second batter from a baseball team of nine players?

d. How many ways are there to place seven distinct pieces of art in a row?

e. How many ways are there to randomly select four balls without replacement from a container of 15 balls numbered 1 to 15?

4. The manager of a large store that sells TV sets wants to set up a display of all the different TV sets that they sell. The manager has seven different TVs that have screen sizes between 37 and 43 inches, nine that have screen sizes between 46 and 52 inches, and twelve that have screen sizes of 55 inches or greater.

   a. In how many ways can the manager arrange the 37–43 inch TV sets?

   b. In how many ways can the manager arrange the 55-inch or greater TV sets?

   c. In how many ways can the manager arrange all the TV sets if he is concerned about the order they were placed in?

5. Seven slips of paper with the digits 1 to 7 are placed in a large jar. After thoroughly mixing the slips of paper, two slips are picked without replacement.

   a. Explain the difference between ${}_7P_2$ and ${}_7C_2$ in terms of the digits selected.

b. Describe a situation in which ${}_7P_2$ is the total number of outcomes.

c. Describe a situation in which ${}_7C_2$ is the total number of outcomes.

d. What is the relationship between ${}_7P_2$ and ${}_7C_2$?

6. If you know ${}_nC_k$, and you also know the value of n and k, how could you find the value of ${}_nP_k$? Explain your answer.

Example 1: Calculating Probabilities

In a high school there are 10 math teachers. The principal wants to form a committee by selecting three math teachers at random. If Mr. H, Ms. B, and Ms. J are among the group of 10 math teachers, what is the probability that all three of them will be on the committee?

Because every different committee of 3 is equally likely,

$$P(\text{these three math teachers will be on the committee}) = \frac{\text{number of ways Mr. H, Ms. B, and Ms. J can be selected}}{\text{total number of 3 math teacher committees that can be formed}}$$

The total number of possible committees is the number of ways that three math teachers can be chosen from 10 math teachers, which is the number of combinations of 10 math teachers taken 3 at a time or ${}_{10}C_3 = 120$. Mr. H, Ms. B, and Ms. J form one of these selections. The probability that the committee will consist of Mr. H, Ms. B, and Ms. J is $\frac{1}{120}$.

Exercises 7–9

7. A high school is planning to put on the musical *West Side Story*. There are 20 singers auditioning for the musical. The director is looking for two singers who could sing a good duet.

   a. What is the probability that Alicia and Juan are the two singers who are selected by the director? How did you get your answer?

   b. The director is also interested in the number of ways to choose a lead singer and a backup singer. What is the probability that Alicia is selected the lead singer and Juan is selected the backup singer? How did you get your answer?

8. For many computer tablets, the owner can set a 4-digit pass code to lock the device.
   a. How many different 4-digit pass codes are possible if the digits cannot be repeated? How did you get your answer?

   b. If the digits of a pass code are chosen at random and without replacement from the digits $0, 1, \dots, 9$, what is the probability that the pass code is 1234? How did you get your answer?

   c. What is the probability that two people, who both chose a pass code by selecting digits at random and without replacement, both have a pass code of 1234? Explain your answer.

9. A chili recipe calls for ground beef, beans, green pepper, onion, chili powder, crushed tomatoes, salt, and pepper. You have lost the directions about the order in which to add the ingredients, so you decide to add them in a random order.
   a. How many different ways are there to add the ingredients? How did you get this answer?

   b. What is the probability that the first ingredient that you add is crushed tomatoes? How did you get your answer?

c. What is the probability that the ingredients are added in the exact order listed above? How did you get your answer?

Example 2: Probability and Combinations

A math class consists of 14 girls and 15 boys. The teacher likes to have the students come to the board to demonstrate how to solve some of the math problems. During a lesson the teacher randomly selects 6 of the students to show their work. What is the probability that all 6 of the students selected are girls?

$$P(\text{all 6 students are girls}) = \frac{\text{number of ways to select 6 girls out of 14}}{\text{number of groups of 6 from the whole class}}$$

The number of ways to select 6 girls from the 14 girls is the number of combinations of 6 from 14 which is ${}_{14}C_6 = 3{,}003$. The total number of groups of 6 is ${}_{29}C_6 = 475{,}020$.

The probability that all 6 students are girls is

$$P(\text{all 6 students are girls}) = \frac{{}_{14}C_6}{{}_{29}C_6} = \frac{3{,}003}{475{,}020} = 0.006$$

Exercises 10–11

10. There are nine golf balls numbered from 1 to 9 in a bag. Three balls are randomly selected without replacement to form a 3-digit number.

    a. How many 3-digit numbers can be formed? Explain your answer.

    b. How many 3-digit numbers start with the digit 1? Explain how you got your answer.

c. What is the probability that the 3-digit number formed is less than 200? Explain your answer.

11. There are eleven seniors and five juniors who are sprinters on the high school track team. The coach must select four sprinters to run the 800-meter relay race.

a. How many 4-sprinter relay teams can be formed from the group of 16 sprinters?

b. In how many ways can two seniors be chosen to be part of the relay team?

c. In how many ways can two juniors be chosen to be part of the relay team?

d. In how many ways can two seniors and two juniors be chosen to be part of the relay team?

e. What is the probability that two seniors and two juniors will be chosen for the relay team?

Lesson Summary

- The number of permutations of n things taken k at a time is
$$_nP_k = \frac{n!}{(n-k)!}.$$
- The number of combinations of k items selected from a set of n distinct items is
$$_nC_k = \frac{_nP_k}{k!} \text{ or } _nC_k = \frac{n!}{k!\,(n-k)!}.$$
- Permutations and combinations can be used to calculate probabilities.

Problem Set

1. For each of the following, indicate whether it is a question that involves permutations, combinations, or neither, and then answer the question posed. Explain your reasoning.
   a. How many ways can a coach choose two co-captains from 16 players in the basketball team?
   b. In how many ways can seven questions out of ten be chosen on an examination?
   c. Find the number of ways that 10 women in the finals of the skateboard street competition can finish first, second, and third in the X Games final.
   d. A postal zip code contains five digits. How many different zip codes can be made with the digits 0–9? Assume a digit can be repeated.

2. Four pieces of candy are drawn at random from a bag containing five orange pieces and seven brown pieces.
   a. How many different ways can four pieces be selected from the 12 colored pieces?
   b. How many different ways can two orange pieces be selected from five orange pieces?
   c. How many different ways can two brown pieces be selected from seven brown pieces?

3. Consider the following:
   a. A game was advertised as having a probability of 0.4 of winning. You know that the game involved five cards with a different digit on each card. Describe a possible game involving the cards that would have a probability of 0.4 of winning.
   b. A second game involving the same five cards was advertised as having a winning probability of 0.05. Describe a possible game that would have a probability of 0.05 or close to 0.05 of winning

4. You have five people who are your friends on a certain social network. You are related to two of the people, but you do not recall who of the five people are your relatives. You are going to invite two of the five people to a special meeting. If you randomly select two of the five people to invite, explain how you would derive the probability of inviting your relatives to this meeting?

5. Charlotte is picking out her class ring. She can select from a ruby, an emerald, or an opal stone, and she can also select silver or gold for the metal.
   a. How many different combinations of one stone and one type of metal can she choose? Explain how you got your answer.
   b. If Charlotte selects a stone and a metal at random, what is the probability that she would select a ring with a ruby stone and gold metal?

6. In a lottery, three numbers are chosen from 0 to 9. You win if the three numbers you pick match the three numbers selected by the lottery machine.
   a. What is the probability of winning this lottery if the numbers cannot be repeated?
   b. What is the probability of winning this lottery if the numbers can be repeated?
   c. What is the probability of winning this lottery if you must match the exact order that the lottery machine picked the numbers?

7. The store at your school wants to stock t-shirts that come in five sizes (small, medium, large, XL, XXL) and in two colors (orange and black).
   a. How many different type t-shirts will the store have to stock?
   b. At the next basketball game, the cheerleaders plan to have a t-shirt toss. If they have one t-shirt of each type in a box and select a shirt at random, what is the probability that the first randomly selected t-shirt is a large orange t-shirt?

8. There are 10 balls in a bag numbered from 1 to 10. Three balls are selected at random without replacement.
   a. How many different ways are there of selecting the three balls?
   b. What is the probability that one of the balls selected is the number 5?

9. There are nine slips of paper in a bag numbered from 1 to 9 in a bag. Four slips are randomly selected without replacement to form a 4-digit number.
   a. How many 4-digit numbers can be formed?
   b. How many 4-digit numbers start with the digit 1?
   c. What is the probability that the 2-digit number formed is less than 20?

10. There are fourteen juniors and twenty-three seniors in the Service Club. The club is to send four representatives to the State Conference.
   a. How many different ways are there to select a group of four students to attend the conference from the 37 Service Club members?
   b. How many ways are there to select exactly two juniors?
   c. How many ways are there to select exactly two seniors?
   d. If the members of the club decide to send two juniors and two seniors, how many different groupings are possible?
   e. What is the probability that two juniors and two seniors are selected to attend the conference?

11. A basketball team of 16 players consists of 6 guards, 7 forwards, and 3 centers. Coach decides to randomly select 5 players to start the game. What is the probability of 2 guards, 2 forwards, and 1 center starting the game?

12. A research study was conducted to estimate the number of white perch (a type of fish) in a Midwestern lake. 300 perch were captured and tagged. After they were tagged, the perch were released back into the lake. A scientist involved in the research estimates there are $1{,}000$ perch in this lake. Several days after tagging and releasing the fish, the scientist caught 50 perch of which 20 were tagged. If this scientist's estimate about the number of fish in the lake is correct, do you think it was likely to get 20 perch out of 50 with a tag? Explain your answer.

Lesson 5: Discrete Random Variables

Classwork

Example 1: Types of Data

Recall that the sample space of a chance experiment is the set of all possible outcomes for the experiment. For example, the sample space of the chance experiment that consists of randomly selecting one of ten apartments in a small building would be a set consisting of the ten different apartments that might have been selected. Suppose that the apartments are numbered from 1 to 10. The sample space for this experiment is $\{1,2,3,4,5,6,7,8,9,10\}$.

Cards with information about these ten apartments will be provided by your teacher. Mix these cards and then select one. Record the following information for the apartment you selected.

Number of bedrooms:

Size (sq. ft.):

Color of walls:

Floor number:

Distance to elevator:

Floor type:

Exercise 1

1. Sort the features of each apartment into three categories:

   a. Describe how the features listed in each category are similar.

b. A random variable associates a number with each outcome of a chance experiment. Which of the features are random variables? Explain.

Example 2: Random Variables

One way you might have sorted these variables is whether they are based on counting (such as the number of languages spoken) or based on measuring (such as the length of a leaf). Random variables are classified into two main types: ***discrete*** and ***continuous***.

- A ***discrete random variable*** is one that has possible values that are isolated points along the number line. Often, discrete random variables involve counting.
- A ***continuous random variable*** is one that has possible values that form an entire interval along the number line. Often, continuous random variables involve measuring.

Exercises 2–3

2. Choose six variables from the list in Exercise 1. For each one, give a specific example of a possible value the variable might have taken on, and identify the variable as discrete or continuous.

3. Suppose you were collecting data about dogs. Give at least two examples of discrete and two examples of continuous data you might collect.

Exercises 4–8: Music Genres

People like different genres of music: country, rock, hip hop, jazz, etc. Suppose you were to give a survey to people asking them how many different music genres they like.

4. What do you think the possible responses might be?

5. The table below shows $11{,}565$ responses to the survey question: How many music genres do you like listening to?

Table 1: Number of music genres survey responders like listening to

Number of music genres	0	1	2	3	4	5	6	7	8
Number of responses	568	$2{,}012$	$1{,}483$	654	749	$1{,}321$	$1{,}233$	608	$2{,}937$

Find the relative frequency for each possible response (each possible value for number of music genres), rounded to the nearest hundredth. (The relative frequency is the proportion of the observations that take on a particular value. For example, the relative frequency for 0 is $\frac{568}{11565}$.)

6. Consider the chance experiment of selecting a person at random from the people who responded to this survey. The table you generated in Exercise 5 displays the probability distribution for the random variable number of music genres liked. Your table shows the different possible values of this variable and the probability of observing each value.
   a. Is the random variable discrete or continuous?

   b. What is the probability that a randomly selected person who responded to the survey said that they like 3 different music genres?

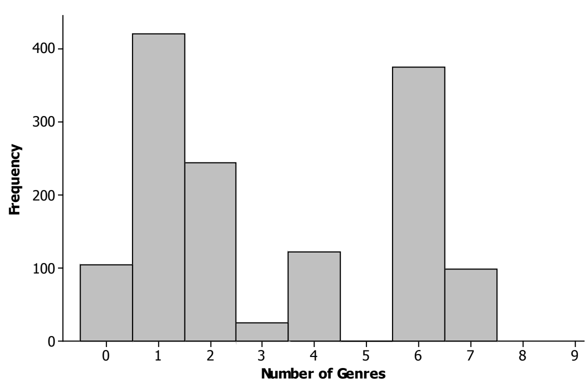
c. Which of the possible values of this variable has the greatest probability of being observed?

d. What is the probability that a randomly selected person who responded to the survey said that they liked 1 or fewer different genres?

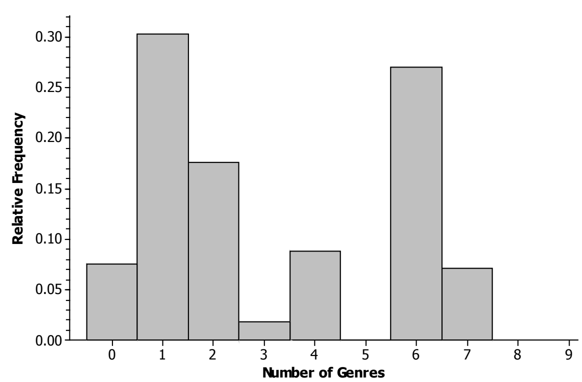
e. What is the sum of the probabilities of all of the possible outcomes? Explain why your answer is reasonable for the situation.

7. The survey data for people age 60 or older are displayed in the graphs below.

Probability Distribution of Frequency of Responses



Probability Distribution of Relative Frequency of Responses



a. What is the difference between the two graphs?

b. What is the probability that a randomly selected person from this group of people age 60 or older chose 4 music genres?

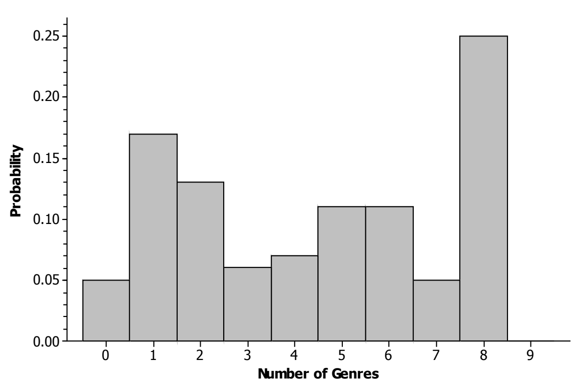
c. Which of the possible values of this variable has the greatest probability of occurring?

d. What is the probability that a randomly selected person from this group of people age 60 or older chose 5 different genres?

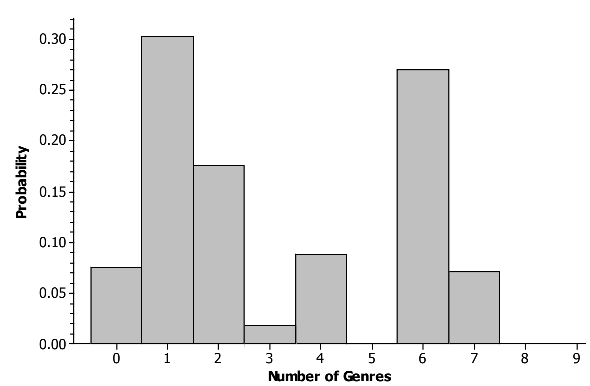
e. Make a conjecture about the sum of the relative frequencies. Then check your conjecture using the values in the table.

8. Below are graphs of the probability distribution based on responses to the original survey and based on responses from those age 60 and older.

Probability Distribution from Original Survey



Probability Distribution for 60 Years and Older



Identify which of the statements are true and which are false. Give a reason for each claim.

a. The probability that a randomly selected person chooses 0 genres is greater for those age 60 and older than for the group that responded to the original survey.

b. The probability that a randomly selected person chooses fewer than 3 genres is smaller for those age 60 and older than for the group that responded to the original survey.

c. The sum of the probabilities for all of the possible outcomes is larger for those age 60 and over than for the group that responded to the original survey.

Exercises 9–11: Family Sizes

The table below displays the distribution of the number of people living in a household according to a recent U.S. Census. This table can be thought of as the probability distribution for the random variable that consists of recording the number of people living in a randomly selected U.S. household. Notice that the table specifies the possible values of the variable, and the relative frequencies can be interpreted as the probability of each of the possible values.

Table 2: Relative frequency of the number of people living in a household

Number of People	Relative Frequency
1	0.24
2	0.32
3	0.17
4	0.16
5	0.07
6	0.02
7 or more	0.02

9. What is the random variable, and is it continuous or discrete? What values can it take on?

10. Use the table to answer each of the following.
    a. What is the probability that a randomly selected household would have 5 or more people living there?

b. What is the probability that 1or more people live in a household? How does the table support your answer?

c. What is the probability that a randomly selected household would have fewer than 6 people living there? Find your answer in two different ways.

11. The probability distributions for the number of people per household in 1790, 1890, and 1990 are below.

Number of people per household	1	2	3	4	5	6	7 or more
1790: Probability	0.03	0.08	0.12	0.14	0.14	0.13	0.36
1890: Probability	0.04	0.13	0.17	0.17	0.15	0.12	0.23
1990: Probability	0.24	0.32	0.17	0.16	0.07	0.02	0.01

Source: U.S. Census Bureau (www.census.gov)

a. Describe the change in the probability distribution of the number of people living in a randomly selected household over the years.

b. What are some factors that might explain the shift?

Lesson Summary

- Random variables can be classified into two types: discrete and continuous.
- A discrete random variable is one that has possible values that are isolated points along the number line. Often, discrete random variables involve counting.
- A continuous random variable is one that has possible values that form an entire interval along the number line. Often, continuous random variables involve measuring.
- Each of the possible values can be assigned a probability and the sum of those probabilities is 1.
- Discrete probability distributions can be displayed graphically or in a table.

Problem Set

1. Each person in a large group of children with cell phones was asked, "How old were you when you first received a cell phone?"

   The responses are summarized in the table below.

Age in years	Probability
9	0.03
10	0.06
11	0.11
12	0.23
13	0.23
14	0.14
15	0.11
16	0.08
17	0.01

   a. Make a graph of the probability distribution.
   b. The bar centered at 12 in your graph represents the probability that a randomly selected person in this group responded that they first received a cell phone at age 12. What is the area of the bar representing age 12? How does this compare to the probability corresponding to 12 in the table?
   c. What do you think the sum of the areas of all of the bars will be? Explain your reasoning.
   d. What is the probability that a randomly selected person from this group first received a cell phone at age 12 or 13?
   e. Is the probability that a randomly selected person from this group first received a cell phone at an age older than 15 greater than or less than the probability that a randomly selected person from this group first received a cell phone at an age younger than 12?

2. The following table represents a discrete probability distribution for a random variable. Fill in the missing values so that the results make sense; then, answer the questions.

Possible value	4	5	10	12	15
Probability	0.08	$???$	0.32	0.27	$???$

   a. What is the probability that this random variable takes on a value of 4 or 5?
   b. What is the probability that the value of the random variable is not 15?
   c. Which possible value is least likely?

3. Identify the following as true or false. For those that are false, explain why they are false.
   a. The probability of any possible value in a discrete random probability distribution is always greater than or equal to 0 and less than or equal to 1.
   b. The sum of the probabilities in a discrete random probability distribution varies from distribution to distribution.
   c. The total number of times someone has moved is a discrete random variable.

4. Suppose you plan to collect data on your classmates. Identify three discrete random variables and three continuous random variables you might observe.

5. Which of the following are not possible for the probability distribution of a discrete random variable? For each one you identify, explain why it is not a legitimate probability distribution.

Possible value	1	2	3	4	5
Probability	0.1	0.4	0.3	0.2	0.2

Possible value	1	2	3	4
Probability	0.8	0.2	0.3	-0.2

Possible value	1	2	3	4	5
Probability	0.2	0.2	0.2	0.2	0.2

6. Suppose that a fair coin is tossed 2 times, and the result of each toss (H or T) is recorded.
   a. What is the sample space for this chance experiment?
   b. For this chance experiment, give the probability distribution for the random variable of total number of heads observed.

7. Suppose that a fair coin is tossed 3 times.
   a. How are the possible values of the random variable of total number of heads observed different from the possible values in the probability distribution of Problem 6(b)?
   b. Is the probability of observing a total of 2 heads greater when the coin is tossed 2 times or when the coin is tossed 3 times? Justify your answer.

Lesson 6: Probability Distribution of a Discrete Random Variable

Classwork

Exercises 1–3: Credit Cards

Credit bureau data from a random sample of adults indicating the number of credit cards is summarized in the table below.

Table 1: Number of credit cards carried by adults

Number of Credit Cards	Relative Frequency
0	0.26
1	0.17
2	0.12
3	0.10
4	0.09
5	0.06
6	0.05
7	0.05
8	0.04
9	0.03
10	0.03

1. Consider the chance experiment of selecting an adult at random from the sample. The number of credit cards is a discrete random variable. The table above sets up the probability distribution of this variable as a relative frequency. Make a histogram of the probability distribution of the number of credit cards per person based on the relative frequencies.

2. Answer the following questions based on the probability distribution.
   a. Describe the distribution.

   b. Is a randomly selected adult more likely to have 0 credit cards or 7 or more credit cards?

   c. Find the area of the bar representing 0 credit cards.

   d. What is the area of all of the bars in the histogram? Explain your reasoning.

3. Suppose you asked each person in a random sample of 500 people how many credit cards he or she has. Would the following surprise you? Explain why or why not in each case.
   a. Everyone in the sample owned at least one credit card.

   b. 65 people had 2 credit cards.

   c. 300 people had at least 3 credit cards.

d. 150 people had more than 7 credit cards.

Exercises 4–7: Male and Female Pups

4. The probability that certain animals will give birth to a male or a female is generally estimated to be equal, or approximately 0.50. This estimate, however, is not always the case. Data are used to estimate the probability that the offspring of certain animals will be a male or a female. Scientists are particularly interested about the probability that an offspring will be a male or a female for animals that are at a high risk of survival. In a certain species of seals, two females are born for every male. The typical litter size for this species of seals is six pups.

   a. What are some statistical questions you might want to consider about these seals?

   b. What is the probability that a pup will be a female? A male? Explain your answer.

   c. Assuming that births are independent, which of the following can be used to find the probability that the first two pups born in a litter will be male? Explain your reasoning.

      i. $\frac{1}{3}+\frac{1}{3}$

      ii. $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$

      iii. $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$

      iv. $2\left(\frac{1}{3}\right)$

5. The probability distribution for the number of males in a litter of six pups is given below.

Table 2: Probability distribution of number of male pups per litter*

Number of male pups	Probability
0	0.088
1	0.243
2	0.33
3	0.22
4	0.075
5	0.018
6	0.001

*The sum of the probabilities in the table is not equal to 1 due to rounding.

Use the probability distribution to answer the following questions.

a. How many male pups will typically be in a litter?

b. Is a litter more likely to have six male pups or no male pups?

6. Based on the probability distribution of the number of male pups in a litter of six given above, indicate whether you would be surprised in each of the situations. Explain why or why not.

a. In every one of a female's five litters of pups, there were fewer males than females.

b. A female had only one male in two litters of pups.

c. A female had two litters of pups that were all males.

d. In a certain region of the world, scientists found that in 100 litters born to different females, 25 of them had four male pups.

7. How would the probability distribution change if the focus was the number of females rather than the number of males?

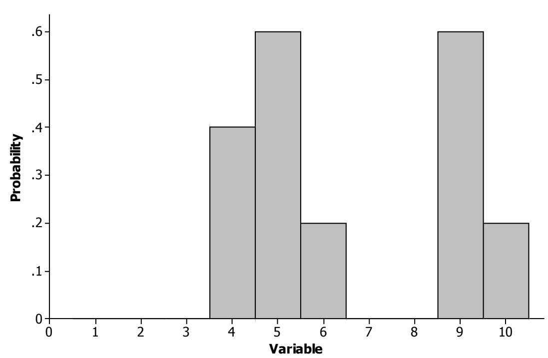
Lesson Summary

The probability distribution of a discrete random variable in table or graphical form describes the long-run behavior of a random variable.

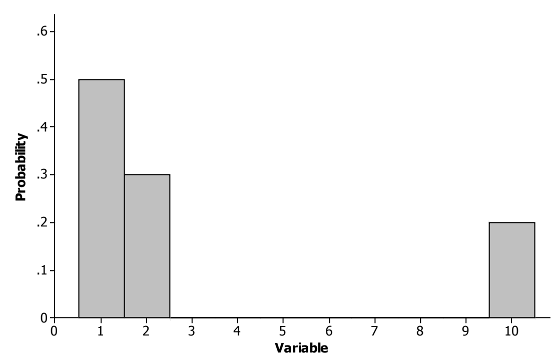
Problem Set

1. Which of the following could be graphs of a probability distribution? Explain your reasoning in each case.

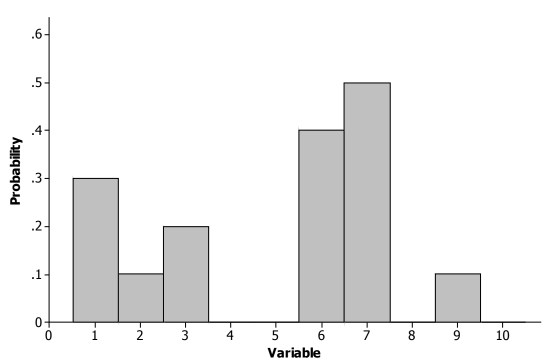
a.



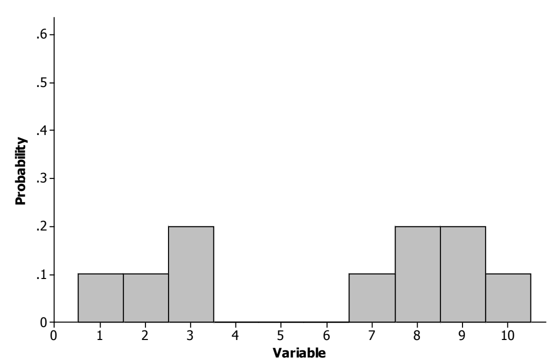
b.



c.



d.



2. Consider randomly selecting a student from New York City schools and recording the value of the random variable number of languages in which the student can carry on a conversation. A random sample of $1{,}000$ students produced the following data.

Table 3: Number of languages spoken by random sample of students in New York City

Number of languages	1	2	3	4	5	6	7
Number of students	542	280	71	40	34	28	5

a. Create a probability distribution of the relative frequencies of the number of languages students can use to carry on a conversation.

b. If you took a random sample of 650 students, would it be likely that 350 of them only spoke one language? Why or why not?

c. If you took a random sample of 650 students, would you be surprised if 100 of them spoke exactly 3 languages? Why or why not?

d. Would you be surprised if 448 students spoke at least two languages? Why or why not?

3. Suppose someone created a special six-sided die. The probability distribution for the number of spots on the top face when the die is rolled is given in the table.

Table 5: Probability distribution of the top face when rolling a die

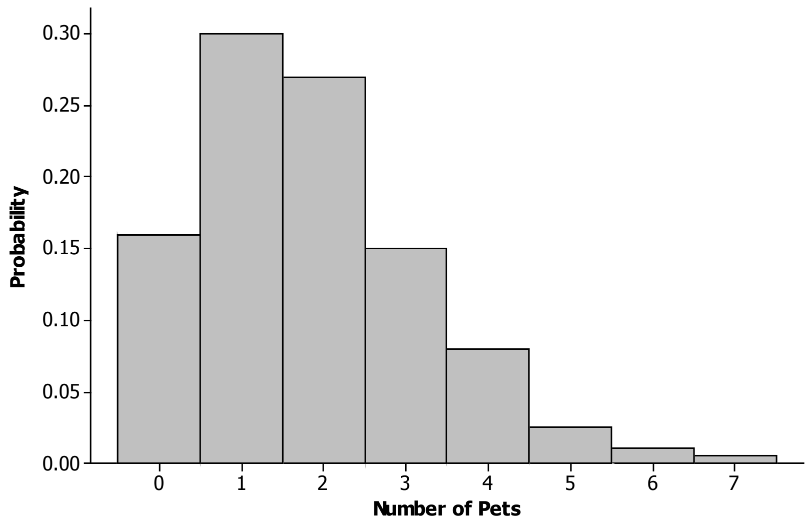
Face	1	2	3	4	5	6
Probability	$\frac{1-x}{6}$	$\frac{1-x}{6}$	$\frac{1-x}{6}$	$\frac{1+x}{6}$	$\frac{1+x}{6}$	$\frac{1+x}{6}$

a. If x is an integer, what does x have to be in order for this to be a valid probability distribution?

b. Find the probability of getting a 4.

c. What is the probability of rolling an even number?

4. The graph shows the relative frequencies of the number of pets for households in a particular community.



a. If a household in the community is selected at random, what is the probability that a household would have at least 1 pet?

b. Do you think it would be likely to have 25 households with 4 pets in a random sample of 225 households? Why or why not?

c. Suppose the results of a survey of 350 households in a section of a city found 175 of them did not have any pets. What comments might you make?

Lesson 7: Expected Value of a Discrete Random Variable

Classwork

Exploratory Challenge 1/Exercises 1–5

A new game, Six Up, involves two players. Each player rolls his or her die, counting the number of times a "six" is rolled. The players roll their die for up to one minute. The first person to roll 15 sixes wins. If no player rolls 15 sixes in the one-minute time limit, then the player who rolls the greatest number of sixes wins that round. The player who wins the most rounds wins the game.

Suppose that your class will play this game. Your teacher poses the following question:

How many sixes would you expect to roll in one round?

1. How would you answer this question?

2. What discrete random variable should you investigate to answer this question?

3. What are the possible values for this discrete random variable?

4. Do you think these possible values are all equally likely to be observed?

5. What might you do to estimate the probability of observing each of the different possible values?

Exploratory Challenge 1/Exercises 6–8

You and your partner will play the Six Up game. Roll the die until the end of one minute or until you or your partner rolls 15 sixes. Count the number of sixes you rolled. Remember to stop rolling if either you or your partner rolls 15 sixes before the end of one minute.

6. Play five rounds. After each round, record the number of sixes that you rolled.

Round 1	Round 2	Round 3	Round 4	Round 5

7. On the board, put a tally mark for the number of sixes rolled in each round.

Number of sixes rolled	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Frequency																

8. Using the data summarized in the frequency chart on the board, find the mean number of sixes rolled in a round.

Exploratory Challenge 1/Exercises 9–13

9. Calculate the relative frequency (*proportion)* for each value of the discrete random variable (i.e., the number of sixes rolled) by dividing the frequency for each possible value of the number of sixes rolled by the total number of rounds (the total number of tally marks). (The relative frequencies can be interpreted as estimates of the probabilities of observing the different possible values of the discrete random variable.)

Number of sixes rolled	0	1	2	3	4	5	6	7
Relative frequency								

Number of sixes rolled	8	9	10	11	12	13	14	15
Relative frequency								

10. Multiply each possible value for the number of sixes rolled by the corresponding probability (relative frequency).

11. Find the sum of the calculated values in Exercise 10. This number is called the expected value of the discrete random variable.

12. What do you notice about the sum in Exercise 11 and the mean that you calculated in Exercise 8?

13. The ***expected value*** of a random variable, x, is also called the ***mean*** of the distribution of that random variable. Why do you think it is called the mean?

Exploratory Challenge 1/Exercise 14

The expected value for a discrete random variable is computed using the following equation:

$$\text{expected value} = \sum \big(\text{each value of the random variable}(x)\big) \times \big(\text{the corresponding probability}(p)\big)$$

or

$$\text{expected value} = \sum xp$$

where x is a possible value of the random variable, and p is the corresponding probability.

The following table provides the probability distribution for the number of heads occurring when two coins are flipped.

Number of heads	0	1	2
Probability	0.25	0.5	0.25

14. If two coins are flipped many times, how many heads would you expect to occur, on average?

Exploratory Challenge 1/Exercises 15–16

15. The estimated expected value for the number of sixes rolled in one round of the Six Up game was 10.4. Write a sentence interpreting this value.

16. Suppose that you plan to change the rules of the Six Up game by increasing the one-minute time limit for a round. You would like to set the time so that most rounds will end by a player reaching 15 sixes. Considering the estimated expected number of sixes rolled in a one minute round, what would you recommend for the new time limit. Explain your choice.

Exploratory Challenge 2/Exercises 17–19

Suppose that we convert the above table to two vectors displaying the discrete distribution for the number of heads occurring when two coins are flipped.

Let vector A be the number of heads occurring.

Let vector B be the corresponding probabilities.

$$A = \langle 0,1,2 \rangle \qquad\qquad B = \langle 0.25,0.5,0.25 \rangle$$

17. Find the dot product of these two vectors.

18. Explain how the dot product computed in Exercise 13 compares to the expected value computed in Exercise 12.

19. How do these two processes, finding the expected value of a discrete random variable and finding the dot product of two vectors, compare?

Lesson Summary

The ***expected value*** of a random variable is the ***mean*** of the distribution of that random variable.

The expected value of a discrete random variable is the ***sum*** of the ***products*** of each possible value (x) and the corresponding probability.

The process of computing the expected value of a discrete random variable is similar to the process of computing the dot product of two vectors.

Problem Set

1. The number of defects observed in the paint of a newly manufactured car is a discrete random variable. The probability distribution of this random variable is shown in the table below.

Number of defects	0	1	2	3	4	5
Probability	0.02	0.15	0.40	0.35	0.05	0.03

   If large numbers of cars were inspected, what would you expect to see for the average number of defects per car?

2.
   a. Interpret the expected value calculated in Problem 1. Be sure to give your interpretation in context.
   b. Explain why it is not reasonable to say that every car will have the expected number of defects.

3. Students at a large high school were asked how many books they read over the summer. The number of books read is a discrete random variable. The probability distribution of this random variable is shown in the table below.

Number of books read	0	1	2	3
Probability	0.12	0.33	0.48	0.07

   If a large number of students were asked how many books they read over the summer, what would you expect to see for the average number of books read?

4. Suppose two dice are rolled. The sum of the two numbers showing is a discrete random variable. The following table displays the probability distribution of this random variable.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

   If you rolled two dice a large number of times, what would you expect the average of the sum of the two numbers showing to be?

5. Explain why it is not possible for a random variable whose only possible values are 3, 4, and 5 to have an expected value greater than 6.

6. Consider a discrete random variable with possible values 1, 2, 3, and 4. Create a probability distribution for this variable so that its expected value would be greater than 3 by entering probabilities into the table below. Then calculate the expected value to verify that it is greater than 3.

Value of variable	1	2	3	4
Probability				

Lesson 8: Interpreting Expected Value

Classwork

Exploratory Challenge 1/Exercises 1–8

Recall the following problem from the Problem Set in Lesson 7.

Suppose two dice are rolled. The sum of the two numbers showing is a discrete random variable. The following table displays the probability distribution of this random variable.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

1. If you rolled two dice and added the numbers showing a large number of times, what would you expect the average sum to be? Explain why.

2. Roll two dice. Record the sum of the numbers on the two dice in the table below. Repeat this nine more times for a total of 10 rolls.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks											
Relative frequency											

3. What is the average sum of these 10 rolls?

4. How does this average compare to the expected value in Exercise 1? Are you surprised? Why or why not?

5. Roll the two dice 10 more times, recording the sums. Combine the sums of these 10 rolls with the sums of the previous 10 rolls for a total of 20 sums.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks											
Relative frequency											

6. What is the average sum for these 20 rolls?

7. How does the average sum for these 20 rolls compare to the expected value in Exercise 1?

8. Combine the sums of your 20 rolls with those of your partner. Find the average of the sum for these 40 rolls.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency											
Relative frequency											

Exploratory Challenge 2/Exercises 9–12

9. Combine the sums of your 40 rolls above with those of another pair for a total of 80 rolls. Find the average value of the sum for these 80 rolls.

10. Combine the sums of your 80 rolls with those of the rest of the class. Find the average sum for all the rolls.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency											
Relative frequency											

11. Think about your answer to Exercise 1. What do you notice about the averages you have calculated as the number of rolls increase? Explain why this happens.

The expected value of a discrete random variable is the long-run mean value of the discrete random variable. Refer back to Exercise 1 where two dice were rolled and the sum of the two dice was recorded. The interpretation of the expected value of a sum of 7 would be:

When two dice are rolled over and over for a long time, the mean sum of the two dice is 7.

Notice that the interpretation includes the context of the problem, which is the random variable *sum of two dice*, and also includes the concept of *long-run average*.

12. Suppose a cancer charity in a large city wanted to obtain donations to send children with cancer to a circus appearing in the city. Volunteers were asked to call residents from the city's telephone book and to request a donation. Volunteers would try each phone number twice (at different times of day). If there was no answer, then a donation of $0 was recorded. Residents who declined to donate were also recorded as $0. The table below displays the results of the donation drive.

Donation	$\$0$	$\$10$	$\$20$	$\$50$	$\$100$
Probability	0.11	0.35	0.25	0.20	0.09

Find the expected value for the amount donated, *and* write an interpretation of the expected value in context.

Lesson Summary

The expected value of a discrete random variable is interpreted as the long-run mean of that random variable.

The interpretation of the expected value should include the context related to the discrete random variable.

Problem Set

1. Suppose that a discrete random variable is the number of broken eggs in a randomly selected carton of one dozen eggs. The expected value for the number of broken eggs is 0.48 eggs. Which of the following statements is a correct interpretation of this expected value? Explain why the others are wrong.
   a. The probability that an egg will break in one dozen cartons is 0.48, on average.
   b. When a large number of one dozen cartons of eggs are examined, the average number of broken eggs in a one dozen carton is 0.48 eggs.
   c. The mean number of broken eggs in one dozen cartons is 0.48 eggs.

2. Due to state funding, attendance is mandatory for students registered at a large community college. Students cannot miss more than eight days of class before being withdrawn from a course. The number of days a student is absent is a discrete random variable. The expected value of this random variable for students at this college is 3.5 days. Write an interpretation of this expected value.

3. The students at a large high school were asked to respond anonymously to the question:

   How many speeding tickets have you received?

   The table below displays the distribution of the number of speeding tickets received by students at this high school.

Number of tickets	0	1	2	3	4	5
Probability	0.55	0.28	0.09	0.04	0.03	0.01

   Compute the expected number of speeding tickets received. Interpret this mean in context.

4. Employees at a large company were asked to respond to the question:

   How many times do you bring your lunch to work each week?

   The table below displays the distribution of the number of times lunch was brought to work each week by employees at this company.

Number of times lunch brought to work each week	0	1	2	3	4	5
Probability	0.30	0.12	0.12	0.10	0.06	0.30

   Compute the expected number of times lunch was brought to work each week. Interpret this mean in context.

5. Graduates from a large high school were asked the following:

How many total AP courses did you take from Grade 9 through Grade 12?

The table below displays the distribution of the total number of AP courses taken by graduates while attending this high school.

Number of AP courses	0	1	2	3	4	5	6	7	8
Probability	0.575	0.06	0.09	0.12	0.04	0.05	0.035	0.025	0.005

Compute the expected number of total AP courses taken per graduate. Interpret this mean in context.

6. At an inspection center in a large city, the tires on the vehicles are checked for damage. The number of damaged tires is a discrete random variable. Create two different distributions for this random variable that have the same expected number of damaged tires. What is the expected number of damaged tires for the two distributions? Interpret the expected value.

Distribution 1:

Number of damaged tires	0	1	2	3	4
Probability					

Distribution 2:

Number of damaged tires	0	1	2	3	4
Probability					

Lesson 9: Determining Discrete Probability Distributions

Classwork

Exercises

A chance experiment consists of flipping a penny and a nickel at the same time. Consider the random variable of the number of heads observed.

1. Create a discrete probability distribution for the number of heads observed.

2. Explain how the discrete probability distribution is useful.

3. What is the probability of observing at least one head when you flip a penny and a nickel?

Suppose that on a particular island 60% of the eggs of a certain type of bird are female. You spot a nest of this bird and find three eggs. You are interested in the number of male eggs. Assume the gender of each egg is independent of the other eggs in the nest.

4. Create a discrete probability distribution for the number of male eggs in the nest.

5. What is the probability that no more than two eggs are male?

6. Explain the similarities and differences between this probability distribution and the one in the first part of the lesson.

7. The manufacturer of a certain type of tire claims that only 5% of the tires are defective. All four of your tires need to be replaced. What is the probability you would be a satisfied customer if you purchased all four tires from this manufacturer? Would you purchase from this manufacturer? Explain your answer using a probability distribution.

Lesson Summary

- To derive a probability distribution for a discrete random variable, you must consider all possible outcomes of the chance experiment.
- A discrete probability distribution displays all possible values of a random variable and the corresponding probabilities.

Problem Set

1. About 11% of adult Americans are left-handed. Suppose that two people are randomly selected from this population.
   a. Create a discrete probability distribution for the number of left-handed people in a sample of two randomly selected adult Americans.
   b. What is probability that at least one person in the sample is left-handed?

2. In a large batch of M&M candies, about 24% of the candies are blue. Suppose that three candies are randomly selected from the large batch.
   a. Create a discrete probability distribution for the number of blue candies out of the three randomly selected candies.
   b. What is probability that at most two candies are blue? Explain how you know.

3. In the 21st century, about 3% of mothers give birth to twins. Suppose three mothers-to-be are chosen at random.
   a. Create a discrete probability distribution for the number of sets of twins born from the sample.
   b. What is the probability that all three mothers do not give birth to twins?

4. About three in 500 people have type O-negative blood. Though it is one of the least frequently-occurring blood types, it is one of the most sought-after because it can be donated to people who have any blood type.
   a. Create a discrete probability distribution for the number of people who have type O-negative blood in a sample of two randomly selected adult Americans.
   b. Suppose two samples of two people are taken. What is the probability that at least one person in each sample has type O-negative blood?

5. The probability of being struck by lightning in one's lifetime is approximately 1 in $3{,}000$.
   a. What is the probability of being struck by lightning twice in one's lifetime?
   b. In a random sample of three adult Americans, how likely is it that at least one has been struck by lightning exactly twice?

Lesson 10: Determining Discrete Probability Distributions

Classwork

Exercises

Recall this example from Lesson 9:

A chance experiment consists of flipping a penny and a nickel at the same time. Consider the random variable of the number of heads observed.

The probability distribution for the number of heads observed is as follows.

Number of Heads	0	1	2
Probability	0.25	0.50	0.25

1. What is the probability of observing exactly 1 head when flipping a penny and a nickel?

2. Suppose you will flip two pennies instead of flipping a penny and a nickel. How will the probability distribution for the number of heads observed change?

3. Flip two pennies and record the number of heads observed. Repeat this chance experiment three more times for a total of four flips.

4. What proportion of the four flips resulted in exactly 1 head?

5. Is the proportion of the time you observed exactly 1 head in Exercise 4 the same as the probability of observing exactly 1 head when two coins are flipped (given in Exercise 1)?

6. Is the distribution of the number of heads observed in Exercise 3 the same as the actual probability distribution of the number of heads observed when two coins are flipped?

7. In Exercise 6, some students may have answered, "Yes, they are the same." But, many may have said, "No, they are different." Why might the distributions be different?

Number of Heads	0	1	2
Tally			

8. Combine your four observations from Exercise 3 with those of the rest of the class on the chart on the board. Complete the table below.

9. How well does the distribution in Exercise 8 estimate the actual probability distribution for the random variable number of heads observed when flipping two coins?

The probability of a possible value is the ***long-run proportion of the time*** that that value will occur. In the above scenario, after flipping two coins MANY times, the proportion of the time each possible number of heads is observed will be close to the probabilities in the probability distribution. This is an application of the law of large numbers, one of the fundamental concepts of statistics. The law says that the more times an event occurs, the closer the experimental outcomes naturally get to the theoretical outcomes.

A May 2000 Gallup Poll found that 38% of the people in a random sample of $1{,}012$ adult Americans said that they believe in ghosts. Suppose that three adults will be randomly selected with replacement from the group that responded to this poll, and the number of adults (out of the three) who believe in ghosts will be observed.

10. Develop a discrete probability distribution for the number of adults in the sample who believe in ghosts.

11. Calculate the probability that at least one adult but at most two adults in the sample believe in ghosts. Interpret this probability in context.

12. Out of the three randomly selected adults, how many would you expect to believe in ghosts? Interpret this expected value in context.

Lesson Summary

- To derive a discrete probability distribution, you must consider all possible outcomes of the chance experiment.
- The interpretation of probabilities from a probability distribution should mention that it is the *long-run proportion of the time* that the corresponding value will be observed.

Problem Set

1. A high school basketball player makes 70% of the free-throws she attempts. Suppose she attempts seven free-throws during a game. The probability distribution for the number of made free-throws out of seven attempts is displayed below.

Number of Completed Free-throws	0	1	2	3	4	5	6	7
Probability	0.00022	0.00357	0.02501	0.09725	0.22689	0.31765	0.24706	0.08235

   a. What is the probability that she completes at least three free-throws? Interpret this probability in context.
   b. What is the probability that she completes more than two but less than six free-throws? Interpret this probability in context.
   c. How many free-throws will she complete, on average? Interpret this expected value in context.

2. In a certain county, 30% of the voters are Republicans. Suppose that four voters are randomly selected.
   a. Develop the probability distribution for the random variable number of Republicans out of the four randomly selected voters.
   b. What is the probability that no more than two voters out of the four randomly selected voters will be Republicans? Interpret this probability in context.

3. An archery target of diameter 122 cm has a bulls-eye with diameter 12.2 cm.
   a. What is the probability that an arrow hitting the target hits the bulls-eye?
   b. Develop the probability distribution for the random variable number of bulls-eyes out of three arrows shot.
   c. What is the probability of an archer getting at least one bulls-eye? Interpret this probability in context.
   d. On average, how many bulls-eyes should an archer expect out of three arrows? Interpret this expected value in context.

4. The probability that two people have the same birthday in a room of 20 people is about 41.1%. It turns out that your math, science, and English classes all have 20 people in them.
   a. Develop the probability distribution for the random variable number of pairs of people who share birthdays out of three classes.
   b. What is the probability that one or more pairs of people share a birthday in your three classes? Interpret the probability in context.

5. You go to the warehouse of the computer company you work for because you need to send eight motherboards to a customer. You realize that someone has accidentally reshelved a pile of motherboards you had set aside as defective. Thirteen motherboards were set aside and 172 are known to be good. You're in a hurry, so you pick eight at random. The probability distribution for the number of defective motherboards is below.

Number of Defective Motherboards	0	1	2	3	4	5	6	7	8
Probability	0.5596	0.3370	0.0888	0.0134	0.0013	0.3177	7.6×10^{-5}	6.1×10^{-8}	5.76×10^{-10}

   a. If more than one motherboard is defective, your company may lose the customer's business. What is the probability of that happening?
   b. You are in a hurry and get nervous, so you pick eight motherboards, then second-guess yourself and put them back on the shelf. You then pick eight more. You do this a few times then decide it's time to bite the bullet and send eight motherboards to the customer. On average, how many defective motherboards are you choosing each time? Is it worth the risk of blindly picking motherboards?

Lesson 11: Estimating Probability Distributions Empirically

Classwork

Exploratory Challenge 1/Exercise 1

In this lesson, you will use empirical data to estimate probabilities associated with a discrete random variable and interpret probabilities in context.

1. Collect the responses to the following questions from your class.

Question 1: Estimate to the nearest whole number the number of hours per week you spend playing games on computers or game consoles.

Question 2: If you rank each of the following subjects in terms of your favorite (number 1), where would you put mathematics: $1, 2, 3, 4, 5$, or 6?
English, foreign languages, mathematics, music, science, and social studies

Exploratory Challenge 1/Exercises 2–5: Computer Games

2. Create a dot plot of the data from Question 1 in the poll: the number of hours per week students in class spend playing computer or video games.

3. Consider the chance experiment of selecting a student at random from the students at your school. You are interested in the number of hours per week a student spends playing games on computers or game consoles.

   a. Identify possible values for the random variable number of hours spent playing games on computers or game consoles.

   b. Which do you think will be more likely: a randomly chosen student at your school will play games for less than 9 hours per week or for more than 15 hours per week? Explain your thinking.

   c. Assume that your class is representative of students at your school. Create an estimated probability distribution for the random variable number of hours per week a randomly selected student at your school spends playing games on computers or game consoles.

   d. Use the estimated probability distribution to check your answer to part (b).

4. Use the data your class collected to answer the following questions.
   a. What is the expected value for the number of hours students at your school play video games on a computer or game console?

   b. Interpret the expected value you calculated in part (a).

5. Again, assuming that the data from your class is representative of students at your school, comment on each of the following statements.
   a. It would not be surprising to have 20 students in a random sample of 200 students from the school who do not play computer or console games.

   b. It would be surprising to have 60 students in a random sample of 200 students from the school spend more than 10 hours per week playing computer or console games.

   c. It would be surprising if more than half of the students in a random sample of 200 students from the school played less than 9 hours of games per week.

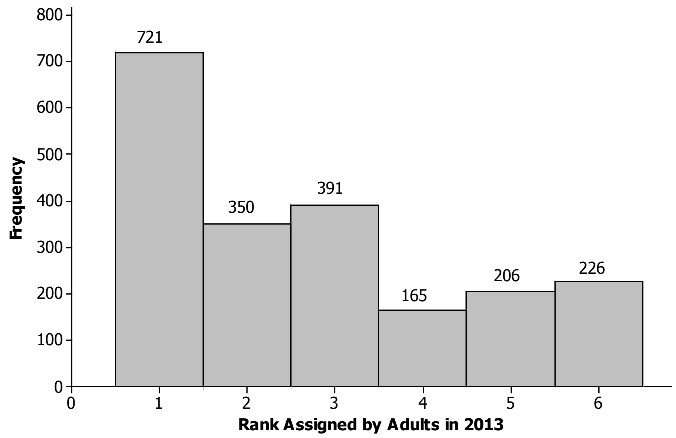
Exploratory Challenge 2/Exercises 6–10: Favorite Subject

6. Create a dot plot of your responses to Question 2 in the poll.

   a. Describe the distribution of rank assigned.

   b. Do you think it is more likely that a randomly selected student in your class would rank mathematics high (1 or 2) or that he or she would rank it low (5 or 6)? Explain your reasoning.

7. The graph displays the results of a 2013 poll taken by a polling company of a large random sample of $2{,}059$ adults 18 and older responding to Question 2 about ranking mathematics.



a. Describe the distribution of the rank assigned to mathematics for this poll.

b. Do you think the proportion of students who would rank mathematics 1 is greater than the proportion of adults who would rank mathematics 1? Explain your reasoning.

8. Consider the chance experiment of randomly selecting an adult and asking them what rank they would assign to mathematics. The variable of interest is the rank assigned to mathematics.

a. What are possible values of the random variable?

b. Using the data from the large random sample of adults, create an estimated probability distribution for the rank assigned to mathematics by adults in 2013.

c. Assuming that the students in your class are representative of students in general, use the data from your class to create an estimated probability distribution for the rank assigned to mathematics by students.

9. Use the two estimated probability distributions from Exercise 2 to answer the following questions.

a. Do the results support your answer to Exercise 1, part (b)? Why or why not?

b. Compare the probability distributions for the rank assigned to mathematics for adults and students.

c. Do adults or students have a greater probability of ranking mathematics in the middle (either a 3 or 4)?

10. Use the probability distributions from Exercise 2 to answer the following questions.
    a. Find the expected value for the estimated probability distribution of rank assigned by adults in 2013.

    b. Interpret the expected value calculated in part (a).

    c. How does the expected value for the rank students assign to mathematics compare to the expected value for the rank assigned by adults?

Lesson Summary

In this lesson you learned that

- You can estimate probability distributions for discrete random variables using data collected from polls or other sources.
- Probabilities from a probability distribution for a discrete random variable can be interpreted in terms of long-run behavior of the random variable.
- An expected value can be calculated from a probability distribution and interpreted as a long-run average.

Problem Set

1. The results of a 1989 poll in which each person in a random sample of adults ranked mathematics as a favorite subject are in the table below. The poll was given in the same city as the poll in Exercise 6.
   a. Create an estimated probability distribution for the random variable that is the rank assigned to mathematics.

   Table: Rank assigned to mathematics by adults in 1989

Rank	1	2	3	4	5	6
Frequency	56	43	12	19	39	61

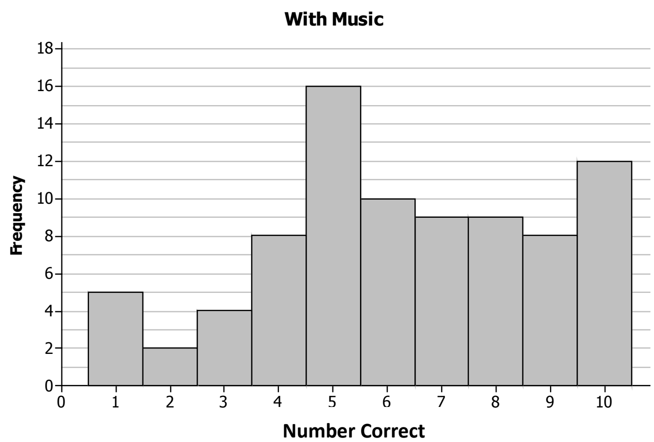
   b. An article about the poll reported, "Americans have a bit of a love-hate relationship with mathematics." Do the results support this statement? Why or why not?
   c. How is the estimated probability distribution of the rank assigned to mathematics by adults in 1989 different from the estimated probability distribution for adults in 2013?

   Table: Rank assigned to mathematics by adults in 2013

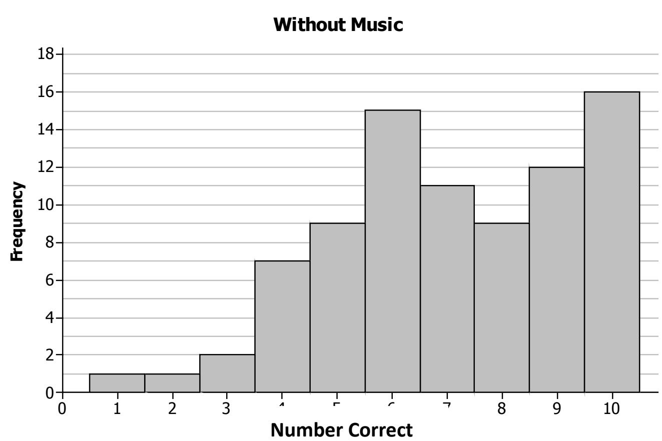
Rank	1	2	3	4	5	6
Frequency	0.35	0.17	0.19	0.08	0.1	0.11

2. A researcher investigated whether listening to music made a difference in people's ability to memorize the spelling of words. A random sample of 83 people memorized the spelling of 10 words with music playing, and then they were tested to see how many of the words they could spell. These people then memorized 10 different words without music playing and were tested again. The results are given in the two displays below.

Number of words correctly spelled when memorized with music



Number of words correctly spelled when memorized without music



a. What do you observe from comparing the two distributions?

b. Identify the variable of interest. What are possible values it could take on?

c. Assume that the group of people that participated in this study is representative of adults in general. Create both a table and a graph of the estimated probability distributions for number of words spelled correctly when memorized with music and number of words spelled correctly when memorized without music. What are the advantages and disadvantages of using a table? A graph?

d. Compare the probability that a randomly chosen person who memorized words with music will be able to correctly spell at least eight of the words to the probability for a randomly chosen person who memorized words without music.

e. Make a conjecture about which of the two estimated probability distributions will have the largest expected value. Check your conjecture by finding the expected values. Explain what each expected value means in terms of memorizing with and without music.

3. A random variable takes on the values $0, 2, 5$, and 10. The table below shows a frequency distribution based on observing values of the random variable and the estimated probability distribution for the random variable based on the observed values. Fill in the missing cells in the table.

Table: Distribution of observed values of a random variable

Variable	0	2	5	10
Frequency	18	12	2	$?$
Probability	0.3	0.2	$??$	0.47

Lesson 12: Estimating Probability Distribution Empirically

Classwork

Exploratory Challenge 1/Exercises 1–2: Moving Along

In a certain game, you toss 2 dice and find the difference of the numbers showing on the faces. You move along a number line according to the absolute value of the difference. For example, if you toss a 6 and a 3 on the first toss, then you move 3 spaces from your current position on the number line. You begin on the number 0, and the game ends when you move past 20 on the number line.

1. How many rolls would you expect it to take for you to get to 20? Explain how you would use simulation to answer this question.

2. Perform the simulation with your partner.
   a. What is the expected value for the distance moved on 1 toss of 2 dice? Interpret your answer in terms of playing the game.

   b. Use your expected value from part (a) to find the expected number of tosses that would put you past 20 on the number line.

Exploratory Challenge 2/Exercises 3–4: Lemon Flavor

Cough drops come in a roll with 2 different flavors, lemon and cherry. The same number of lemon and cherry cough drops are produced. Assume the cough drops are randomly packed with 30 per roll and that the flavor of a cough drop in the roll is independent of the flavor of the others.

3. Suppose you really liked the lemon flavor. How many cough drops would you expect to go through before finding 2 lemon cough drops in a row? Explain how you would use simulation to answer this question.

4. Carry out the simulation and use your data to estimate the average number of cough drops you would expect to go through before you found 2 lemon-flavored ones in a row. Explain what your answer indicates about 2 lemon-flavored cough drops in a row.

Lesson Summary

In this lesson, you learned that

- You can estimate probability distributions for discrete random variables using data from simulating experiments.
- Probabilities from an estimated probability distribution for a discrete random variable can be interpreted in terms of long-run behavior of the random variable.
- An expected value can be calculated from an estimated probability distribution and interpreted as a long-run average.

Problem Set

1. Suppose the rules of the game in Exploratory Challenge 1 changed.

   If you got an absolute difference of

   - 3 or more, you move forward a distance of 1;
   - 1 or 2, you move forward a distance of 2;
   - 0, you do not move forward.

   a. Use your results from Exploratory Challenge 1/Exercise 2 to estimate the probabilities for the distance moved on 1 toss of 2 dice in the new game.
   b. Which distance moved is most likely?
   c. Find the expected value for distance moved if you tossed 2 dice 10 times.
   d. If you tossed the dice 20 times, where would you expect to be on the number line, on average?

2. Suppose you were playing the game of Monopoly, and you got the Go to Jail card. You cannot get out of jail until you toss a double (the same number on both dice when 2 dice are tossed) or pay a fine.
   a. If the random variable is the number of tosses you make before you get a double, what are possible values for the random variable?
   b. Create an estimated probability distribution for how many times you would have to toss a pair of dice to get out of jail by tossing a double. (You may toss actual dice or use technology to simulate tossing dice.)
   c. What is the expected number of tosses of the dice before you would get out of jail with a double?

3. The shuttle company described in the Exit Ticket found that when they make 11 reservations, the average number of people denied a seat per shuttle is about 1 passenger per trip, which leads to unhappy customers. The manager suggests they take reservations for only 10 seats. But his boss says that might leave too many empty seats.
   a. Simulate 50 trips with 10 reservations, given that in the long run, 10% of those who make reservations do not show up. (You might let the number 1 represent a no-show and a 0 represent someone who does show up. Generate 10 random numbers from the set that contains one 1 and nine 0s to represent the 10 reservations and then count the number of 1s.)
   b. If 3 people do not show up for their reservation, how many seats are empty? Explain your reasoning.
   c. Use the number of empty seats as your random variable and create an estimated probability distribution for the number of empty seats.
   d. What is the expected value for the estimated probability distribution? Interpret your answer from the perspective of the shuttle company.
   e. How many reservations do you think the shuttle company should accept and why?

Lesson 13: Games of Chance and Expected Value

Classwork

Example 1: Ducks at the Charity Carnival

One game that is popular at some carnivals and amusement parks involves selecting a floating plastic duck at random from a pond full of ducks. In most cases, the letters S, M, or L appear on the bottom of the duck signifying that the winner receives a small, medium, or large prize, respectively. The duck is then returned to the pond for the next game.

Although the prizes are typically toys, crafts, etc., suppose that the monetary values of the prizes are as follows:
Small = $\$0.50$, Medium = $\$1.50$, and Large = $\$5.00$.

The probabilities of winning an item on 1 duck selection are as follows: Small 60%, Medium 30%, and Large 10%.

Suppose a person plays the game 4 times. What is the expected monetary value of the prizes won?

Exercises 1–4

1. Let X = the monetary value of the prize that you win playing this game 1 time. Complete the table below and calculate $E(X)$.

Event	X	Probability of X	X ·Probability of X
Small	$\$0.50$	0.6	
Medium	$\$1.50$	0.3	
Large	$\$5.00$	0.1	
		Sum:	$= E(X)$

2. Regarding the $E(X)$ value you computed above, could you win exactly that amount on any 1 play of the game?

3. What is the least you could win in 4 attempts? What is the most you could win in 4 attempts?

4. How would you explain the $E(X)$ value in context to someone who had never heard of this measurement? What would you expect for the total monetary value of your prizes from 4 attempts at this game? (To answer this question, check the introduction to this lesson; do not do anything complicated like developing a tree diagram for all of the outcomes from 4 attempts at the game.)

The question above gets at an important feature of expected value. If the probability distribution remains the same for each trial (also called *independent trials*), you can determine the expected value for N trials by computing the expected value of 1 trial and multiplying that $E(X)$ value by N. You do not have to develop a complicated probability distribution covering all the results of all N trials. This rule makes it far easier to compute expected value for situations where several hundred independent trials occur, such as a carnival, lottery, etc.

Example 2: Expected Value for Repeated Trials

In a laboratory experiment, 3 mice will be placed in a simple maze one at a time. In the maze, there is 1 decision point where the mouse can turn either left (L) or right (R). When the 1^{st} mouse arrives at the decision point, the direction the mouse chooses is recorded. The same is done for the 2^{nd} and the 3^{rd} mouse. The researchers conducting the experiment add food in the simple maze such that the long-run relative frequency of each mouse turning left is believed to be 0.7 or 70%.

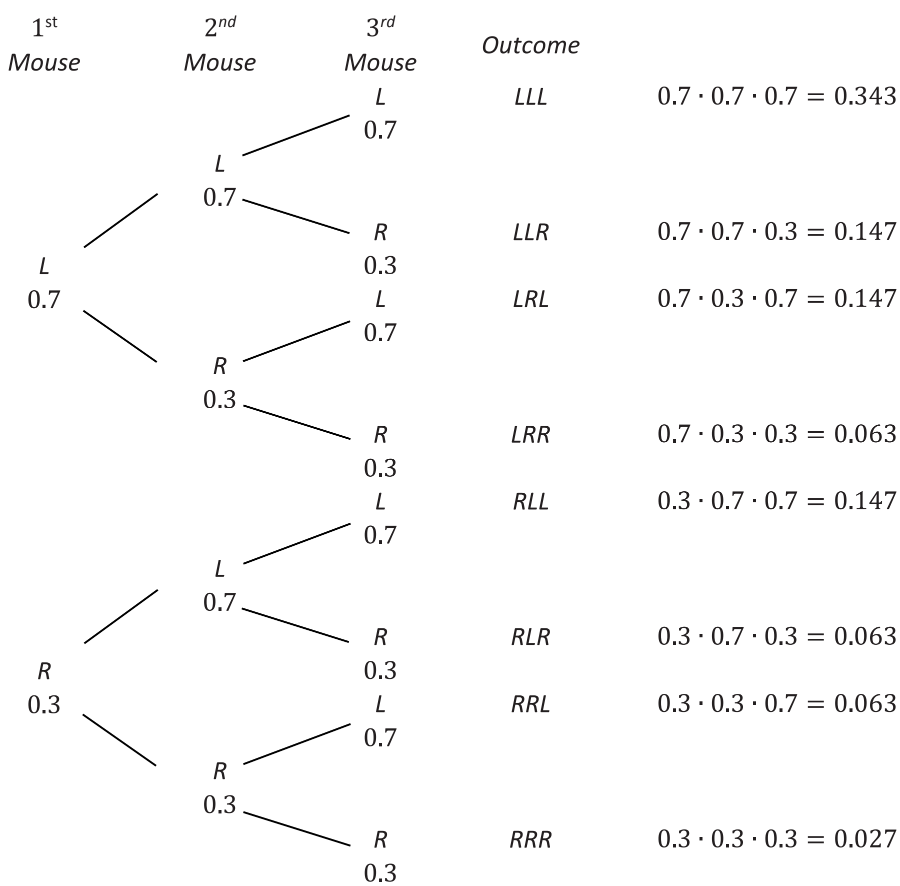
Exercises 5–8

5. Examining the outcomes for just 1 mouse, define the random variable and complete the following table:

Event	X	Probability of X	$X \cdot$ Probability of X
Left	1	0.7	
Right	0	0.3	
		Sum:	$= E(X)$

6. Using this value and the rule mentioned above, determine the expected number of left turns for 3 mice. (Remember, the value you compute may not be an exact, attainable value for 1 set of 3 mice. Rather, it is the average number of left turns by 3 mice based on many sets of 3 mice.)

The tree diagram below demonstrates the 8 possible outcomes for 3 mice where the first stage of the tree represents the decision made by the 1^{st} mouse and the second stage represents the decision made by the 2^{nd} mouse, and so on.



7. Use the tree diagram to answer the following questions:

   a. Complete the following table and compute $E(Y)$, the expected number of left turns for three mice.

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
3 Lefts	3	0.343	
2 Lefts	2		
1 Left	1		
0 Lefts	0	0.027	
		Sum:	$= E(Y)$

   b. Verify that the expected number of left turns for three mice, $E(Y)$, is the same as three times the expected number of left turns for 1 mouse, $3 \cdot E(X)$.

8. Imagine that 200 mice are sent through the maze one at a time. The researchers believed that the probability of a mouse turning left due to the food is 0.7. How many left turns would they expect from 200 mice?

Example 3: So How Does the Charity Make Money?

Revisiting the charity carnival of Example 1, recall that when selecting a duck, the average monetary value of the prizes you win per game is $\$1.25$. How can the charity running the carnival make any money if it is paying out $\$1.25$ to each player on average for each game?

To address this, in most cases a player must pay to play a game, and that is where the charity (or any other group running such a game) would earn its money.

Imagine that the cost to play the game is $\$2.00$. What are the expected net earnings for the charity? What are the expected net winnings for a player?

Exercises 9–13

9. Compute a player's net earnings for each of the three outcomes: small, medium, and large.

10. For 2 of the outcomes, the net earnings result is negative. What does a negative value of net earnings mean in context as far as the player is concerned?

11. Let $Y =$ the *net* amount that you win (or lose) playing the duck game 1 time. Complete the table below and calculate $E(Y)$.

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
Small		0.6	
Medium		0.3	
Large		0.1	
		Sum:	$= E(Y)$

12. How would you explain the $E(Y)$ value in context to someone who had never heard of this measurement? Write a sentence explaining this value from the perspective of a player; then write a sentence explaining this value from the perspective of the charity running the game.

13. How much money should the charity expect to earn from the game being played 100 times?

Lesson Summary

By computing the expected value, $E(X)$, for the earnings from a game of chance, one can determine the expected average pay off per game.

When this value is positive, the player can expect to come out ahead in the long run. However, in most games of chance, this value is negative and represents how much the group operating the game takes in on average per game. From a player's perspective, a negative expected value means that the player is expected to lose that $E(X)$ amount on average with each trial. Businesses and establishments that intend to make money from players, customers, etc., are counting on situations where the player's expected value is negative.

As long as the probabilities remain the same for each instance of a game or trial, you can compute the expected value of N games as N times the expected value of 1 game.

Problem Set

1. The Maryland Lottery Pick 3 game described in the Exit Ticket has a variety of ways in which a player can bet. Instead of the Front Pair bet of $\$0.50$ described above with a payout of $\$25.00$, a player could make a Front Pair bet of $\$1.00$ on a single ticket for a payout of $\$50.00$.

   Let $Y =$ a player's NET gain or loss from playing 1 game in this manner.

   a. Compute $E(Y)$.
   b. On average, how much does the Maryland Lottery make on each such bet?
   c. Assume that for a given time period, 100,000 bets like the one described above were placed. How much money should the Maryland Lottery expect to earn on average from 100,000 bets?
   d. Compare your answers to the three questions above with your Exit Ticket answers. How are the answers to these questions and the answers to the Exit Ticket questions related?

2. Another type of carnival or arcade game is a spinning wheel game. Imagine that someone playing a spinning wheel game earns points (pay off) as follows for each spin:

   - You gain 2 points 50% of the time.
   - You lose 3 points 25% of the time.
   - You neither gain nor lose any points 25% of the time.

   The results of each spin are added to one another, and the object is for a player to accumulate 5 or more points. Negative total point values are possible.

   a. Develop a model of a spinning wheel that would reflect the probabilities and point values.
   b. Compute $E(X)$ where $X =$ the number of points earned in a given spin.
   c. Based on your computation, how many spins on average do you think it might take to reach 5 points?
   d. Use the spinning wheel you developed in part (a) (or some other randomization device) to take a few spins. See how many spins it takes to reach 5 or more points. Comment on whether this was fewer spins, more spins, or the same number of spins you expected in part (c) above.
   e. Let $Y =$ the number of spins needed to reach 5 or more points (like the number of spins it took you to reach 5 points in part (d) above), and repeat the simulation process from part (d) many times. Record on a dot plot the various values of Y you obtain. After several simulations, comment on the distribution of Y.

Lesson 14: Game of Chance and Expected Value

Classwork

Example 1: Which Game to Play?

As mentioned in the previous lesson, games of chance are very popular. Some towns, amusement parks, themed restaurants, etc., have arcades that contain several games of chance. In many cases, tickets are awarded as a form of currency so that players can obtain tickets and eventually exchange them for a large prize at a prize center located within the arcade.

Suppose you are asked to give advice to a child who is interested in obtaining a prize that costs $1{,}000$ tickets. The child can choose from the following three games: a spinning wheel, a fishing game (very similar to the duck pond game described in the previous lesson), and a slot machine–style game with cartoon characters. Again, each of these is a game of *chance*; no skill is involved. Each game costs $\$0.50$ per play. The child will play only one of these three games but will play the game as many times as it takes to earn $1{,}000$ tickets.

Below are the ticket payout distributions for the three games:

Spinning Wheel	
Tickets	Probability
1	0.51
2	0.35
5	0.07
10	0.04
100	0.03

Fishing Game	
Tickets	Probability
1	0.50
5	0.20
10	0.15
30	0.14
150	0.01

Slot Machine	
Tickets	Probability
1	0.850
2	0.070
10	0.060
100	0.019
500	0.001

Which game would you recommend to the child?

Exercises 1–3

1. At first, glance of the probability distributions of the three games, without performing any calculations, which do you think might be the best choice and why?

2. Perform necessary calculations to determine which game to recommend to the child. Explain your choice in terms of both tickets and price. Is this the result you anticipated?

Spinning Wheel	
Tickets	Probability
1	0.51
2	0.35
5	0.07
10	0.04
100	0.03

Fishing Game	
Tickets	Probability
1	0.50
5	0.20
10	0.15
30	0.14
150	0.01

Slot Machine	
Tickets	Probability
1	0.850
2	0.070
10	0.060
100	0.019
500	0.001

3. The child states that she would like to play the slot machine game because it offers a chance of winning 500 tickets per game, and that means she might only have to play twice to reach her goal, and none of the other games offer that possibility. Using both the information from the distributions above and your expected value calculations, explain to her why this might not be the best strategy.

Example 2: Insurance

Insurance companies consider expected value when developing insurance products and determining the pricing structure of these products. From the perspective of the insurance company, the company "gains" each time it earns more money from a customer than it needs to pay out to the customer.

An example of this would be a customer paying a one-time premium (that's the cost of insurance) of $\$500.00$ to purchase a one-year, $\$10{,}000.00$ casualty policy on an expensive household item that ends up never being damaged, stolen, etc., in that one-year period. In that case, the insurance company gained $\$500.00$ from that transaction.

However, if something catastrophic did happen to the household item during that one-year period (such that it was stolen, or damaged so badly that it could not be repaired, etc.), the customer could then ask the insurance company for the $\$10{,}000.00$ of insurance money per the agreement, and the insurance company would lose $\$9{,}500.00$ from the transaction.

Imagine that an insurance company is considering offering two coverage plans for two major household items that owners would typically want to insure (or are required to insure by law). Based on market analysis, the company believes that it could sell the policies as follows:

- Plan A: Customer pays a one-year premium of $\$600.00$ and gets $\$10{,}000.00$ of insurance money if Item A is ever stolen, or damaged so badly that it could not be repaired, etc., that year.
- Plan B: Customer pays a one-year premium of $\$900.00$ and gets $\$8{,}000.00$ of insurance money if Item B is ever stolen, or damaged so badly that it could not be repaired, etc., that year.

It is estimated that the chance of the company needing to pay out on a Plan A policy is 0.09%, and the chance of the company needing to pay out on a Plan B policy is 3.71%.

Which plan should the company offer?

Exercise 4

4. The company can market and maintain only one of the two policy types, and some people in the company feel it should market Plan B since it earns the higher premium from the customer and has the lower claim payout amount. Assuming that the cost of required resources for the two types of policies is the same (for the advertising, selling, maintaining, etc., of the policies) and that the same number of policies would be sold for either Plan A or Plan B. In terms of earning the most money for the insurance company, do you agree with the Plan B decision? Explain your decision.

Lesson Summary

The application of expected value is very important to many businesses, lotteries, and others. It helps to determine the average gain or loss that can be expected for a given iteration of a probability trial.

By comparing the expected value, $E(X)$, for different games of chance (or situations that closely mirror games of chance), one can determine the most effective strategy to reach one's goal.

Problem Set

1. In the previous lesson, a duck pond game was described with the following payout distribution to its players:

Event	Y	Probability of Y
Small	$-\$1.50$	0.60
Medium	$-\$0.50$	0.30
Large	$\$3.00$	0.10

where Y = the *net* amount that a player won (or lost) playing the duck game 1 time.

This led to a situation where the people running the game could expect to gain $\$0.75$ on average per attempt.

Someone is considering changing the probability distribution as follows:

Event	Y	Probability of Y
Small	$-\$1.50$	0.70
Medium	$-\$0.50$	0.18
Large	$\$3.00$	0.12

Will this adjustment favor the players, favor the game's organizers, or will it make no difference at all in terms of the amount the organization can expect to gain on average per attempt?

2. In the previous lesson's Problem Set, you were asked to make a model of a spinning wheel with a point distribution as follows:
   - You gain 2 points 50% of the time.
   - You lose 3 points 25% of the time.
   - You neither gain nor lose any points 25% of the time.

   When X = the number of points earned in a given spin, $E(X) = 0.25$ points.

   Suppose you change the probabilities by "moving" 10% of the distribution as follows:
   - You gain 2 points 60% of the time.
   - You lose 3 points 15% of the time.
   - You neither gain nor lose any points 25% of the time.

a. Without performing any calculations, make a guess as to whether or not this new distribution will lead to a player needing a fewer number of attempts than before on average to attain 5 or more points. Explain your reasoning.

b. Determine the expected value of points earned from 1 game based on this new distribution. Based on your computation, how many spins on average do you think it might take to reach 5 points?

c. Does this value from part (b) support your guess in part (a)? (Remember that with the original distribution and its expected value of 0.25 points per play, it would have taken 20 spins on average to reach 5 points.)

3. You decide to invest $\$1{,}000.00$ in the stock market. After researching, you estimate the following probabilities:
   - Stock A has a 73% chance of earning a 20% profit in 1 year, an 11% chance of earning no profit, and a 16% chance of being worthless.
   - Stock B has a 54% chance of earning a 75% profit in 1 year, a 23% chance of earning no profit, and a 23% chance of being worthless.

   a. At first glance, which seems to be the most appealing?

   b. Which stock should you decide to invest in and why? Is this what you predicted?

4. The clock is winding down in quarter 4 of the basketball game. The scores are close. It is still anyone's game. As the team's coach, you need to quickly decide which player to put on the court to help ensure your team's success. Luckily, you have the historical data for Player A and Player B in front of you.
   - 80% of Player A's shot attempts have been 2-point field goals, and 60% of them have hit their marks. The remaining shots have been 3-point field goals, and 20% of them have hit their marks.
   - 85% of Player B's shot attempts have been 2-point field goals, and 62% of them have hit their marks. The remaining shots have been 3-point field goals, and 22% of them have hit their marks.

   a. At first glance, whom would you put in and why?

   b. Based on these statistics, which player might be more likely to help lead your team to victory and why?

5. Prior versions of College Board examinations (SAT, AP) awarded the test taker with 1 point for each correct answer and deducted $\frac{1}{4}$ point for each incorrect answer. Current versions have eliminated the point deduction for incorrect responses (test takers are awarded 0 points).

   The math section of the SAT contains 44 multiple-choice questions, with choices A–E. Suppose you answer all the questions but end up guessing on eight questions. How might your math score look different on your score report using each point system? Explain your answer.

Lesson 15: Using Expected Values to Compare Strategies

Classwork

Example 1

A math club has been conducting an annual fundraiser for many years that involves selling products. The club advisors have kept records of revenue for the products that they have made and sold over the years and have constructed the following probability distributions for the three most popular products. (Revenue has been rounded to the nearest hundred dollars.)

Candy			Magazine Subscriptions			Wrapping Paper	
Revenue	Probability		Revenue	Probability		Revenue	Probability
\$100.00	0.10						
\$200.00	0.10		\$200.00	0.4			
\$300.00	0.25		\$300.00	0.4		\$300.00	1.0
\$400.00	0.45		\$400.00	0.2			
\$500.00	0.05						
\$600.00	0.05						

Exercises 1–2

1. The club advisors only want to offer one product this year. They have decided to let the club members choose which product to offer and have shared the records from past years. Assuming that these probability distributions were to hold for the coming fundraiser, which product should the club members recommend they sell? Explain.

2. The club advisors forgot to include overhead costs with the past revenue data. The overhead costs are $\$80.00$ for candy, $\$20.00$ for magazine subscriptions, and $\$40.00$ for wrapping paper. Will this additional information change the product that the math club members recommend they sell? Why?

Example 2

A game on television has the following rules. There are four identical boxes on a table. One box contains $\$1.00$; the second, $\$15.00$; the third, $\$15{,}000.00$, and the fourth, $\$40{,}000.00$. You are offered the choice between taking $\$5{,}000$ or taking the amount of money in one of the boxes you choose at random.

Exercises 3–4

3. Suppose that you want to buy a $\$7{,}500.00$ pre-owned car. What should you do? Take the $\$5{,}000.00$ or choose a box? Why?

4. What should you do if you want to buy a $\$20{,}000.00$ brand-new car? Take the $\$5{,}000.00$ or choose a box? Why?

Example 3

In a certain two-player game, players accumulate points. One point is earned for a win, half a point is earned for a tie, and zero points are earned for a loss. A match consists of two games. There are two different approaches for how the game can be played—boldly (B) or conservatively (C). Before a match, Henry wants to determine whether to play both games boldly (BB), one game boldly and one game conservatively (BC or CB), or both games conservatively (CC). Based on years of experience, he has determined the following probabilities for a win, a tie, or a loss depending on whether he plays boldly or conservatively.

Approach	Win (W)	Tie (T)	Lose (L)
Bold (B)	0.45	0.0	0.55
Conservative (C)	0.0	0.8	0.2

How should Henry play to maximize the expected number of points earned in the match? The following exercises will help you answer this question.

Exercises 5–14

5. What are the possible values for the total points Henry can earn in a match? For example, he can earn $1\frac{1}{2}$ points by WT or TW (he wins the first game and ties the second game or ties the first game and wins the second game). What are the other possible values?

6. If Henry plays both games boldly (BB), find the probability that Henry will earn
   a. 2 points

   b. $1\frac{1}{2}$ points

   c. 1 point

   d. $\frac{1}{2}$ point

   e. 0 points

7. What is the expected number of points that Henry will earn if he plays using a BB strategy?

8. If Henry plays both games conservatively (CC), find the probability that Henry will earn
   a. 2 points

   b. $1\frac{1}{2}$ points

c. 1 point

d. $\frac{1}{2}$ point

e. 0 points

9. What is the expected number of points that Henry will earn if he plays using a CC strategy?

10. If Henry plays the first game boldly and the second game conservatively (BC), find the probability that Henry will earn

a. 2 points

b. $1\frac{1}{2}$ points

c. 1 point

d. $\frac{1}{2}$ point

e. 0 points

11. What is the expected number of points that Henry will earn if he plays using a BC strategy?

12. If Henry plays the first game conservatively and the second game boldly (CB), find the probability that Henry will earn

a. 2 points

b. $1\frac{1}{2}$ points

c. 1 point

d. $\frac{1}{2}$ point

e. 0 points

13. What is the expected number of points that Henry will earn if he plays using a CB strategy?

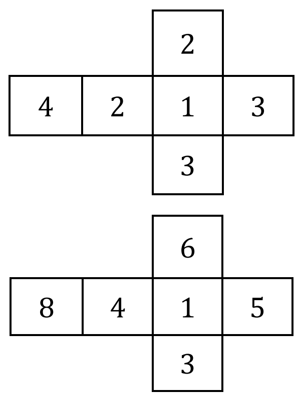
14. Of the four possible strategies, which should Henry play in order to maximize his expected number of points earned in a match?

Lesson Summary

- Making decisions in the face of uncertainty is one of the primary uses of statistics.
- Expected value can be used as one way to decide which of two or more alternatives is best for either maximizing or mininizing an objective.

Problem Set

1. A game allows you to choose what number cubes you would like to use to play. One pair of number cubes is a regular pair in which the sides of each cube are numbered from 1 to 6. The other pair consists of two different cubes as shown below. For all of these number cubes, it is equally likely that the cube will land on any one of its six sides.



   a. Suppose that you want to maximize the expected sum per roll in the long run. Which pair of number cubes should you use? Explain why.

   b. Imagine that you are playing a game in which you earn special privileges by rolling doubles (i.e., the same number on both cubes). Which number cubes would you prefer to use? Explain.

2. Amy is a wedding planner. Some of her clients care about whether the wedding is held indoors or outdoors depending on weather conditions as well as respective costs. Over the years, Amy has compiled the following data for June weddings. (Costs are in thousands of dollars.)

Weather	Cost Indoors	Cost Outdoors	Probability
Cold and sunny	$29	$33	0.15
Cold and rainy	$30	$40	0.05
Warm and sunny	$22	$27	0.45
Warm and rainy	$24	$30	0.35

   a. What is the expected cost of a June wedding held indoors?

   b. What is the expected cost of a June wedding held outdoors?

   c. A new client has her heart set on an outdoor wedding. She has at most $25,000.00 available. What do you think Amy told the client and why?

3. A venture capitalist is considering two investment proposals. One proposal involves investing $\$100{,}000.00$ in a green alternative energy source. The probability that it will succeed is only 0.05, but the gain on investment would be $\$2{,}500{,}000.00$. The other proposal involves investing $\$300{,}000.00$ in an existing textile company. The probability that it will succeed is 0.5 and the gain on investment would be $\$725{,}000.00$. In which proposal should the venture capitalist invest? Explain.

4. A student is required to purchase injury insurance in order to participate on his high school football team. The insurance will cover all expenses incurred if the student is injured during a football practice or game, but the student must pay a deductible for submitting a claim. There is also an up-front cost to purchase the injury insurance.
   - Plan A costs $\$75.00$ up front. If the student is injured and files a claim, the deductible is $\$100.00$.
   - Plan B costs $\$100.00$ up front. If the student is injured and files a claim, the deductible is $\$50.00$.

   Suppose there is a 1 in 5 chance of the student making a claim on the insurance policy. Which plan should the student choose? Explain.

Lesson 16: Making Fair Decisions

Classwork

Exploratory Challenge 1: What Is a Fair Decision?

Andre, Bobby, and Chris are competing in a 3-on-3 basketball tournament where a set number of teams compete to determine a winning team. Each basketball team plays with three players on the court at the same time.

The team of three wins the tournament. Part of the prize package is a pair of new basketball shoes. All three players want the shoes, but there is only one pair. The boys need to figure out a fair way to determine who gets to keep the new shoes.

- Chris wants to flip a coin two times to decide who will get the shoes. If two heads appear, then Andre keeps the shoes. If two tails appear, then Bobby keeps the shoes. And if one head and one tail appear (in either order), then Chris will keep the shoes.
- Bobby wants to write each of their names 10 times on torn pieces of paper and put all 30 pieces in a hat. He will give the hat a good shake, and then Bobby will choose one piece of paper from the hat to determine who gets the shoes.
- Andre wants to roll a fair six-sided die to decide who will keep the shoes. If a 1 or 2 appears, Andre will keep the shoes. If a 3 or 4 appears, Bobby will keep the shoes. If a 5 or 6 appears, then Chris keeps the shoes.

Which player's method is the most fair?

Exploratory Challenge 2

Work with your group to explore each of the decision-making methods. Your teacher will assign one or more methods to each group and specify the number of times each decision should be simulated. Record the outcomes in the table.

Method	Probability Andre keeps shoes	Probability Bobby keeps shoes	Probability Chris keeps shoes
Drawing a name out of a hat (Selecting at Random)			
Fair Coin			
Fair Six-Sided Die			
Random Number Generator (Technology Based)			
Spinner			

Which method do you think should be used to make a fair decision in this case? Explain.

Lesson Summary

- A decision can be considered fair if it does not favor one outcome over another.
- Fair decisions can be made by using several methods like selecting names from a hat or using a random number generator.

Problem Set

1. You and your sister each want to sit in the front seat of your mom's car. For each of the following, decide if the decision would be fair or unfair and explain your answer.
   a. You flip a two-sided coin.
   b. Both you and your sister try to pick a number closest to one randomly generated on a smartphone.
   c. You let your mom decide.

2. Janice, Walter, and Brooke are siblings. Their parents need them to divide the chores around the house. The one task no one wants to volunteer for is cleaning the bathroom. Janice sees a deck of 52 playing cards sitting on the table and convinces her brother and sister to use the cards to decide who will clean the bathroom.
   - Janice thinks they should draw one card. If a heart is drawn, Janice cleans the bathroom. If a spade is drawn, then Walter cleans. If a diamond is drawn, then Brooke cleans. All of the club cards will be removed from the deck before they begin drawing a card.
   - Walter wants to draw two cards at a time. If both cards are red, then Janice cleans. If both cards are black, then Walter cleans. If one card is red and one card is black, then Brooke cleans the bathroom.
   - Brooke thinks they should draw cards until they get a heart. If the first card drawn is a heart, then Janice cleans the bathroom. If the second card drawn is a heart, then Walter cleans the bathroom. If it takes three or more times to draw a heart, then Brooke cleans the bathroom.

   Whose method is fair? Explain using probabilities.

3. A large software company is moving into new headquarters. Although the workspace is larger, there is not enough space for each of the 239 employees to have his own office. It turns out that two of the employees will need to share an office. Explain how to use a random number generator to make a fair decision as to which employees will share an office.

4. Leslie and three of her friends each want to eat dinner at different restaurants. Describe a fair way to decide to which restaurant the four friends should go to eat dinner.

Lesson 17: Fair Games

Classwork

Example 1: What Is a Fair Game?

An instant lottery game card consists of six disks labeled A, B, C, D, E, F. The game is played by purchasing a game card and scratching off two disks. Each of five of the disks hides $\$1.00$, and one of the disks hides $\$10.00$. The total of the amounts on the two disks that are scratched off is paid to the person who purchased the card.

Exercises 1–5

1. What are the possible total amounts of money you could win if you scratch off two disks?

2. If you pick two disks at random:
   a. How likely is it that you win $\$2.00$?

   b. How likely is it that you win $\$11.00$?

3. Based on Exercise 3, how much should you expect to win on average per game if you played this game a large number of times?

4. To play the game, you must purchase a game card. The price of the card is set so that the game is fair. What do you think is meant by a fair game in the context of playing this instant lottery game?

5. How much should you be willing to pay for a game card if the game is to be a fair one? Explain.

Example 2: Deciding Between Two Alternatives

You have a chore to do around the house for which your mom plans to pay you $\$10.00$. When you are done, your mom, being a mathematics teacher, gives you the opportunity to change the amount that you are paid by playing a game. She puts three $\$2.00$ bills in a bag along with two $\$5.00$ bills and one $\$20.00$ bill. She says that you can take the $\$10.00$ she offered originally or you can play the game by reaching into the bag and selecting two bills without looking. You get to keep these two bills as your payment

Exercises 6–7

6. Do you think you should take your mom's original payment of $\$10.00$ or play the "bag" game? In other words, is this game a fair alternative to getting paid $\$10.00$? Use a probability distribution to help answer this question.

7. Alter the contents of the bag in Example 2 to create a game that would be a fair alternative to getting paid $\$10.00$. You must keep six "bill" in the bag, but you can choose to include bill-sized pieces of paper that are marked as $\$0.00$ to represent a $\$0.0$ bill.

Example 3: Is an Additional Year of Warranty Worth Purchasing?

Suppose you are planning to buy a computer. The computer comes with a one-year warranty, but you can purchase a waranty for an additional year for $\$24.95$. Your research indicates that in the second year, there is a 1 in 20 chance of incurring a major repair that costs $\$180.00$ and a probability of 0.15 of a minor repair that costs $\$65.00$.

Exercises 8–9

8. Is it worth purchasing the additional year warranty? Why or why not?

9. If the cost of the additional year warranty is too high, what would be a fair price to charge?

Example 4: Spinning a Pentagon

Your math club is sponsoring a game tournament to raise money for the club. The game is to spin a fair pentagon spinner twice and add the two outcomes. The faces of the spinner are numbered $1, 2, 3, 4$, and 5. If the sum is odd, you win; if the sum is even, the club wins.

Exercises 10–11

10. The math club is trying to decide what to charge to play the game and what the winning payoff should be per play to make it a fair game. Give an example.

11. What should the math club charge per play to make $\$0.25$ on average for each game played? Justify your answer.

Lesson Summary

- The concept of fairness in statistics requires that one outcome is not favored over another.
- In a game that involves a fee to play, a game is fair if the amount paid for one play of the game is the same as the expected winnings in one play.

Problem Set

1. A game is played by drawing a single card from a regular deck of playing cards. If you get a black card, you win nothing. If you get a diamond, you win $\$5.00$. If you get a heart, you win $\$10.00$. How much would you be willing to pay if the game is to be fair? Explain.

2. Suppose that for the game described in Problem 1, you win a bonus for drawing the queen of hearts. How would that change the amount you are willing to pay for the game? Explain.

3. You are trying to decide between playing two different carnival games and want to only play games that are fair.

   One game involves throwing a dart at a balloon. It costs $\$10.00$ to play, and if you break the balloon with one throw, you win $\$75.00$. If you do not break the balloon, you win nothing. You estimate that you have about a 15% chance of breaking the balloon.

   The other game is a ring toss. For $\$5.00$ you get to toss three rings and try to get them around the neck of a bottle. If you get one ring around a bottle, you win $\$3.00$. For two rings around the bottle, you win $\$15.00$. For three rings, you win $\$75.00$. If no rings land around the neck of the bottle, you win nothing. You estimate that you have about a 15% chance of tossing a ring and it landing around the neck of the bottle. Each toss of the ring is independent.

   Which game will you play? Explain.

4. Invent a fair game that involves three fair number cubes. State how the game is played and how the game is won. Explain how you know the game is fair.

5. Invent a game that is not fair that involves three fair number cubes. State how the game is played and how the game is won. Explain how you know the game is not fair.

Lesson 18: Analyzing Decisions and Strategies Using Probability

Classwork

Exercise 1

Suppose that someone is offering to sell you raffle tickets. There are blue, green, yellow, and red tickets available. Each ticket costs the same to purchase regardless of color. The person selling the tickets tells you that 369 blue tickets, 488 green tickets, 523 yellow tickets, and 331 red tickets have been sold. At the drawing, one ticket of each color will be drawn, and four identical prizes will be awarded. Which color ticket would you buy? Explain your answer.

Exercise 2

Suppose that you are taking part in a TV game show. The presenter has a set of 60 cards, 10 of which are red and the rest are blue. The presenter randomly splits the cards into two piles and places one on your left and one on your right. The presenter tells you that there are 32 blue cards in the pile on your right. You look at the pile of cards on your left and estimate that it contains 24 cards. You will be given a chance to pick a card at random, and you know that if you pick a red card you will win $\$5{,}000.00$. If you pick a blue card, you will get nothing. The presenter gives you the choice of picking a card at random from the pile on the left, from the pile on the right, or from the entire set of cards. Which should you choose? Explain your answer.

Exercise 3

The American Lyme Disease Foundation states that a commonly used test called the ELISA test will be positive for virtually all patients who have the disease but that the test is also positive for around 6% of those who do not have the disease. (http://www.aldf.com/faq.shtml#Testing) For the purposes of this question, assume that the ELISA test is positive for all patients who have the disease and for 6% of those who do not have the disease. Suppose the test is performed on a randomly selected resident of Connecticut where, according to the Centers for Disease Control and Prevention, 46 out of every $100{,}000$ people have Lyme disease. (http://www.cdc.gov/lyme/stats/chartstables/reportedcases_statelocality.html)

a. Complete the hypothetical $100{,}000$-person two-way frequency table below for $100{,}000$ Connecticut residents.

	Test is Positive	Test is Negative	Total
Has the disease			
Does not have the disease			
Total			$100{,}000$

b. If a randomly selected person from Connecticut is tested for the disease using the ELISA test and the test is positive, what is the probability that the person has the disease? (Round your answer to the nearest thousandth.)

c. Comment on your answer to part (b). What should the medical response be if a person is tested using the ELISA test and the test is positive?

Example 1

You are playing a game that uses a deck of cards consisting of 10 green, 10 blue, 10 purple, and 10 red cards. You will select four cards at random, and you want all four cards to be the same color. You are given two alternatives. You can randomly select the four cards one at a time, with each card being returned to the deck and the deck being shuffled before you pick the next card. Alternatively, you can randomly select four cards without the cards being returned to the deck. Which should you choose? Explain your answer.

Exercise 4

You are at a stall at a fair where you have to throw a ball at a target. There are two versions of the game. In the first version, you are given three attempts, and you estimate that your probability of success on any given throw is 0.1. In the second version, you are given five attempts, but the target is smaller, and you estimate that your probability of success on any given throw is 0.05. The prizes for the two versions of the game are the same, and you are willing to assume that the outcomes of your throws are independent. Which version of the game should you choose? (Hint: In the first version of the game, the probability that you do not get the prize is the probability that you fail on all three attempts.)

Lesson Summary

If a number of strategies are available and the possible outcomes are success and failure, the best strategy is the one that has the highest probability of success.

Problem Set

1. Jonathan is getting dressed in the dark. He has three drawers of socks. The top drawer contains 5 blue and 5 red socks, the middle drawer contains 6 blue and 4 red socks, and the bottom drawer contains 3 blue and 2 red socks. Jonathan will open one drawer and will select two socks at random.
   a. Which drawer should he choose in order to make it most likely that he will select 2 red socks?
   b. Which drawer should he choose in order to make it most likely that he will select 2 blue socks?
   c. Which drawer should he choose in order to make it most likely that he will select a matching pair?

2. Commuters in London have the problem that buses are often already full and, therefore, cannot take any further passengers. Sarah is heading home from work. She has the choice of going to Bus Stop A, where there are three buses per hour and 30% of the buses are full, or Bus Stop B, where there are four buses per hour and 40% of the buses are full. Which stop should she choose in order to maximize the probability that she will be able to get on a bus within the next hour? (Hint: Calculate the probability, for each bus stop, that she will fail to get on a bus within the next hour. You may assume that the buses are full, or not, independently of each other.)

3. An insurance salesman has been told by his company that about 20% of the people in a city are likely to buy life insurance. Of those who buy life insurance, around 30% own their homes, and of those who do not buy life insurance, around 10% own their homes. In the questions that follow, assume that these estimates are correct.
   a. If a homeowner is selected at random, what is the probability that the person will buy life insurance? (Hint: Use a hypothetical $1{,}000$-person two-way frequency table.)
   b. If a person is selected at random from those who do not own their homes, what is the probability that the person will buy life insurance?
   c. Is the insurance salesman better off trying to sell life insurance to homeowners or to people who do not own their homes?

4. You are playing a game. You are given the choice of rolling a fair six-sided number cube (with faces labeled 1–6) three times or selecting three cards at random from a deck that consists of
   - 4 cards labeled 1
   - 4 cards labeled 2
   - 4 cards labeled 3
   - 4 cards labeled 4
   - 4 cards labeled 5
   - 4 cards labeled 6

If you decide to select from the deck of cards, then you will not replace the cards in the deck between your selections. You will win the game if you get a triple (that is, rolling the same number three times or selecting three cards with the same number). Which of the two alternatives, the number cube or the cards, will make it more likely that you will get a triple? Explain your answer.

5. There are two routes Jasmine can take to work. Route A has five stoplights. The probability distribution of how many lights at which she will need to stop is below. The average amount of time spent at each stoplight on Route A is 30 seconds.

Number of Red Lights	0	1	2	3	4	5
Probability	0.04	0.28	0.37	0.14	0.11	0.06

Route B has three stoplights. The probability distribution of Route B is below. The average wait time for these lights is 45 seconds.

Number of Red Lights	0	1	2	3
Probability	0.09	0.31	0.40	0.20

a. In terms of stopping at the least number of stoplights, which route may be the best for Jasmine to take?

b. In terms of least time spent at stoplights, which route may be the best for Jasmine to take?

6. A manufacturing plant has been shorthanded lately, and one of its plant managers recently gathered some data about shift length and frequency of work-related accidents in the past month (accidents can range from forgetting safety equipment to breaking a nail to other, more serious injuries). Below is the table displaying his findings.

	Number of shifts with 0 accidents	Number of shifts with at least one accident	Total
$0 < x < 8$ hours	815	12	827
$8 \leq x \leq 10$ hours	53	27	80
$x > 10$ hours	52	41	93
Total	920	80	1,000

a. What is the probability that a person had an accident?

b. What happens to the accident likelihood as the number of hours increases?

c. What are some options the plant could pursue in order to try to cut down or eliminate accidents?

Lesson 19: Analyzing Decisions and Strategies Using Probability

Classwork

Example 1: Using Probability to Make a Decision About a Hypothesis

Suppose that a friend lends you an MP3 player containing a large number of songs. Your friend tells you that $\frac{1}{3}$ of the songs are indie rock and the rest are hip-hop. You put the MP3 player in random shuffle mode, listen to eight randomly selected songs, and discover that they are all hip-hop. How could you determine if whether your friend's statement is true or not?

a. Do you believe your friend? Why or why not? Use probability to justify your answer.

b. If it were true that $\frac{1}{3}$ of the songs on the MP3 player are indie rock and the rest are hip-hop, what would the probability be that all eight randomly selected songs are hip-hop? (Round your answer to the nearest thousandth.)

c. Does the result of your random selection lead you to suspect that what your friend told you is false? Explain your answer using the probability in part (a).

Exercises

1. A class consists of 12 boys and 12 girls. The teacher picks five students to present their work to the rest of the class and says that the five students are being selected at random. The students chosen are all girls.

    a. If the teacher were truly selecting the students at random, what would the probability be that all five students selected are girls? (Round your answer to the nearest thousandth.)

    b. Does the fact that all five students selected are girls lead you to suspect that the teacher was not truly selecting the students at random but that the teacher had a preference for choosing girls? Explain your answer using the probability in part (a).

2. School starts at 8:00 a.m.; it is now 7:30 a.m., and you are still at home. You have to decide whether to leave now and ride your bicycle to school or to call a friend and ask the friend to pick you up. Your friend would take 10 minutes to get to your house. You know that when you ride your bicycle, the time it takes to get to school has a mean of 26 minutes and a standard deviation of 3 minutes. When your friend drives you, the time it takes to get from your home to school has a mean of 14 minutes and standard deviation of 5 minutes. Which of the two alternatives will make you more likely to get to school on time? Show the assumptions that must be made and the calculations that lead to your conclusion.

3. Recent polls have shown that 58% of voters in a city support a particular party. However, the party has just entered a new phase of its campaign, and in a new poll of $1{,}000$ randomly selected voters, the proportion of voters who support the party is found to be 0.597.
   a. It is known that if 58% of all voters support the party, the proportion of people in a random sample of $1{,}000$ voters who support the party is approximately normally distributed with mean 0.58 and standard deviation 0.0156. If 58% of voters supported the party, what would be the probability of a sample proportion of 0.597 or more supporting the party? (Round your answer to the nearest thousandth.)

   b. Should the result of the new poll lead the party to think that support for the party has increased? Explain your answer using the probability in part (a).

4. In order to be admitted to a master's degree program, Tim must take a graduate record examination. A graduate record examination is in three sections: verbal reasoning, quantitative reasoning, and analytic writing. The exam is administered by two different companies. In Company A's version of the exam, Tim estimates that his probabilities of passing the three sections are 0.85, 0.95, and 0.90, respectively. In Company B's version, he estimates the probabilities of passing to be 0.92, 0.93, and 0.88 for the three sections. (You may assume that, with either of the companies, Tim's outcomes for the three sections of the exam are independent.)
   a. Which company should Tim choose if he must pass all three sections?

   b. Which company should Tim choose if he must pass at least one of the sections?

Lesson Summary

If a number of strategies are available and the possible outcomes are success and failure, the best strategy is the one that has the highest probability of success.

Probability can be used to decide if there is evidence against a hypothesis. If, assuming that the hypothesis is true, the observed outcome is unlikely (probability < 0.05), then there is strong evidence that the hypothesis is false. If the observed outcome is not particularly unlikely, then there is *not* strong evidence that the hypothesis is false.

Problem Set

1. To qualify for a vegetable-growing competition, onions must have a mass of at least 200 g. A grower has the choice of two different types of onion. The first type (when grown under certain conditions) grows to a mass that is approximately normally distributed with mean 192 g and standard deviation 15.3 g. The second (grown under the same conditions) grows to a mass that is approximately normally distributed with mean 187 g and standard deviation 18.8 g. Which of the two types is more likely to produce an onion that qualifies for the competition?

2. Ron's Joke Store sells both regular (fair) number cubes and weighted number cubes. Unfortunately, some of the number cubes have been mixed up. An employee rolls one of the number cubes 100 times, and the proportion of these rolls that result in sixes (the sample proportion) is 0.24.
   a. It is known that if a fair number cube is rolled 100 times, then the sample proportion of rolls that result in sixes is approximately normally distributed with mean $\frac{1}{6}$ and standard deviation 0.0373. If the number cube were fair, what would the probability be that the sample proportion in 100 rolls that result in sixes would be at least 0.24? (Round your answer to the nearest thousandth.)
   b. Does the employee's result provide strong evidence that the number cube is biased toward sixes? Explain your answer using the probability in part (a).

3. Alex and Max are twins, and they are both about to take exams in math, English, history, and science. Their parents have offered them special privileges (details to be announced) if they get A's in all their exams. For the four exams, Alex's probabilities of getting A's are 0.8, 0.9, 0.8, and 0.95, respectively. The equivalent probabilities for Max are 0.9, 0.9, 0.85, and 0.85. (You can assume that the results of the four tests are independent of one another.)
   a. Which of the twins is more likely to get A's in all the exams?
   b. Which of the twins is more likely to get at least one A?

4. Roxy is a statistics teacher. She has a set of 52 cards, and she tells her class that there are 26 red and 26 black cards. Roxy shuffles the cards and offers cookies to the first student to select a red card. An eager volunteer starts to select cards at random; after each selection, the card is returned and the cards are shuffled. Having picked black cards on each of the first five selections, the volunteer exclaims that his teacher is up to one of her tricks. Does the student have strong evidence that the cards are not as Roxy described? Include a probability calculation in your answer.

5. You decide to lay tile on the floor in one room of your house. The room measures 120 sq. ft., and the tiles themselves each measure one square foot. You want to mix white, gray, and black tile in the ratio $3:2:1$, respectively. When the tile is delivered to your house, you open up the packages and realize that the tiles are completely assorted. You begin to sort them out. As you are sorting, you randomly pick six gray tiles in a row. You immediately look for the tile company's phone number and call them to have your order replaced because the ratio is not correct. Are you justified in doing this?

6. The published universal distribution of M&Ms by color is 13% brown, 13% red, 14% yellow, 24% blue, 20% orange, and 16% green. You find a snack pack, which has about 20 candies; you open it, calculate the distribution, and then eat them all. In that pack, 15% are orange. Curious, you find a single-serve bag, containing about 50 candies. In this bag, 24% are orange. More intrigued, you go for a king-size bag, which ends up containing about 19% orange. You try some more bags, but none of the bags you try contain the actual published percentage of orange M&Ms. Does this make you think that their figures are wrong and that they should publish the correct percentages?