Lesson 15: Solving Equations Involving Linear Transformations of the Coordinate Space

Classwork

Opening Exercise

Mariah was studying currents in two mountain streams. She determined that five times the current in stream A was 8 feet per second stronger than twice the current in stream B. Another day she found that double the current in stream A plus ten times the current in stream B was 3 feet per second. She estimated the current in stream A to be 1.5 feet per second and stream B to be almost still (0 feet per second). Was her estimate reasonable? Explain your answer after completing parts (a)–(c).

a. Write a system of equations to model this problem.

b. Solve the system using algebra.

c. Solve the system by representing it as a linear transformation of the point x and then applying the inverse of the transformation matrix L to the equation. Verify that the solution is the same as that found in part (b).

Example 1

Dillon is designing a card game where different colored cards are assigned point values. Kryshna is trying to find the value of each colored card. Dillon gives him the following hints. If I have 3 green cards, 1 yellow card, and 2 blue cards in my hand, my total is 9. If I discard 1 blue card, my total changes to 7. If I have 1 card of each color (green, yellow, and blue), my cards total 1.

a. Write a system of equations for each hand of cards if $x =$ value of green cards, $y =$ value of yellow cards, and $z =$ value of blue cards.

b. Solve the system using any method you choose.

c. Let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$. Find a matrix L so that the linear transformation equation $Lx = b$ would produce image coordinates that are the same as the solution to the system of equations.

d. Enter matrix L into a software program or app, and try to calculate its inverse. Does L have an inverse? If so, what is it?

e. Calculate $L^{-1}\begin{bmatrix}9\\7\\1\end{bmatrix}$. Verify that the result is equivalent to the solution to the system you calculated in part (b). Why should the solutions be equivalent?

Exercises 1–3

1. The system of equations is given:

$$\begin{aligned}2x - 4y + 6z &= 14\\9x - 3y + z &= 10\\5x + 9z &= 1\end{aligned}$$

a. Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.

b. Verify your solution is correct.

c. Justify your decision to use the method you selected to solve the system.

2. An athletic director at an all-boys high school is trying to find out how many coaches to hire for the football, basketball, and soccer teams. To do this, he needs to know the number of boys that play each sport. He does not have names or numbers but finds a note with the following information listed:

The total number of boys on all three teams is 86.

The number of boys that play football is 7 less than double the total number of boys playing the other two sports.

The number of boys that play football is 5 times the number of boys playing basketball.

a. Write a system of equations representing the number of boys playing each sport where x is the number of boys playing football, y basketball, and z soccer.

b. Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.

c. Verify that your solution is correct.

d. Justify your decision to use the method you selected to solve the system.

3. Kyra had $\$20{,}000$ to invest. She decided to put the money into three different accounts earning 3%, 5%, and 7% simple interest respectively and earned a total of $\$920.00$ in interest. She invested half as much money at 7% as at 3%. How much did she invest in each account?

a. Write a system of equations that models this situation.

b. Find the amount invested in each account.

Problem Set

1. A small town has received funding to design and open a small airport. The airport plans to operate flights from three airlines. The total number of flights scheduled is 100. The airline with the greatest number of flights is planned to have double the sum of the flights of the other two airlines. The plan also states that the airline with the greatest number of flights will have 40 more flights than the airline with the least number of flights.
   a. Represent the situation described with a system of equations. Define all variables.
   b. Represent the system as a linear transformation using the matrix equation $Ax = b$. Define matrices A, x, and b.

      Equation:

      A:
      x:
      b:
   c. Explain how you can determine if the matrix equation has a solution without solving it.
   d. Solve the matrix equation for x.
   e. Discuss the solution in context.

2. A new blockbuster movie opens tonight, and several groups are trying to buy tickets. Three types of tickets are sold: adult, senior (over 65), and youth (under 10). A groups of 3 adults, 2 youths, and 1 senior pays $\$54.50$ for their tickets. Another group of 6 adults and 12 youths pays $\$151.50$. A final group of 1 adult, 4 youths, and 1 senior pays $\$49.00$. What is the price for each type of ticket?
   a. Represent the situation described with a system of equations. Define all variables.
   b. Represent the system as a linear transformation using the matrix equation $Ax = b$.
   c. Explain how you can determine if the matrix equation has a solution without solving it.
   d. Solve the matrix equation for x.
   e. Discuss the solution in context.
   f. How much would it cost your family to attend the movie?

3. The system of equations is given:

$$\begin{gathered} 5w - 2x + y + 3z = 2 \\ 4w - x + 6y + 2z = 0 \\ w - x - y - z = 3 \\ 2w + 7x - 3y + 5z = 12 \end{gathered}$$

   a. Write the system using a matrix equation in the form $Ax = b$.
   b. Write the matrix equation that could be used to solve for x. Then use technology to solve for x.
   c. Verify your solution using back substitution.
   d. Based on your experience solving this problem and others like it in this lesson, what conclusions can you draw about the efficiency of using technology to solve systems of equations compared to using algebraic methods?

4. In three-dimensional space, a point x is reflected over the xz plane resulting in an image point of $\begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$.

   a. Write the transformation as an equation in the form $Ax = b$, where A represents the transformation of point x resulting in image point b.

   b. Use technology to calculate A^{-1}.

   c. Calculate $A^{-1}b$ to solve the equation for x.

   d. Verify that this solution makes sense geometrically.

5. Jamie needed money and decided it was time to open her piggy bank. She had only nickels, dimes, and quarters. The total value of the coins was $\$85.50$. The number of quarters was 39 less than the number of dimes. The total value of the nickels and dimes was equal to the value of the quarters. How many of each type of coin did Jamie have? Write a system of equations and solve.