Lesson 22: Linear Transformations of Lines

Student Outcomes

- Students write parametric equations for a line through two points in $\mathbb{R}^2$ and $\mathbb{R}^3$ and for a line segment between two points in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Students write parametric equations for the image of a line under a given linear transformation in $\mathbb{R}^2$ and $\mathbb{R}^3$ and for the image of a line segment between two points under a given linear transformation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Lesson Notes

In this lesson, students continue their work with parametric equations to see the relationship between their work with functions and vectors (**N-VM.C.11**). This lesson continues the work of understanding the definition of a vector.

The main question of this lesson is whether the image of a line under a linear transformation is again a line. Before we answer this, we need to extend the process of finding parametric equations for a line in $\mathbb{R}^2$ and $\mathbb{R}^3$ introduced in Lesson 21. In the previous lesson, students found vector and parametric equations for a line given a point and a vector; in this lesson, we extend the process to finding parametric equations for the line given two points on the line. We also consider the question of how to parameterize a line segment. In Topic E, students will use linear transformations to emulate 3-dimensional motion on a 2-dimensional screen, and learn that one of the fundamental qualities of linear transformations is that they preserve lines.

Classwork

Opening Exercise (3 minutes)

The Opening Exercise reviews the process from Lesson 21 of finding parametric equations of lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ given a point and a vector. This lesson will extend this process to find parametric equations of lines through two given points and to find parametric equations of line segments.

Opening Exercise

a. **Find parametric equations of the line through point $P(1,1)$ in the direction of vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.**

A vector form of the equation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} t$, which gives parametric equations $x(t) = 1 - 2t$ and $y(t) = 1 + 3t$ for any real number t.

b. **Find parametric equations of the line through point $P(2,3,1)$ in the direction of vector $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$.**

A vector form of the equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} t$, which gives parametric equations $x(t) = 2 + 4t$ and $y(t) = 3 + t$ and $z(t) = 1 - t$ for any real number t.

Discussion (5 minutes)

- In the Opening Exercise we found parametric equations for the line ℓ through $P(1,1)$ with direction vector $\begin{bmatrix}-2\\3\end{bmatrix}$. How could we find parametric equations for this line if all we knew was that points $P(1,1)$ and $Q(-1,4)$ were on line ℓ?
    - *First we find the vector that points from P to Q, and then we apply the process from the last lesson.*
- What is this direction vector?
    - *The direction vector $\vec{\mathbf{v}}$ is the difference between the vectors representing points Q and P:* $\vec{\mathbf{v}} = \begin{bmatrix}-1\\4\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}$ *so* $\vec{\mathbf{v}} = \begin{bmatrix}-2\\3\end{bmatrix}$.
- What are parametric equations for the line ℓ?
    - *This is the same direction vector as we had in Problem 1 of the Opening Exercise, so parametric equations are $x(t) = 1 - 2t$, $y(t) = 1 + 3t$ for any real number t.*
- What would happen if we swapped P and Q? Do we get parametric equations for a different line?
    - *No. If we interchange P and Q, then we get a direction vector $\begin{bmatrix}2\\-3\end{bmatrix}$, and the parametric equations are $x(t) = 1 + 2t$, $y(t) = 1 - 3t$ for any real number t. This describes the same line, but it is being traversed backwards. Instead of moving from P to Q as t increases, this new line locates points from Q to P as t increases.*

Scaffolding:

- For struggling learners, display an image of the point $P(1,1)$ and the line $y = -3/2x + 5/2$. Place a marker on P to indicate when $t = 0$, and slide the marker upward to the left to illustrate the point on the line corresponding to increasing values of t.
- Ask advanced learners to find the parametric equations described in Example 1 in pairs without the guiding questions and then present their work to the class.

Example 1 (8 minutes)

This example is analogous to the one in the previous discussion but in $\mathbb{R}^3$ instead of $\mathbb{R}^2$. It then proceeds to describe how to use parametric equations to describe a line segment $\overline{PQ}$.

- What if we had a line in $\mathbb{R}^3$? Suppose we want to find parametric equations of the line through points $P(1,2,3)$ and $Q(-4,1,0)$. How do we find these equations?
    - *First, we need the vector that points from P to Q:* $\vec{\mathbf{v}} = \begin{bmatrix}-4\\1\\0\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-5\\-1\\-3\end{bmatrix}$. *Then we have the vector form of the equation* $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-4\\1\\0\end{bmatrix} + \begin{bmatrix}-5\\-1\\-3\end{bmatrix}t$, *which gives three parametric equations $x(t) = 1 - 5t$, $y(t) = 2 - t$ and $z(t) = 3 - 3t$ for real numbers t.*

MP.2

- Is there a way to use parametric equations to describe just the line segment $\overline{PQ}$ instead of the entire line $\overleftrightarrow{PQ}$?
- Give students time to figure this out on their own or with a partner and then discuss later with the class as shown below.
- What is the value of t in $x(t) = 1 - 5t$, $y(t) = 2 - t$ and $z(t) = 3 - 3t$ that produces point P?
    - *If $t = 0$, then* $\begin{bmatrix}x(0)\\y(0)\\z(0)\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} = P$.

- What value of t in $x(t) = 1 - 5t$, $y(t) = 2 - t$ and $z(t) = 3 - 3t$ produces point Q?
  - *If $t = 1$, then* $\begin{bmatrix} x(1) \\ y(1) \\ z(1) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} = Q$.
- So, how can we use parametric equations to describe just segment $\overline{PQ}$?
  - *Use the same parametric equations as for line $\overleftrightarrow{PQ}$, but restrict $0 \leq t \leq 1$.*
- In general, describe the process for finding parametric equations of the line through P and Q.
  - *First, find the direction vector $\vec{\mathbf{v}}$ by subtracting the vector associated with P from the vector associated with Q. Then find the vector form of the equation of the line and the parametric form. Let t take on any real number value.*
- In general, describe the process for finding parametric equations of the segment $\overline{PQ}$.
  - *First, find the direction vector $\vec{\mathbf{v}}$ by subtracting the vector associated with P from the vector associated with Q. Then find the vector form of the equation of the line and the parametric form. The segment $\overline{PQ}$ corresponds to the part of the line with $0 \leq t \leq 1$.*

Discussion (12 minutes)

This discussion starts with an example that shows that the image of a particular line in $\mathbb{R}^3$ under a given linear transformation is again a line in $\mathbb{R}^3$. Once this example has been established, the discussion proceeds to establish this fact for any line in $\mathbb{R}^2$ or $\mathbb{R}^3$ and any linear transformation L.

- Now, we want to explore what happens when we transform a line using a linear transformation. What do you expect the image of a line to be under a linear transformation? Why?
  - *I don't know. Linear transformations include things like rotation, dilation, and reflection. All of these operations will transform a line into another line. But, there might be a linear transformation that does something else that might distort or bend a line.*
- Suppose that the line passes through points $P(1,0,1)$ and $Q(3,-3,2)$ and we have a linear transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then what are the transformed points $L(P)$ and $L(Q)$?
  - $L(P) = L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ *and* $L(Q) = L\left(\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 4 \end{bmatrix}$
- How can we describe a point on the line $\overleftrightarrow{PQ}$?
  - *We can use the parametric equations for $\overleftrightarrow{PQ}$: First, a direction vector is* $\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. *Then, a vector form of the equation of $\overleftrightarrow{PQ}$ is* $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} t$. *Finally, parametric equations for $\overleftrightarrow{PQ}$ are $x(t) = 1 + 2t$, $y(t) = -3t$ and $z(t) = 1 + t$ for all real numbers t.*

- Since we know that $\begin{bmatrix} 1+2t \\ -3t \\ 1+t \end{bmatrix}$ is a generic point on the line $\overleftrightarrow{PQ}$, we can transform this point under L:

$$\begin{aligned} L\left(\begin{bmatrix} 1+2t \\ -3t \\ 1+t \end{bmatrix}\right) &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1+2t \\ -3t \\ 1+t \end{bmatrix} \\ &= \begin{bmatrix} 1(1+2t)+2(-3t)+1(1+t) \\ 2(1+2t)+1(-3t)+3(1+t) \\ 1(1+2t)+1(-3t)+2(1+t) \end{bmatrix} \\ &= \begin{bmatrix} 2-3t \\ 5+4t \\ 3+t \end{bmatrix} \end{aligned}$$

  But, this is how we express a line in vector form. So, any point on the line PQ is transformed into a point on the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} t$. We saw earlier that $L(P) = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$. Is this a coincidence?

  - *No, it's probably not a coincidence, because the starting point is when t = 0 and when t = 0 in our parametric equation, we get the initial point.*

- Now, let's generalize this result to any transformation L and any line ℓ through points P and Q in $\mathbb{R}^3$. Let ℓ be a line in either $\mathbb{R}^2$ or $\mathbb{R}^3$, and let L be a linear transformation on that space that can be represented by multiplication by matrix A. Let point P be represented by vector $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and let point Q be represented by vector $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$. Then, we can find the direction vector $\vec{\mathbf{v}}$ by $\vec{\mathbf{v}} = \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix}$. Any point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on line $\overleftrightarrow{PQ}$ is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix} t$ for some real number t. Then the transformed point is given by

$$\begin{aligned} L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \\ &= A\left(\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix} t\right) \\ &= A\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \left(A\begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix}\right) t \\ &= A\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \left(A\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} - A\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}\right) t \\ &= L(P) + (L(Q) - L(P))t \end{aligned}$$

  Since $L(P)$ and $(L(Q) - L(P))$ are vectors that represent points in space, this is the vector form of a line that passes through $L(P)$ and has direction vector $\left(L(Q) - L(P)\right)$. Therefore, the image of any line in $\mathbb{R}^3$ under a linear transformation L is again a line.

Exercises 1–3 (8 minutes)

Have students work in pairs or small groups on these exercises.

Exercises 1–3

1. **Consider points $P(2,1,4)$ and $Q(3,-1,2)$, and define a linear transformation by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find parametric equations to describe the image of line $\overleftrightarrow{PQ}$ under the transformation L.**

   Direction vector: $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} t$ *for all real numbers t.*

   Parametric Equations: $x(t) = 2 + t$, $y(t) = 1 - 2t$, *and* $z(t) = 4 - 2t$ *for all real numbers t.*

2. **The process that we developed for images of lines in $\mathbb{R}^3$ also applies to lines in $\mathbb{R}^2$. Consider points $P(2,3)$ and $Q(-1,4)$. Define a linear transformation by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Find parametric equations to describe the image of line $\overleftrightarrow{PQ}$ under the transformation L.**

   Direction vector: $\begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} t$ *for all real numbers t*

   Parametric Equations: $x(t) = 2 - 3t$ *and* $y(t) = 3 + t$ *for all real numbers t.*

3. **Not only is the image of a line under a linear transformation another line, but the image of a line segment under a linear transformation is another line segment. Let P, Q, and L be as specified in Exercise 2. Find parametric equations to describe the image of segment $\overline{PQ}$ under the transformation L.**

   Direction vector: $\begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} t$ *for* $0 \le t \le 1$

   Parametric Equations: $x(t) = 2 - 3t$ *and* $y(t) = 3 + t$ *for* $0 \le t \le 1$.

Closing (4 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

Lesson Summary

We can find vector and parametric equations of a line in the plane or in space if we know two points that the line passes through, and we can find parametric equations of a line segment in the plane or in space by restricting the values of t in the parametric equations for the line.

- **Let ℓ be a line in the plane that contains points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then a direction vector is given by $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$, and an equation in vector form that represents line ℓ is**

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} t, \text{ for all real numbers } t.$$

Parametric equations that represent line ℓ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \text{ for all real numbers } t. \end{aligned}$$

Parametric equations that represent segment $\overline{PQ}$ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \text{ for } t \le t \le 1. \end{aligned}$$

- **Let ℓ be a line in space that contains points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Then a direction vector is given by $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$, and an equation in vector form that represents line ℓ is**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} t, \text{ for all real numbers } t.$$

Parametric equations that represent line ℓ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \\ z(t) &= z_1 + (z_2 - z_1)t \text{ for all real numbers } t. \end{aligned}$$

Parametric equations that represent segment $\overline{PQ}$ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \\ z(t) &= z_1 + (z_2 - z_1)t \text{ for } 0 \le t \le 1. \end{aligned}$$

- **The image of a line $\overleftrightarrow{PQ}$ in the plane under a linear transformation L is given by**

$$\begin{bmatrix} x \\ y \end{bmatrix} = L(P) + \big(L(Q) - L(P)\big)t, \text{ for all real numbers } t.$$

- **The image of a line $\overleftrightarrow{PQ}$ in space under a linear transformation L is given by**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = L(P) + \big(L(Q) - L(P)\big)t, \text{ for all real numbers } t.$$

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 22: Linear Transformations of Lines

Exit Ticket

1. Consider points $P(2,1)$ and $Q(2,5)$. Find parametric equations that describe points on the line segment $\overline{PQ}$.

2. Suppose that points $P(2,1)$ and $Q(2,5)$ are transformed under the linear transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Find parametric equations that describe the image of line $\overleftrightarrow{PQ}$ under this transformation.

Exit Ticket Sample Solutions

1. Consider points $P(2,1)$ and $Q(2,5)$. Find parametric equations that describe points on the line segment $\overline{PQ}$.

A direction vector $\vec{v}$ is given by $\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, so a vector form of the segment $\overline{PQ}$ is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} t$ for $0 \le t \le 1$. This gives the parametric equations $x(t) = 2$ and $y(t) = 1 + 4t$ for $0 \le t \le 1$.

2. Suppose that points $P(2,1)$ and $Q(2,5)$ are transformed under the linear transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Find parametric equations that describe the image of line $\overleftrightarrow{PQ}$ under this transformation.

The images of P and Q are

$$L(P) = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$L(Q) = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -13 \\ 5 \end{bmatrix}.$$

The direction vector $\vec{v}$ is then $\vec{v} = \begin{bmatrix} -13 \\ 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 4 \end{bmatrix}$, so the vector form of the image of $\overleftrightarrow{PQ}$ is

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -12 \\ 4 \end{bmatrix} t$ for all real numbers t.

Parametric equations that represent the limit of line $\overleftrightarrow{PQ}$ are $x(t) = -1 - 12t$ and $y(t) = 1 + 4t$ for all real numbers t.

Problem Set Sample Solutions

1. Find parametric equations of the line $\overleftrightarrow{PQ}$ through points P and Q in the plane.

   a. $P(1,3)$, $Q(2,-5)$

   Direction vector: $\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -8 \end{bmatrix} t$ for all real numbers t.

   Parametric Equations: $x(t) = 1 + t$ and $y(t) = 3 - 8t$ for all real numbers t.

   b. $P(3,1)$, $Q(0,2)$

   Direction vector: $\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} t$ for all real numbers t.

   Parametric Equations: $x(t) = 3 - 3t$ and $y(t) = 1 + t$ for all real numbers t.

   c. $P(-2,2)$, $Q(-3,-4)$

   Direction vector: $\vec{v} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -6 \end{bmatrix} t$ for all real numbers t.

   Parametric Equations: $x(t) = -2 - t$ and $y(t) = 2 - 6t$ for all real numbers t.

2. **Find parametric equations of the line $\overleftrightarrow{PQ}$ through points P and Q in space.**

   a. $P(1,0,2)$, $Q(4,3,1)$

   Direction vector: $\vec{v} = \begin{bmatrix}4\\3\\1\end{bmatrix} - \begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}3\\3\\-1\end{bmatrix}$

   Vector equation: $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix} + \begin{bmatrix}3\\3\\-1\end{bmatrix} t$ *for all real numbers* t.

   Parametric Equations: $x(t) = 1 + 3t$, $y(t) = 3t$, *and* $z(t) = 2 - t$ *for all real numbers* t.

   b. $P(3,1,2)$, $Q(2,8,3)$

   Direction vector: $\vec{v} = \begin{bmatrix}2\\8\\3\end{bmatrix} - \begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}-1\\7\\1\end{bmatrix}$

   Vector equation: $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\1\\2\end{bmatrix} + \begin{bmatrix}-1\\7\\1\end{bmatrix} t$ *for all real numbers* t.

   Parametric Equations: $x(t) = 3 - t$, $y(t) = 1 + 7t$, *and* $z(t) = 2 + t$ *for all real numbers* t.

   c. $P(1,4,0)$, $Q(-2,1,-1)$

   Direction vector: $\vec{v} = \begin{bmatrix}-2\\1\\-1\end{bmatrix} - \begin{bmatrix}1\\4\\0\end{bmatrix} = \begin{bmatrix}-3\\-3\\-1\end{bmatrix}$

   Vector equation: $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\4\\0\end{bmatrix} + \begin{bmatrix}-3\\-3\\-1\end{bmatrix} t$ *for all real numbers* t.

   Parametric Equations: $x(t) = 1 - 3t$, $y(t) = 4 - 3t$, *and* $z(t) = -t$ *for all real numbers* t.

3. **Find parametric equations of segment $\overline{PQ}$ through points P and Q in the plane.**

   a. $P(2,0)$, $Q(2,10)$

   Direction vector: $\vec{v} = \begin{bmatrix}2\\10\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}0\\10\end{bmatrix}$

   Vector equation: $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\10\end{bmatrix} t$ *for* $0 \le t \le 1$

   Parametric Equations: $x(t) = 2$ *and* $y(t) = 10t$ *for* $0 \le t \le 1$

   b. $P(1,6)$, $Q(-3,5)$

   Direction vector: $\vec{v} = \begin{bmatrix}-3\\5\end{bmatrix} - \begin{bmatrix}1\\6\end{bmatrix} = \begin{bmatrix}-4\\-1\end{bmatrix}$

   Vector equation: $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix} + \begin{bmatrix}-4\\-1\end{bmatrix} t$ *for* $0 \le t \le 1$

   Parametric Equations: $x(t) = 1 - 4t$ *and* $y(t) = 6 - t$ *for* $0 \le t \le 1$

c. $P(-2,4)$, $Q(6,9)$

Direction vector: $\vec{v} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 8 \\ 5 \end{bmatrix} t$ *for* $0 \le t \le 1$

Parametric Equations: $x(t) = -2 + 8t$ *and* $y(t) = 4 + 5t$ *for* $0 \le t \le 1$

4. Find parametric equations of segment $\overline{PQ}$ through points P and Q in space.

a. $P(1,1,1)$, $Q(0,0,0)$

Direction vector: $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} t$ *for* $0 \le t \le 1$

Parametric Equations: $x(t) = 1 - t$, $y(t) = 1 - t$ *and* $z(t) = 1 - t$ *for* $0 \le t \le 1$

b. $P(2,1,-3)$, $Q(1,1,4)$

Direction vector: $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} t$ *for* $0 \le t \le 1$

Parametric Equations: $x(t) = 1 + 3t$, $y(t) = 3t$ *and* $z(t) = 2 - t$ *for* $0 \le t \le 1$

c. $P(3,2,1)$, $Q(1,2,3)$

Direction vector: $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} t$ *for* $0 \le t \le 1$

Parametric Equations: $x(t) = 3 - 2t$, $y(t) = 2$, *and* $z(t) = 1 + 2t$ *for* $0 \le t \le 1$

5. Jeanine claims that the parametric equations $x(t) = 3 - t$ and $y(t) = 4 - 3t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is she correct? Explain how you know.

Yes, she is correct. If $t = 1$, *then* $x(t) = 2$ *and* $y(t) = 1$, *so the line passes through point* P. *If* $t = 0$, *then* $x(t) = 3$ *and* $y(t) = 4$, *so the line passes through point* Q.

6. Kelvin claims that the parametric equations $x(t) = 3 + t$ and $y(t) = 4 + 3t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.

Yes, he is correct. If $t = -1$, *then* $x(t) = 2$ *and* $y(t) = 1$, *so the line passes through point* P. *If* $t = 0$, *then* $x(t) = 3$ *and* $y(t) = 4$, *so the line passes through point* Q.

7. LeRoy claims that the parametric equations $x(t) = 1 + 3t$ and $y(t) = -2 + 9t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.

Yes, he is correct. If $= \frac{1}{3}$, then $x(t) = 2$ and $y(t) = 1$, so the line passes through point P. If $t = \frac{2}{3}$, then $x(t) = 3$ and $y(t) = 4$, so the line passes through point Q.

8. Miranda claims that the parametric equations $x(t) = -2 + 2t$ and $y(t) = 3 - t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is she correct? Explain how you know.

No, she is not correct. If $t = 2$, then $x(t) = 2$ and $y(t) = 1$, so the line passes through point P. However, when we solve $-2 + 2t = 3$ we find $t = \frac{5}{2}$ and when we solve $3 - t = 4$, we find that $t = -1$. Thus, there is no value of t so that $(x(t), y(t)) = (3,4)$ so this line does not pass through point Q.

9. Find parametric equations of the image of the line $\overleftrightarrow{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ for the given points P, Q, and matrix A.

a. $P(2,4)$, $Q(5,-1)$, $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

$L(P) = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$ *and* $L(Q) = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ *so* $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} -12 \\ -7 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix} + \begin{bmatrix} -12 \\ -7 \end{bmatrix} t$ *for all real numbers t.*

Parametric equations: $x(t) = 14 - 12t$ *and* $y(t) = 10 - 7t$ *for all real numbers t.*

b. $P(1,-2)$, $Q(0,0)$, $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$L(P) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$ *and* $L(Q) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *so* $\vec{v} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix} t$ *for all real numbers t.*

Parametric equations: $x(t) = 1 - 3t$ *and* $y(t) = -2 - 4t$ *for all real numbers t.*

c. $P(2,3)$, $Q(1,10)$, $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$L(P) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \end{bmatrix}$ *and* $L(Q) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 10 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$ *so* $\vec{v} = \begin{bmatrix} 14 \\ 10 \end{bmatrix} - \begin{bmatrix} 14 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \end{bmatrix} t$ *for all real numbers t.*

Parametric equations: $x(t) = 14$ *and* $y(t) = 3 + 7t$ *for all real numbers t.*

10. **Find parametric equations of the image of the line $\overleftrightarrow{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given points P, Q, and matrix A.**

   a. $P(1,-2,1)$, $Q(-1,1,3)$, $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

   $L(P) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ *and* $L(Q) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix}$ *so* $\vec{v} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 10 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 10 \end{bmatrix}t$ *for all real numbers t.*

   Parametric equations: $x(t) = -t$ *and* $y(t) = -1 + 5t$ *and* $z(t) = 10t$ *for all real numbers t.*

   b. $P(2,1,4)$, $Q(1,-1,-3)$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

   $L(P) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$ *and* $L(Q) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ -2 \end{bmatrix}$ *so* $\vec{v} = \begin{bmatrix} -3 \\ -6 \\ -2 \end{bmatrix} - \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} -10 \\ -17 \\ -8 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} + \begin{bmatrix} -10 \\ -17 \\ -8 \end{bmatrix}t$ *for all real numbers t.*

   Parametric equations: $x(t) = 7 - 10t$ *and* $y(t) = 11 - 17t$ *and* $z(t) = 6 - 8t$ *for all real numbers t.*

   c. $P(0,0,1)$, $Q(4,2,3)$, $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

   $L(P) = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ *and* $L(Q) = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 7 \end{bmatrix}$ *so* $\vec{v} = \begin{bmatrix} 10 \\ 9 \\ 7 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 6 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 9 \\ 8 \\ 6 \end{bmatrix}t$ *for all real numbers t.*

   Parametric equations: $x(t) = 9$ $y(t) = 1 + 8t$ *and* $z(t) = 1 + 6t$ *for all real numbers t.*

11. **Find parametric equations of the image of the segment $\overline{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ for the given points P, Q, and matrix A.**

   a. $P(2,1)$, $Q(-1,-1)$, $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

   $L(P) = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ *and* $L(Q) = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$ *so* $\vec{v} = \begin{bmatrix} -4 \\ -3 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \end{bmatrix}$

   Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \begin{bmatrix} -9 \\ -7 \end{bmatrix}t$ *for* $0 \le t \le 1$

   Parametric equations: $x(t) = 5 - 9t$ *and* $y(t) = 4 - 7t$ *for* $0 \le t \le 1$

b. $P(0,0)$, $Q(4,2)$, $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$L(P) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $L(Q) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$ so $\vec{v} = \begin{bmatrix} 8 \\ -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ -6 \end{bmatrix} t$ for $0 \le t \le 1$

Parametric equations: $x(t) = 8t$ and $y(t) = -6t$ for $0 \le t \le 1$

c. $P(3,1)$, $Q(1,-2)$, $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$L(P) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $L(Q) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \end{bmatrix}$ so $\vec{v} = \begin{bmatrix} -7 \\ -2 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -14 \\ -3 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} + \begin{bmatrix} -14 \\ -3 \end{bmatrix} t$ for $0 \le t \le 1$

Parametric equations: $x(t) = 7 - 14t$ and $y(t) = 1 - 3t$ for $0 \le t \le 1$

12. Find parametric equations of the image of the segment $\overline{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given points P, Q and matrix A.

a. $P(0,1,1)$, $Q(-1,1,2)$, $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$L(P) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $L(Q) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$ so $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} t$ for $0 \le t \le 1$

Parametric equations: $x(t) = 1 - 2t$, $y(t) = 2 + t$, and $z(t) = 5 + 2t$ for $0 \le t \le 1$

b. $P(2,1,1)$, $Q(1,1,2)$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$L(P) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$ and $L(Q) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ so $\vec{v} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t$ for $0 \le t \le 1$

Parametric equations: $x(t) = 4$, $y(t) = 5 + t$, and $z(t) = 3$ for $0 \le t \le 1$

c. $P(0,0,1)$, $Q(1,0,0)$, $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$L(P) = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $L(Q) = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ so $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Vector equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t$ for $0 \le t \le 1$

Parametric equations: $x(t) = t$, $y(t) = 1$, and $z(t) = 1 - t$ for $0 \le t \le 1$