Lesson 15: Solving Equations Involving Linear Transformations of the Coordinate Space

Student Outcomes

- Students will represent systems of three or more simultaneous equations as a linear transformation in the form $Ax = b$, where A represents the linear transformation, x represents the coordinates of the pre-image, and b represents the coordinates of the image point.
- Students will apply inverse matrix operations to solve systems of linear equations with three equations.

Lesson Notes

The students will expand on their previous work from Lesson 14 to represent more complicated systems of equations as linear transformations in coordinate space. They will use software to calculate inverse matrices for systems of degree 3 and will apply the inverse of the coefficient matrix to the linear transformation equation to solve systems. Students will model real-world situations that can be written as systems of equations and solve the systems in a specific context including finding the intersection points of lines, designing a card game, and determining the number of coaches needed for a school athletic program (MP.4).

Classwork

Opening Exercise (5 minutes)

The students should complete the Opening Exercise in pairs. They should find the answer independently and then verify the solution with a partner. One pair could display the algebraic solution process on the board while another pair displays the matrix method of solving the system.

Opening Exercise

Mariah was studying currents in two mountain streams. She determined that five times the current in stream A was 8 feet per second stronger than twice the current in stream B. Another day she found that double the current in stream A plus ten times the current in stream B was 3 feet per second. She estimated the current in stream A to be 1.5 feet per second and stream B to be almost still (0 feet per second). Was her estimate reasonable? Explain your answer after completing parts (a)–(c).

a. **Write a system of equations to model this problem.**

$$\mathbf{5x = 8 + 2y}$$
$$\mathbf{2x + 10y = 3}$$

Scaffolding:

Provide a simpler system:

$$x + 2y = -1$$
$$2x - y = 3$$

Use scaffolded questions such as:

- Is $(1, 2)$ a solution to this system of equations? Explain how you know.
- Is $\left(\frac{43}{27}, -\frac{1}{54}\right)$ a solution to this system of equations? Explain how you know.

b. Solve the system using algebra.

Procedures may vary. An example of an appropriate algebraic procedure is shown:

Multiply the first equation by 5*:* $5(5x - 2y = 8) \rightarrow 25x - 10y = 40$

Add this equation to the second equation:

$$\begin{aligned} 25x - 10y &= 40 \\ +\ 2x + 10y &= 3 \\ 27x &= 43 \\ x &= \frac{43}{27} \end{aligned}$$

Back substitute the x*-value into the first equation and isolate* y*.*

$$\begin{aligned} 5 \cdot \frac{43}{27} - 2y &= 8 \\ -2y &= \frac{1}{27} \\ y &= -\frac{1}{54} \end{aligned}$$

Solution:

$$\begin{bmatrix} \frac{43}{27} \\ -\frac{1}{54} \end{bmatrix}$$

c. Solve the system by representing it as a linear transformation of the point x and then applying the inverse of the transformation matrix L to the equation. Verify that the solution is the same as that found in part (b).

$$\begin{bmatrix} 5 & -2 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 2 & 10 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{54}\begin{bmatrix} 10 & 2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{43}{27} \\ -\frac{1}{54} \end{bmatrix}$$

which is the same solution as in part (b).

Mariah's estimate was reasonable because $x = \frac{43}{27} \approx 1.59\ \mathrm{ft/sec}$ *and* $y = -\frac{1}{54} \approx -0.02\ \mathrm{ft/sec}$*. Since the signs are opposite, the currents are moving in opposite directions.*

Discussion (3 minutes)

- Do you prefer the algebraic or matrix method for solving systems of two linear equations? Explain why you prefer one method over the other. Share your thoughts with a partner.
  - *Answers will vary. An example of an appropriate response would be to state a preference for the algebraic method because it does not require calculating the inverse of the transformation matrix.*

- Describe when it might be more efficient to use algebra to solve systems of equations. Discuss your ideas with your partner.
  - *Answers will vary. An example of an appropriate response would be that it would be more efficient to use algebra to solve systems of equations when there are a few equations or when it is clear that the system is either inconsistent or dependent.*
- Describe when it might be more efficient to use matrices to solve systems of equations. Discuss your ideas with your partner.
  - *Answers will vary. An example of an appropriate response would be that using matrices might be more efficient to solve systems of equations when the system consists of several equations with several variables.*

Scaffolding:

- Students above grade level can complete the entire example with a partner with no leading questions from the teacher.
- Struggling students may need help setting up equations. Consider giving them questions to lead to the system such as:

Look at the first hand. If x is the value of green cards, y the value of yellow cards, and z the value of blue cards, let's write an equation for just the first hand.

Example 1 (12 minutes)

Students should complete parts (a)–(c) in pairs during the first few minutes. One pair could display their procedure and algebraic solution on the board, while another pair displays the linear transformation equation. Alternatively, each pair could display first their algebraic solution and then the transformation equation on white boards for a quick check. For part (d), a free software program or graphing calculator app should be used to demonstrate how to input matrices and calculate their inverses. The program should be projected on the board for student viewing. Alternatively, the screen of a graphing calculator could be projected to demonstrate the use of technology to calculate matrix inverses. Part (e) should be completed as a teacher-led exercise. The calculation could first be performed using technology, and then students could multiply A^{-1} by hand to verify that the software calculation is accurate.

- What features of the system of equations lend it to being solved using algebra?
  - *Answers will vary but might include that elimination could be used to find the value of the variables quickly because some of the equations in the system have corresponding coefficients that are either identical or opposites, which means that a variable could be eliminated by adding or subtracting two of the equations as they are written. Also, the values of the variables are integers, which facilitate computations when performing back substitution.*
- How did you find matrix L?
  - *The matrix is constructed of the coefficients of the variables of the system.*
- Can you always construct matrix L using the coefficients of the variables in the order in which they appear in the system? Why?
  - *No. The equations must be written in standard form first.*
- When we looked at two-dimensional space, how did we represent a system of equations using matrices?
  - *We wrote the system using the equation $Lx = b$.*
- What did the equation represent?
  - *A linear transformation of the point x to the image point b.*
- Could all systems be represented as a linear transformation of a point to a desired image point in two-dimensional space?
  - *No*

- How can you tell when systems could be solved using the linear transformation matrix equation?
  - *The determinant of the matrix L must be nonzero.*
- How did we solve the equation $Lx = b$ when the determinant of L is nonzero?
  - *We applied L^{-1} to both sides of the equation.*
- What is the result?
  - $x = L^{-1}b$
- And what does x represent geometrically?
  - *The pre-image point that, once L is applied to it, results in image point b.*
- Why can't we solve the equation $Lx = b$ if the determinant is 0?
  - *L does not have an inverse when the determinant is 0.*
- What does this imply geometrically?
  - *The transformation L cannot be reversed to produce point x.*
- Now, for three-dimensional space, how can we represent a system of equations as a linear transformation of a point?
  - *The same way that we represent two-dimensional systems, with the equation $Lx = b$.*
- And what condition do you think should be met to find the point x using this equation?
  - *The determinant must be nonzero.*
- Right. And how can you determine if a 3×3 matrix has a nonzero determinant?
  - *Answers may vary but might include using the diagonal method as shown:*

    If $L = \begin{bmatrix} l_{1,1} & l_{1,2} & l_{1,3} \\ l_{2,1} & l_{2,2} & l_{2,3} \\ l_{3,1} & l_{3,2} & l_{3,3} \end{bmatrix}$, $\begin{vmatrix} l_{1,1} & l_{1,2} & l_{1,3} \\ l_{2,1} & l_{2,2} & l_{2,3} \\ l_{3,1} & l_{3,2} & l_{3,3} \end{vmatrix} = l_{1,1}l_{2,2}l_{3,3} + l_{1,2}l_{2,3}l_{3,1} + l_{1,3}l_{2,1}l_{3,2} - l_{1,2}l_{2,1}l_{3,3} -$ $l_{1,1}l_{2,3}l_{3,2} - l_{1,3}l_{2,2}l_{3,1}$
- How can we use technology to determine if a matrix L has an inverse?
  - *Answers will vary but will probably include entering the matrix into the software program and applying the inverse function to it.*
- How do we know matrix L for the system in our example has an inverse?
  - *We can calculate it directly using the software.*
- How can we use technology to solve the equation $Lx = b$?
- Why does it make sense that our solution to part (e) was the same as the solution we found in part (b)?
  - *In part (e) we calculated $L^{-1}b$, which is equal to x, and x represents the point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, which is the solution to the system.*
- In this problem, which method of solving did you prefer and why? Discuss your reasoning with a partner.
  - *Answers will vary but might include that the algebraic method was preferable for this system because the elimination method was easy to use given the coefficients of the variables in the equations.*

- What factors would you consider when trying to determine whether to solve a system of equations in three-dimensional space using algebraic methods versus using matrices? Share your ideas with your partner.
  - *Answers may vary but might include the size and relationship between the corresponding coefficients in the equations, e.g., whether elimination could be used easily to find the values of the variables.*

Example 1

Dillon is designing a card game where different colored cards are assigned point values. Kryshna is trying to find the value of each colored card. Dillon gives him the following hints. If I have 3 green cards, 1 yellow card, and 2 blue cards in my hand, my total is 9. If I discard 1 blue card, my total changes to 7. If I have 1 card of each color (green, yellow, and blue), my cards total 1.

a. **Write a system of equations for each hand of cards if $x =$ value of green cards, $y =$ value of yellow cards, and $z =$ value of blue cards.**

$$3x + y + 2z = 9$$

$$3x + y + z = 7$$
$$x + y + z = 1$$

b. **Solve the system using any method you choose.**

Answers will vary. An example of an appropriate response is shown.

Subtract the second equation from the first.

$$\begin{aligned} 3x \mp y + 2z &= 9 \\ -\,(3x \mp y + z &= 7) \\ z &= 2 \end{aligned}$$

Subtract the third equation from the second.

$$\begin{aligned} 3x \mp y + z &= 7 \\ -(x + y + z &= 1) \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Back substitute values of x and z into the first equation, and isolate y.

$$\begin{aligned} 3(3) \mp y + 2(2) &= 9 \\ 13 - y &= 9 \\ y &= -4 \end{aligned}$$

Solution: $\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$ *Green cards are worth 3 points, yellow cards -4 points, and blue cards 2 points.*

c. **Let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$. Find a matrix L so that the linear transformation equation $Lx = b$ would produce image coordinates that are the same as the solution to the system of equations.**

$$L = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Scaffolding:

- Provide students with a flow chart that outlines the steps to solving systems of equations using inverse matrix operations.
- Provide written directions or printed screen shots to students to aid them in entering matrices and calculating their inverses using software.

d. **Enter matrix L into a software program or app, and try to calculate its inverse. Does L have an inverse? If so, what is it?**

$$L^{-1} = \begin{bmatrix} 0 & 0.50 & -0.50 \\ -1 & 0.50 & 1.50 \\ 1 & -1 & 0 \end{bmatrix}$$

e. **Calculate $L^{-1}\begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$. Verify that the result is equivalent to the solution to the system you calculated in part (b). Why should the solutions be equivalent?**

$$L^{-1}\begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.50 & -0.50 \\ -1 & 0.50 & 1.50 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0.50 & -0.50 \\ -1 & 0.50 & 1.50 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

which is equivalent to the solution from part (b). This makes sense because the system of equations can be represented using the linear transformation equation $Lx = b$. From what we learned in Lesson 14, $x = L^{-1}b$, so the solution found using inverse matrix operations should be the same as the solution set when solving the system of equations using algebra.

Exercises 1–3 (15 minutes)

The students should be placed into small groups. Each group should solve one of the problems. The group should spend about 5 minutes determining whether to solve the system using algebra or matrices, assisted by technology. They should solve the system using the method they chose and then verify the solution using back substitution. For the next few minutes, they should meet with the other groups assigned the same problem and verify their solutions, as well as discuss their arguments for the solution method they selected. During the last few minutes, each problem should be presented to the entire class. Presenters should display the problem, state the solution method chosen, justify the method selected, and display the solution. Each small group will need access to software that can be used to determine the inverse of matrices.

Exercises 1–3

1. **The system of equations is given:**

$$\begin{aligned} 2x - 4y + 6z &= 14 \\ 9x - 3y + z &= 10 \\ 5x + 9z &= 1 \end{aligned}$$

a. **Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.**

Answers will vary. An appropriate response is included.

Matrix method:

$$A = \begin{bmatrix} 2 & -4 & 6 \\ 9 & -3 & 1 \\ 5 & 0 & 9 \end{bmatrix}$$

Scaffolding:

Challenge advanced students by having them write a system of equations which cannot be solved using inverse matrix operations, and have them explain the process they used in constructing their system.

$$A^{-1} = \begin{bmatrix} -\frac{27}{340} & \frac{9}{85} & \frac{7}{170} \\ -\frac{19}{85} & -\frac{3}{85} & \frac{13}{85} \\ \frac{3}{68} & -\frac{1}{17} & \frac{3}{34} \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} -\frac{27}{340} & \frac{9}{85} & \frac{7}{170} \\ -\frac{19}{85} & -\frac{3}{85} & \frac{13}{85} \\ \frac{3}{68} & -\frac{1}{17} & \frac{3}{34} \end{bmatrix}\begin{bmatrix} 14 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{85} \\ -\frac{283}{85} \\ \frac{2}{17} \end{bmatrix}$$

b. **Verify your solution is correct.**

$$2\left(-\frac{1}{85}\right) - 4\left(-\frac{283}{85}\right) + 6\left(\frac{2}{17}\right) = 14$$
$$9\left(-\frac{1}{85}\right) - 3\left(-\frac{283}{85}\right) + 1\left(\frac{2}{17}\right) = 10$$
$$5\left(-\frac{1}{85}\right) + 9\left(\frac{2}{17}\right) = 1$$

c. **Justify your decision to use the method you selected to solve the system.**

Answers will vary. An example of an appropriate response is shown: The coefficients of the variables in the system did not contain corresponding variables that were identical or opposites, which might indicate that elimination might be a slower method than using the matrix method.

2. **An athletic director at an all-boys high school is trying to find out how many coaches to hire for the football, basketball, and soccer teams. To do this, he needs to know the number of boys that play each sport. He does not have names or numbers but finds a note with the following information listed:**

The total number of boys on all three teams is 86.

The number of boys that play football is 7 less than double the total number of boys playing the other two sports.

The number of boys that play football is 5 times the number of boys playing basketball.

a. **Write a system of equations representing the number of boys playing each sport where x is the number of boys playing football, y basketball, and z soccer.**

$$x + y + z = 86$$
$$2(y + z) - 7 = x$$
$$x = 5y$$

b. **Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.**

Answers will vary. An appropriate response is shown.

Matrix method:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \\ 1 & -5 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 86 \\ 7 \\ 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{2}{15} & -\frac{1}{15} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{2}{15} & -\frac{1}{15} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 86 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 55 \\ 11 \\ 20 \end{bmatrix}$$

c. **Verify that your solution is correct.**

$$1(55) + 1(11) + 1(20) = 86$$
$$2(11 + 20) - 7 = 55$$
$$55 = 5(11)$$

d. **Justify your decision to use the method you selected to solve the system.**

Answers will vary. An example of an appropriate response is shown: Using technology to solve the system using matrices generally takes less time than using algebra, especially when the solution set contains fractions.

3. **Kyra had $\$20,000$ to invest. She decided to put the money into three different accounts earning 3%, 5%, and 7% simple interest respectively and earned a total of $\$920.00$ in interest. She invested half as much money at 7% as at 3%. How much did she invest in each account?**

   a. **Write a system of equations that models this situation.**

$$x + y + z = 20000$$
$$0.03x + 0.05y + 0.07z = 920$$
$$x - 2z = 0$$

   b. **Find the amount invested in each account.**

   She invested $\$8,000$ at 3%, $\$8,000$ at 5%, and $\$4,000$ at 7%.

Closing (5 minutes)

Have students summarize in writing the process of solving a problem using systems of equations in three-dimensional space and matrices. Students could individually write a bulleted list including important steps in the procedure along with suggestions for recognizing special cases and recommendations for when representing a system as a linear transformation and applying inverse matrix operations is an efficient method of finding the solution. Have students share their results with a partner, and if time permits, students can display a compilation of their bulleted ideas on chart paper or on the board.

- *Create a system of equations.*
- *Systems of equations in three-dimensional space can be represented as linear transformations using the matrix equation $Ax = b$, where A represents the linear transformation of point x and b represents the image point after the linear transformation.*
- *The entries of matrix A are the coefficients of the system, and b is the column matrix representing the constants for the system.*
- *By applying inverse matrix operations, the solution to the system can be found by calculating $A^{-1}b$.*
- *Software programs can be used to calculate the inverse matrices for 3×3 systems and can be an efficient way to solve systems in three-dimensional space.*
- *As with systems in two-dimensional space, systems in three-dimensional space can only be represented as linear transformations of x when A has an inverse, e.g., the determinant of A is nonzero.*
- *If the determinant of A is 0, there is no single solution to the system.*
- *When the coefficients of corresponding variables are not identical or opposites, which expedite the process of elimination, it may be quicker to use matrices to solve the system rather than algebraic methods.*

Exit Ticket (5 minutes)

Name ____ Date ____________________

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Exit Ticket

The lemonade sales at a baseball game were described as follows:

The number of small lemonades purchased was the number of mediums sold plus double the number of larges sold.

The total number of all sizes sold was 70.

One and a half times the number of smalls purchased plus twice the number of mediums sold was 100.

Use a system of equations and its matrix representation to determine the number of small, medium, and large lemonades sold.

Exit Ticket Sample Solutions

The lemonade sales at a baseball game were described as follows:

The number of small lemonades purchased was the number of mediums sold plus double the number of larges sold.

The total number of all sizes sold was 70.

One and a half times the number of smalls purchased plus twice the number of mediums sold was 100.

Use a system of equations and its matrix representation to determine the number of small, medium, and large lemonades sold.

$$2l + m = s$$
$$s + m + l = 70$$
$$1.5s + 2m = 100$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \\ 1.5 & 2 & 0 \end{bmatrix} \begin{bmatrix} s \\ m \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ 70 \\ 100 \end{bmatrix}$$

$$\det(A) = 6$$

$$A^{-1}: \begin{bmatrix} \frac{-4}{9} & \frac{8}{9} & \frac{-2}{9} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{1}{9} & \frac{7}{9} & \frac{-4}{9} \end{bmatrix}$$

$$A^{-1}b: \begin{bmatrix} 40 \\ 20 \\ 10 \end{bmatrix}$$

Based on the system, projected sales are 40 small lemonades, 20 medium lemonades, and 10 large lemonades.

Problem Set Sample Solutions

1. **A small town has received funding to design and open a small airport. The airport plans to operate flights from three airlines. The total number of flights scheduled is 100. The airline with the greatest number of flights is planned to have double the sum of the flights of the other two airlines. The plan also states that the airline with the greatest number of flights will have 40 more flights than the airline with the least number of flights.**

   a. **Represent the situation described with a system of equations. Define all variables.**

$$a + b + c = 100$$
$$a = 2(b + c)$$
$$a = c + 40$$

a = number of flights for the airline with the most flights

b = number of flights for the airline with the second greatest number of flights

c = number of flights for the airline with the least number of flights

Note: variables used may differ.

b. **Represent the system as a linear transformation using the matrix equation $Ax = b$. Define matrices A, x, and b.**

Equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 40 \end{bmatrix}$$

A:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

x:

$$\begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix}$$

b:

$$\begin{bmatrix} 100 \\ 0 \\ 40 \end{bmatrix}$$

c. **Explain how you can determine if the matrix equation has a solution without solving it.**

Answers will vary but should indicate that technology can be used to verify that the matrix A has a nonzero determinant or that it has an inverse.

d. **Solve the matrix equation for x.**

$$x = A^{-1}b = \begin{bmatrix} 66\frac{2}{3} \\ 6\frac{2}{3} \\ 26\frac{2}{3} \end{bmatrix}$$

e. **Discuss the solution in context.**

The solution indicates that the airlines, from greatest to least number of flights, should have $66\frac{2}{3}$, $26\frac{2}{3}$, and $6\frac{2}{3}$ flights, respectively. This does not make sense given that the number of flights must be a whole number. Therefore, the number of flights granted should be approximately 67, 27, and 7, which would satisfy the second and third conditions and would results in only 1 more than the total number of flights planned for the airport.

2. **A new blockbuster movie opens tonight, and several groups are trying to buy tickets. Three types of tickets are sold: adult, senior (over 65), and youth (under 10). A groups of 3 adults, 2 youths, and 1 senior pays $\$54.50$ for their tickets. Another group of 6 adults and 12 youths pays $\$151.50$. A final group of 1 adult, 4 youths, and 1 senior pays $\$49.00$. What is the price for each type of ticket?**

a. **Represent the situation described with a system of equations. Define all variables.**

$$3a + 2y + 1s = 54.50$$
$$6a + 12y = 151.50$$
$$1a + 4y + 1s = 49$$

a = price of an adult ticket

y= price of a youth ticket

s = price of a senior ticket

Note: variables used may differ.

b. **Represent the system as a linear transformation using the matrix equation $Ax = b$.**

$$Ax = b \begin{bmatrix} 3 & 2 & 1 \\ 6 & 12 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ y \\ s \end{bmatrix} = \begin{bmatrix} 54.50 \\ 151.50 \\ 49 \end{bmatrix}$$

c. **Explain how you can determine if the matrix equation has a solution without solving it.**

Answers will vary but should indicate that technology can be used to verify that the matrix A has a nonzero determinant or that it has an inverse.

d. **Solve the matrix equation for x.**

$$x = A^{-1}b = \begin{bmatrix} 10.25 \\ 7.50 \\ 8.75 \end{bmatrix}$$

e. **Discuss the solution in context.**

An adult ticket costs $\$10.25$*, a youth ticket costs* $\$7.50$*, and a senior ticket costs* $\$8.75$.

f. **How much would it cost your family to attend the movie?**

Answers will vary.

3. **The system of equations is given:**

$$5w - 2x + y + 3z = 2$$
$$4w - x + 6y + 2z = 0$$
$$w - x - y - z = 3$$
$$2w + 7x - 3y + 5z = 12$$

a. **Write the system using a matrix equation in the form $Ax = b$.**

$$\begin{bmatrix} 5 & -2 & 1 & 3 \\ 4 & -1 & 6 & 2 \\ 1 & -1 & -1 & -1 \\ 2 & 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 12 \end{bmatrix}$$

b. **Write the matrix equation that could be used to solve for x. Then use technology to solve for x.**

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 & 3 \\ 4 & -1 & 6 & 2 \\ 1 & -1 & -1 & -1 \\ 2 & 7 & -3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{-8}{159} & \frac{22}{159} & \frac{85}{159} & \frac{13}{159} \\ \frac{-38}{159} & \frac{25}{159} & \frac{46}{159} & \frac{22}{159} \\ \frac{-11}{106} & \frac{17}{106} & \frac{-9}{106} & \frac{-1}{53} \\ \frac{31}{106} & \frac{-19}{106} & \frac{-71}{106} & \frac{-2}{53} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{395}{159} \\ \frac{326}{159} \\ \frac{-73}{106} \\ \frac{-199}{106} \end{bmatrix}$$

c. **Verify your solution using back substitution.**

$$5\left(\frac{395}{159}\right)-2\left(\frac{326}{159}\right)+\frac{-73}{106}+3\left(\frac{-199}{106}\right)=2$$
$$4\left(\frac{395}{159}\right)-\frac{326}{159}+6\left(\frac{-73}{106}\right)+2\left(\frac{-199}{106}\right)=0$$
$$\frac{395}{159}-\frac{326}{159}-\frac{-73}{106}-\frac{-199}{106}=3$$
$$2\left(\frac{395}{159}\right)+7\left(\frac{326}{159}\right)-3\left(\frac{-73}{106}\right)+5\left(\frac{-199}{106}\right)=12$$

d. **Based on your experience solving this problem and others like it in this lesson, what conclusions can you draw about the efficiency of using technology to solve systems of equations compared to using algebraic methods?**

Answers will vary. An example of an appropriate response would be that, in general, as the number of equations and variables in a system increases, the more efficient it is to use matrices and technology to solve systems when compared to using algebraic methods.

4. **In three-dimensional space, a point x is reflected over the xz plane resulting in an image point of $\begin{bmatrix}-3\\1\\-2\end{bmatrix}$.**

a. **Write the transformation as an equation in the form $Ax = b$, where A represents the transformation of point x resulting in image point b.**

$$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-3\\1\\-2\end{bmatrix}$$

b. **Use technology to calculate A^{-1}.**

$$A^{-1}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

c. **Calculate $A^{-1}b$ to solve the equation for x.**

$$x=\begin{bmatrix}-3\\-1\\-2\end{bmatrix}$$

d. **Verify that this solution makes sense geometrically.**

Reflecting a point over the xz plane effectively reflects it over the plane $y = 0$, which will change the sign of the y-coordinate and will leave the x- and z-coordinates unchanged.

5. **Jamie needed money and decided it was time to open her piggy bank. She had only nickels, dimes, and quarters. The total value of the coins was $\$85.50$. The number of quarters was 39 less than the number of dimes. The total value of the nickels and dimes was equal to the value of the quarters. How many of each type of coin did Jamie have? Write a system of equations and solve.**

   Let $x =$ the number of quarters, $y =$ the number of dimes, and $z =$ the number of nickels.

   $0.25x + 0.10y + 0.05z = 85.50$

   $x = y - 39$

   $0.05z + 0.10y = 0.25x$

   $x = 171, y = 210, z = 435$

   Jamie has 171 quarters, 210 dimes, and 435 nickels.