Lesson 22: Linear Transformations of Lines

Classwork

Opening Exercise

a. Find parametric equations of the line through point $P(1,1)$ in the direction of vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

b. Find parametric equations of the line through point $P(2,3,1)$ in the direction of vector $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$.

Exercises 1–3

1. Consider points $P(2,1,4)$ and $Q(3,-1,2)$, and define a linear transformation by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find parametric equations to describe the image of line $\overleftrightarrow{PQ}$ under the transformation L.

2. The process that we developed for images of lines in $\mathbb{R}^3$ also applies to lines in $\mathbb{R}^2$. Consider points $P(2,3)$ and $Q(-1,4)$. Define a linear transformation by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Find parametric equations to describe the image of line $\overleftrightarrow{PQ}$ under the transformation L.

3. Not only is the image of a line under a linear transformation another line, but the image of a line segment under a linear transformation is another line segment. Let P, Q, and L be as specified in Exercise 2. Find parametric equations to describe the image of segment $\overline{PQ}$ under the transformation L.

Lesson Summary

We can find vector and parametric equations of a line in the plane or in space if we know two points that the line passes through, and we can find parametric equations of a line segment in the plane or in space by restricting the values of t in the parametric equations for the line.

- Let ℓ be a line in the plane that contains points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then a direction vector is given by $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$, and an equation in vector form that represents line ℓ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} t, \text{ for all real numbers } t.$$

Parametric equations that represent line ℓ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \quad \text{for all real numbers } t. \end{aligned}$$

Parametric equations that represent segment $\overline{PQ}$ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \quad \text{for } t \leq t \leq 1. \end{aligned}$$

- Let ℓ be a line in space that contains points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Then a direction vector is given by $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$, and an equation in vector form that represents line ℓ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} t, \text{ for all real numbers } t.$$

Parametric equations that represent line ℓ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \\ z(t) &= z_1 + (z_2 - z_1)t \quad \text{for all real numbers } t. \end{aligned}$$

Parametric equations that represent segment $\overline{PQ}$ are

$$\begin{aligned} x(t) &= x_1 + (x_2 - x_1)t \\ y(t) &= y_1 + (y_2 - y_1)t \\ z(t) &= z_1 + (z_2 - z_1)t \quad \text{for } 0 \leq t \leq 1. \end{aligned}$$

- The image of a line $\overleftrightarrow{PQ}$ in the plane under a linear transformation L is given by

  $\begin{bmatrix} x \\ y \end{bmatrix} = L(P) + \big(L(Q) - L(P)\big)t$, for all real numbers t.

- The image of a line $\overleftrightarrow{PQ}$ in space under a linear transformation L is given by

  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = L(P) + \big(L(Q) - L(P)\big)t$, for all real numbers t.

Problem Set

1. Find parametric equations of the line $\overleftrightarrow{PQ}$ through points P and Q in the plane.
   a. $P(1,3)$, $Q(2,-5)$
   b. $P(3,1)$, $Q(0,2)$
   c. $P(-2,2)$, $Q(-3,-4)$

2. Find parametric equations of the line $\overleftrightarrow{PQ}$ through points P and Q in space.
   a. $P(1,0,2)$, $Q(4,3,1)$
   b. $P(3,1,2)$, $Q(2,8,3)$
   c. $P(1,4,0)$, $Q(-2,1,-1)$

3. Find parametric equations of segment $\overline{PQ}$ through points P and Q in the plane.
   a. $P(2,0)$, $Q(2,10)$
   b. $P(1,6)$, $Q(-3,5)$
   c. $P(-2,4)$, $Q(6,9)$

4. Find parametric equations of segment $\overline{PQ}$ through points P and Q in space.
   a. $P(1,1,1)$, $Q(0,0,0)$
   b. $P(2,1,-3)$, $Q(1,1,4)$
   c. $P(3,2,1)$, $Q(1,2,3)$

5. Jeanine claims that the parametric equations $x(t) = 3 - t$ and $y(t) = 4 - 3t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is she correct? Explain how you know.

6. Kelvin claims that the parametric equations $x(t) = 3 + t$ and $y(t) = 4 + 3t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.

7. LeRoy claims that the parametric equations $x(t) = 1 + 3t$ and $y(t) = -2 + 9t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.

8. Miranda claims that the parametric equations $x(t) = -2 + 2t$ and $y(t) = 3 - t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is she correct? Explain how you know.

9. Find parametric equations of the image of the line $\overleftrightarrow{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ for the given points P, Q, and matrix A.
   a. $P(2,4)$, $Q(5,-1)$, $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

b. $P(1,-2)$, $Q(0,0)$, $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

c. $P(2,3)$, $Q(1,10)$, $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

10. Find parametric equations of the image of the line $\overleftrightarrow{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given points P, Q, and matrix A.

    a. $P(1,-2,1)$, $Q(-1,1,3)$, $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

    b. $P(2,1,4)$, $Q(1,-1,-3)$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

    c. $P(0,0,1)$, $Q(4,2,3)$, $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

11. Find parametric equations of the image of the segment $\overline{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ for the given points P, Q, and matrix A.

    a. $P(2,1)$, $Q(-1,-1)$, $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

    b. $P(0,0)$, $Q(4,2)$, $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

    c. $P(3,1)$, $Q(1,-2)$, $A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

12. Find parametric equations of the image of the segment $\overline{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given points P, Q and matrix A.

    a. $P(0,1,1)$, $Q(-1,1,2)$, $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

    b. $P(2,1,1)$, $Q(1,1,2)$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

    c. $P(0,0,1)$, $Q(1,0,0)$, $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$