Lesson 18: Vectors and Translation Maps

Classwork

Opening Exercise

Write each vector described below in component form and find its magnitude. Draw an arrow originating from $(0,0)$ to represent each vector's magnitude and direction.

a. Translate 3 units right and 4 units down.

b. Translate 6 units left.

c. Translate 2 units left and 2 units up.

d. Translate 5 units right and 7 units up.

Exercises 1–3

1. Write a translation map defined by each vector from the opening.

   Consider the vector $\mathbf{v} = \langle -2,5 \rangle$, and its associated translation map:

   $T_{\mathrm{v}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-2 \\ y+5 \end{bmatrix}$

2. Suppose we apply the translation map T_{v} to each point on the circle $(x+4)^2+(y-3)^2=25$.
   a. What is the radius and center of the original circle?

   b. Show that the image points satisfy the equation of another circle.

   c. What is center and radius of this image circle?

3. Suppose we apply the translation map T_{v} to each point on the line $2x-3y=10$.
   a. What are the slope and y-intercept of the original line?

   b. Show that the image points satisfy the equation of another line.

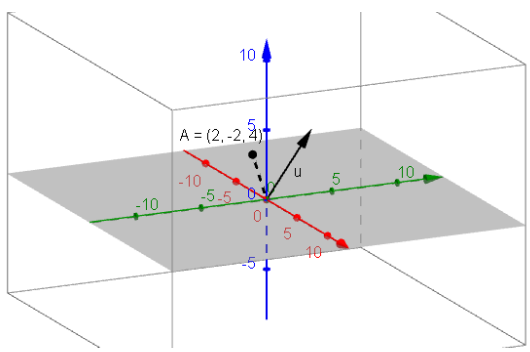
c. What are the slope and y-intercept of this image line?

Example 1: Vectors and Translation Maps in $\mathbb{R}^3$

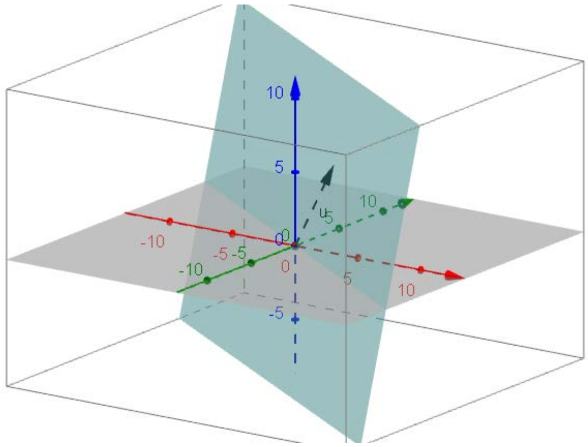
Translate by the vector $\mathrm{v} = \langle 1,3,5 \rangle$ by applying the translation map T_{v} to the following objects in $\mathbb{R}^3$. A sketch of the original object and the vector is shown. Sketch the image.

$$T_{\mathrm{v}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y+3 \\ z+5 \end{bmatrix}$$

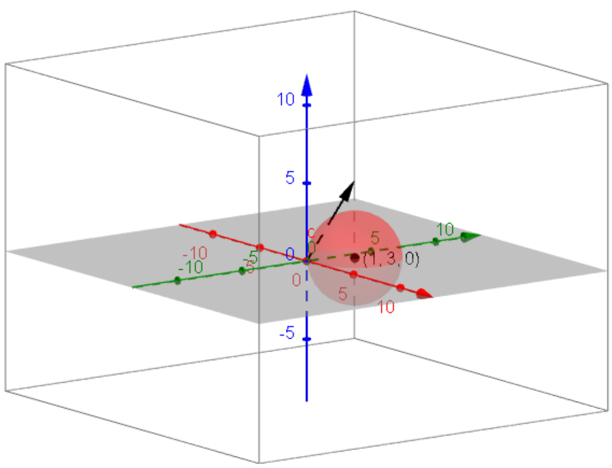
a. The point $A(2,-2,4)$



b. The plane $2x + 3y - z = 0$



c. The sphere $(x-1)^2 + (y-3)^2 + z^2 = 9$.



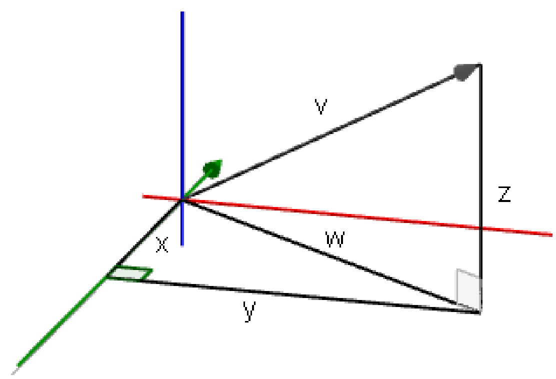
Exercise 4

4. Given the sphere $(x+3)^2 + (y-1)^2 + (z-3)^2 = 10$.

   a. What are its center and radius?

b. Write a vector and its associated translation map that would take this sphere to its image centered at the origin.

Example 2: What is the Magnitude of a Vector in $\mathbb{R}^3$?



a. Find a general formula for $\|\mathbf{v}\|^2$.

b. Solve this equation for $\|\mathbf{v}\|$ to find the magnitude of the vector.

Exercises 5–8

5. Which vector has greater magnitude, $\mathbf{v} = \langle 0,5,-4 \rangle$ or $\mathbf{u} = \langle 3,-4,4 \rangle$? Show work to support your answer.

6. Explain why vectors can have equal magnitude but not be the same vector.

7. Vector arithmetic in $\mathbb{R}^3$ is analogous to vector arithmetic in $\mathbb{R}^2$. Complete the graphic organizer to illustrate these ideas.

	Vectors in $\mathbb{R}^2$	Vectors in $\mathbb{R}^3$
Component Form	$\langle a, b \rangle$	$\langle a, b, c \rangle$
Column Form	$\begin{bmatrix} a \\ b \end{bmatrix}$	
Magnitude	$\lVert\mathbf{v}\rVert = \sqrt{a^2 + b^2}$	
Addition	If $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{u} = \langle c, d \rangle$, Then $\mathbf{v} + \mathbf{u} = \langle a + c, b + d \rangle$	
Subtraction	If $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{u} = \langle c, d \rangle$, Then $\mathbf{v} - \mathbf{u} = \langle a - c, b - d \rangle$	
Scalar Multiplication	If $\mathbf{v} = \langle a, b \rangle$ and k is a real number $k\mathbf{v} = \langle ka, kb \rangle$	

8. Given $\mathbf{v} = \langle 2,0,-4 \rangle$ and $\mathbf{u} = \langle -1,5,3 \rangle$.
   a. Calculate the following.
      i. $\mathbf{v} + \mathbf{u}$

ii. $2\mathbf{v} - \mathbf{u}$

iii. $\|\mathbf{v}\|$

b. Suppose the point $(1,3,5)$ is translated by $\mathbf{v}$ and then by $\mathbf{u}$. Determine a vector $\mathbf{w}$ that would return the point back to its original location $(1,3,5)$.

Lesson Summary

A vector $\mathbf{v}$ can define a translation map T_{v} that takes a point to its image under the translation. Applying the map to the set of all points that make up a geometric figure serves to translate the figure by the vector.

Problem Set

1. Myishia says that when applying the translation map $T_{\mathrm{v}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y-2 \end{bmatrix}$ to a set of points given by an equation relating x and y, we should replace every x that is in the equation by $x+1$, and y by $y-2$. For example, the equation of the parabola $y = x^2$ would become $y - 2 = (x+1)^2$. Is she correct? Explain your answer.

2. Given the vector $\mathbf{v} = \langle -1, 3 \rangle$, find the image of the line $x + y = 1$ under the translation map T_{v}. Graph the original line and its image, and explain the geometric effect of the map T_{v} on the line.

3. Given the vector $\mathbf{v} = \langle 2,1 \rangle$, find the image of the parabola $y - 1 = x^2$ under the translation map T_{v} . Draw a graph of the original parabola and its image, and explain the geometric effect of the map T_{v} on the parabola. Find the vertex and x-intercepts of the graph of the image.

4. Given the vector $\mathbf{v} = \langle 3,2 \rangle$, find the image of the graph of $y + 1 = (x+1)^3$ under the translation map T_{v}. Draw the original graph and its image, and explain the geometric effect of the map T_{v} on the graph. Find the x-intercepts of the graph of the image.

5. Given the vector $\mathbf{v} = \langle 3, -3 \rangle$, find the image of the graph of $y + 2 = \sqrt{x+1}$ under the translation map T_{v}. Draw the original graph and its image, and explain the geometric effect of the map T_{v} on the graph. Find the x-intercepts of the graph of the image.

6. Given the vector $\mathbf{v} = \langle -1, -2 \rangle$, find the image of the graph of $y = \sqrt{9 - x^2}$ under the translation map T_{v}. Draw the original graph and its image, and explain the geometric effect of the map T_{v} on the graph. Find the x-intercepts of the graph of the image.

7. Given the vector $\mathbf{v} = \langle 1,3 \rangle$, find the image of the graph of $y = \frac{1}{x+2} + 1$ under the translation map T_{v}. Draw the original graph and its image, and explain the geometric effect of the map T_{v} on the graph. Find the equations of the asymptotes of the graph of the image.

8. Given the vector $\mathbf{v} = \langle -1,2 \rangle$, find the image of the graph of $y = |x+2| + 1$ under the translation map T_{v}. Draw the original graph and its image, and explain the geometric effect of the map T_{v} on the graph. Find the x-intercepts of the graph of the image.

9. Given the vector $\mathbf{v} = \langle 1, -2 \rangle$, find the image of the graph of $y = 2^x$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

10. Given the vector $\mathbf{v} = \langle -1, 3 \rangle$, find the image of the graph of $y = \log_2 x$, under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

11. Given the vector $\mathbf{v} = \langle 2, -3 \rangle$, find the image of the graph of $\frac{x^2}{4} + \frac{y^2}{16} = 1$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the new center, major and minor axis of the graph of the image.

12. Given the vector $\mathbf{v}$, find the image of the given point P under the translation map $T_\mathbf{v}$. Graph P and its image.

    a. $\mathbf{v} = \langle 3,2,1 \rangle$, $P = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,

    b. $\mathbf{v} = \langle -2,1,-1 \rangle$, $P = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$

13. Given the vector $\mathbf{v}$, find the image of the given plane under the translation map $T_\mathbf{v}$. Sketch the original vector and its image.

    a. $\mathbf{v} = \langle 2, -1,3 \rangle$, $3x - 2y - z = 0$.

    b. $\mathbf{v} = \langle -1,2,-1 \rangle$, $2x - y + z = 1$.

14. Given the vector $\mathbf{v}$, find the image of the given sphere under the translation map $T_\mathbf{v}$. Sketch the original sphere and its image.

    a. $\mathbf{v} = \langle -1,2,3 \rangle$, $x^2 + y^2 + z^2 = 9$.

    b. $\mathbf{v} = \langle -3, -2,1 \rangle$, $(x+2)^2 + (y-3)^2 + (z+1)^2 = 1$.

15. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the line $x - y = 1$ to the line $x - y = -3$. Sketch the original vector and its image.

16. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the parabola $y = x^2 + 4x + 1$ to the parabola $y = x^2$

17. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the circle with equation $x^2 + y^2 - 4x + 2y - 4 = 0$ to the circle with equation $(x+3)^2 + (y-4)^2 = 9$

18. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the graph of $y = \sqrt{x-3} + 2$ to the graph of $y = \sqrt{x+2} - 3$.

19. Find a vector $\mathbf{v}$ and translation map $T_{\mathbf{v}}$ that will translate the sphere $(x+2)^2 + (y-3)^2 + (z+1)^2 = 1$ to the sphere $(x-3)^2 + (y+1)^2 + (z+2)^2 = 1$

20. Given vectors $\mathbf{u} = \langle 2, -1, 3 \rangle$, $\mathbf{v} = \langle 2, 0, -2 \rangle$, and $\mathbf{w} = \langle -3, 6, 0 \rangle$, find the following.
    a. $3\mathbf{u} + \mathbf{v} + \mathbf{w}$
    b. $\mathbf{w} - 2\mathbf{v} - \mathbf{u}$
    c. $3\left(2\mathbf{u} - \frac{1}{2}\mathbf{v}\right) - \frac{1}{3}\mathbf{w}$
    d. $-2\mathbf{u} - 3(5\mathbf{v} - 3\mathbf{w})$.
    e. $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$.
    f. Show that $\mathbf{2}\|\mathbf{v}\| = \|\mathbf{2v}\|$.
    g. Show that $\|\mathbf{u} + \mathbf{v}\| \neq \|\mathbf{u}\| + \|\mathbf{v}\|$.
    h. Show that $\|\mathbf{v} - \mathbf{w}\| \neq \|\mathbf{v}\| - \|\mathbf{w}\|$.
    i. $\frac{\mathbf{1}}{\|\mathbf{u}\|}\mathbf{u}$ and $\left\|\frac{\mathbf{1}}{\|\mathbf{u}\|}\mathbf{u}\right\|$.