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Lesson 22: Linear Transformations of Lines

Student Outcomes

* Students write parametric equations for a line through two points in and and for a line segment between two points in and
* Students write parametric equations for the image of a line under a given linear transformation in and and for the image of a line segment between two points under a given linear transformation in and

Lesson Notes

In this lesson, students continue their work with parametric equations to see the relationship between their work with functions and vectors (**N-VM.C.11**). This lesson continues the work of understanding the definition of a vector.

The main question of this lesson is whether the image of a line under a linear transformation is again a line. Before we answer this, we need to extend the process of finding parametric equations for a line in and introduced in   
Lesson 21. In the previous lesson, students found vector and parametric equations for a line given a point and a vector; in this lesson, we extend the process to finding parametric equations for the line given two points on the line. We also consider the question of how to parameterize a line segment. In Topic E, students will use linear transformations to emulate 3-dimensional motion on a 2-dimensional screen, and learn that one of the fundamental qualities of linear transformations is that they preserve lines.

Classwork

Opening Exercise (3 minutes)

The Opening Exercise reviews the process from Lesson 21 of finding parametric equations of lines in and given a point and a vector. This lesson will extend this process to find parametric equations of lines through two given points and to find parametric equations of line segments.

Opening Exercise

* 1. Find parametric equations of the line through point in the direction of vector .

A vector form of the equation is , which gives parametric equations and  
 for any real number .

* 1. Find parametric equations of the line through point in the direction of vector .

A vector form of the equation is , which gives parametric equations and and for any real number .

**Discussion (5 minutes)**

* In the Opening Exercise we found parametric equations for the line through with direction vector . How could we find parametric equations for this line if all we knew was that points and were on line ?
  + *First we find the vector that points from to , and then we apply the process from the last lesson.*
* What is this direction vector?
  + *The direction vector is the difference between the vectors representing points and :   
     so*
* What are parametric equations for the line
  + *This is the same direction vector as we had in Problem 1 of the Opening Exercise, so parametric equations are , for any real number .*
* What would happen if we swapped and Do we get parametric equations for a different line?

*Scaffolding:*

* For struggling learners, display an image of the point and the line Place a marker on to indicate when , and slide the marker upward to the left to illustrate the point on the line corresponding to increasing values of .
* Ask advanced learners to find the parametric equations described in Example 1 in pairs without the guiding questions and then present their work to the class.
  + *No. If we interchange and then we get a direction vector and the parametric equations are , for any real number . This describes the same line, but it is being traversed backwards. Instead of moving from to as increases, this new line locates points from to as increases.*

Example 1 (8 minutes)

This example is analogous to the one in the previous discussion but in instead of It then proceeds to describe how to use parametric equations to describe a line segment .

* What if we had a line in ? Suppose we want to find parametric equations of the line through points and . How do we find these equations?
  + *First, we need the vector that points from to Then we have the vector form of the equation , which gives three parametric equations  
    , and for real numbers .*

**MP.2**

* Is there a way to use parametric equations to describe just the line segment instead of the entire line ?
* Give students time to figure this out on their own or with a partner and then discuss later with the class as shown below.
* What is the value of in , and that produces point ?
  + *If , then*
* What value of in , and produces point
  + *If , then .*
* So, how can we use parametric equations to describe just segment ?
  + *Use the same parametric equations as for line , but restrict .*
* In general, describe the process for finding parametric equations of the line through and.
  + *First, find the direction vector by subtracting the vector associated with from the vector associated with . Then find the vector form of the equation of the line and the parametric form. Let take on any real number value.*
* In general, describe the process for finding parametric equations of the segment
  + *First, find the direction vector by subtracting the vector associated with from the vector associated with . Then find the vector form of the equation of the line and the parametric form. The segment corresponds to the part of the line with*

**Discussion (12 minutes)**

This discussion starts with an example that shows that the image of a particular line in under a given linear transformation is again a line in Once this example has been established, the discussion proceeds to establish this fact for any line in or and any linear transformation

* Now, we want to explore what happens when we transform a line using a linear transformation. What do you expect the image of a line to be under a linear transformation? Why?
  + *I don’t know. Linear transformations include things like rotation, dilation, and reflection. All of these operations will transform a line into another line. But, there might be a linear transformation that does something else that might distort or bend a line.*
* Suppose that the line passes through points and and we have a linear transformation . Then what are the transformed points and
  + *and*
* How can we describe a point on the line ?
  + *We can use the parametric equations for : First, a direction vector is . Then, a vector form of the equation of is . Finally, parametric equations for are , and for all real numbers .*
* Since we know that is a generic point on the line , we can transform this point under

But, this is how we express a line in vector form. So, any point on the line is transformed into a point on the line . We saw earlier that Is this a coincidence?

* + *No, it’s probably not a coincidence, because the starting point is when t = 0 and when t = 0 in our parametric equation, we get the initial point.*
* Now, let’s generalize this result to any transformation and any line through points and in Let be a line in either or , and let be a linear transformation on that space that can be represented by multiplication by matrix Let point be represented by vector and let point be represented by vector . Then, we can find the direction vector by . Any point on line is given by  
   for some real number . Then the transformed point is given by

Since and are vectors that represent points in space, this is the vector form of a line that passes through and has direction vector Therefore, the image of any line in under a linear transformation is again a line.

Exercises 1–3 (8 minutes)

Have students work in pairs or small groups on these exercises.

Exercises 1–3

1. Consider points and , and define a linear transformation by Find parametric equations to describe the image of line under the transformation .

Direction vector:

Vector equation: for all real numbers t.

Parametric Equations: , , and for all real numbers t.

1. The process that we developed for images of lines in also applies to lines in . Consider points and Define a linear transformation by Find parametric equations to describe the image of line under the transformation .

Direction vector:

Vector equation: for all real numbers

Parametric Equations: and for all real numbers .

1. Not only is the image of a line under a linear transformation another line, but the image of a line segment under a linear transformation is another line segment. Let ,, and be as specified in Exercise 2. Find parametric equations to describe the image of segment under the transformation.

Direction vector:

Vector equation: for 0

Parametric Equations: and for .

Closing (4 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

Lesson Summary

We can find vector and parametric equations of a line in the plane or in space if we know two points that the line passes through, and we can find parametric equations of a line segment in the plane or in space by restricting the values of in the parametric equations for the line.

* **Let be a line in the plane that contains point and . Then a direction vector is given by , and an equation in vector form that represents line is**

for all real numbers.

Parametric equations that represent line are

Parametric equations that represent segment are

* **Let be a line in space that contains points and Then a direction vector is given by , and an equation in vector form that represents line is**

for all real numbers .

Parametric equations that represent line are

Parametric equations that represent segment are

* **The image of a line in the plane under a linear transformation is given by**

, for all real numbers .

* **The image of a line in space under a linear transformation is given by**

for all real numbers

Exit Ticket (5 minutes)

Name Date

Lesson 22: Linear Transformations of Lines

Exit Ticket

1. Consider points and . Find parametric equations that describe points on the line segment
2. Suppose that points and are transformed under the linear transformation . Find parametric equations that describe the image of line under this transformation.

Exit Ticket Sample Solutions

1. Consider points and . Find parametric equations that describe points on the line segment

*A direction vector is given by ,so a vector form of the segment is   
for This gives the parametric equations and for*

1. Suppose that points and are transformed under the linear transformation . Find parametric equations that describe the image of line under this transformation.

*The images of and are*

*The direction vector* *is then , so the vector form of the image of is*

*for all real numbers .*

*Parametric equations that represent the limit of line are and for all real numbers .*

Problem Set Sample Solutions

1. Find parametric equations of the line through points and in the plane.
   1. ,

*Direction vector:*

Vector equation: for all real numbers

Parametric Equations: and for all real numbers .

* 1. ,

*Direction vector:*

Vector equation: for all real numbers

Parametric Equations: and for all real numbers .

* 1. ,

*Direction vector:*

Vector equation: for all real numbers

Parametric Equations: and for all real numbers .

1. Find parametric equations of the line through points and in space.
   1. ,

***Direction vector:***

Vector equation: for all real numbers

Parametric Equations: , , and for all real numbers .

**Direction vector:**

Vector equation: for all real numbers

Parametric Equations: , , and for all real numbers .

* 1. ,

**Direction vector:**

Vector equation: for all real numbers

Parametric Equations: , , and for all real numbers .

1. Find parametric equations of segment through points and in the plane.
   1. ,

**Direction vector:**

Vector equation: for

Parametric Equations: and for

* 1. ,

**Direction vector:**

Vector equation: for

Parametric Equations: and for

* 1. ,

**Direction vector:**

Vector equation: for

Parametric Equations: and for

1. Find parametric equations of segment through points and in space.
   1. ,

**Direction vector:**

Vector equation: for

Parametric Equations: and for

* 1. ,

**Direction vector:**

Vector equation: for

Parametric Equations: and for

* 1. ,

**Direction vector:**

Vector equation: for

Parametric Equations: , , and for

1. Jeanine claims that the parametric equations and describe the line through points and . Is she correct? Explain how you know.

Yes, she is correct. If , then and , so the line passes through point . If , then and , so the line passes through point

1. Kelvin claims that the parametric equations and describe the line through points and . Is he correct? Explain how you know.

Yes, he is correct. If , then and , so the line passes through point . If , then and , so the line passes through point

1. LeRoy claims that the parametric equations and describe the line through points and . Is he correct? Explain how you know.

Yes, he is correct. If , then and , so the line passes through point . If , then and , so the line passes through point

1. Miranda claims that the parametric equations and describe the line through points and . Is she correct? Explain how you know.

No, she is not correct. If , then and , so the line passes through point . However, when we solve we find and when we solve , we find that . Thus, there is no value of so that , so this line does not pass through point

1. Find parametric equations of the image of the line under the transformation for the given points , and matrix
   1. , ,

*and so*

Vector equation: for all real numbers .

Parametric equations: and for all real numbers .

* 1. , ,

*and so*

Vector equation: for all real numbers

Parametric equations: and for all real numbers .

* 1. , ,

*and so*

Vector equation: for all real numbers .

Parametric equations: and for all real numbers .

1. Find parametric equations of the image of the line under the transformation for the given points , , and matrix
   1. , ,

*and so*

Vector equation: for all real numbers .

Parametric equations: and and for all real numbers .

* 1. , ,

*and so*

Vector equation: for all real numbers .

Parametric equations: and and for all real numbers .

* 1. , ,

*and so*

Vector equation: for all real numbers .

Parametric equations: and for all real numbers .

1. Find parametric equations of the image of the segment under the transformation for the given points , , and matrix
   1. , ,

*and so*

Vector equation: for

Parametric equations: and for

* 1. , ,

*and so*

Vector equation: for

Parametric equations: and for

* 1. , ,

*and so*

Vector equation: for

Parametric equations: and for

1. Find parametric equations of the image of the segment under the transformation for the given points , and matrix
   1. , ,

*and so*

Vector equation: for

Parametric equations: , , and for

* 1. , ,

*and so*

Vector equation: for

Parametric equations: , , and for

* 1. , ,

*and so*

Vector equation: for

Parametric equations: , , and for