Lesson 14: Solving Equations Involving Linear Transformations of the Coordinate Plane

Student Outcomes

- Students will represent systems of equations as linear transformations of the form $Lx = b$, using matrix notation.
- Students will discover that the systems of equations written in the form $Lx = b$ can be solved by computing $x = L^{-1}b$ for all invertible matrices, L and they will apply this process to find solutions to systems of two linear equations in two variables.

Lesson Notes

In this lesson, students will explore the relationship between linear transformations of points in two-dimensional space and systems of equations. They will represent systems of equations as linear transformations represented by matrix equations and will apply inverse matrix multiplication to find the solutions to systems of equations, which will establish a foundation for solving systems of three or more equations using inverse matrix operations.

Classwork

Opening Exercise (3 minutes)

Students briefly review how to represent linear transformations of points in the coordinate plane as matrices in preparation for representing systems of equations as linear transformations. Students analyze and critique the reasoning in the given problem. Students can perform the matrix multiplication for each problem to verify that the matrices correctly represent the transformations. Debrief with the class after students work individually.

MP.3

Opening Exercise

Ahmad says the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ applied to the point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ will reflect the point to $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Randelle says that applying the matrix to the given point will produce a rotation of $180°$ about the origin. Who is correct? Explain your answer, and verify the result.

Randelle is correct. Applying the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ to the given point produces a rotation of $180°$ about the origin of the point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ to the image point $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$. Applying the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to the given point would produce a reflection to the image point $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Example 1 (20 minutes)

Students should complete part (a) with a partner. After a few minutes, each pair can share its example with another pair. The students should complete part (b) in small groups, e.g., with the pairs that exchanged examples for part (a). After a few minutes, a different group should describe each example from part (b) and justify why the linear transformation will not produce an image point of $\begin{bmatrix}4\\1\end{bmatrix}$. Students should be allowed to question the reasoning of other groups. Students should recognize that not all linear transformations will produce a desired image point. In particular, linear transformations represented by matrices with a determinant of 0 will not transform a point $\begin{bmatrix}x\\y\end{bmatrix}$ to any desired image point. Part (c) should be completed as a teacher-led discussion. Students will discover that systems of equations in two-dimensional space can be represented as a linear transformation represented by the equation $Lx = b$, where L represents the linear transformation of point x resulting in image point b. The coordinates of the pre-image can be found by applying the reverse of the linear transformation to the equation $Lx = b$, resulting in the equation $x = L^{-1}b$ for all invertible matrices L.

- How can you verify that the transformation matrix you found in part (a) accurately represents the transformation you described?
  - *Multiply the pre-image by the matrix to verify that the result is the column matrix* $\begin{bmatrix}4\\1\end{bmatrix}$.
- If we calculate the determinants of the transformation matrices you found for part (a), what do they have in common?
  - *They are all nonzero.*
- Why can't the transformation matrix $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ produce an image point of $\begin{bmatrix}4\\1\end{bmatrix}$?
  - *It collapses the points to the origin.*
- Why can't the transformation matrix $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ produce an image point of $\begin{bmatrix}4\\1\end{bmatrix}$?
  - *It produces only ordered pairs that have the same x- and y- coordinates.*
- Why can't the transformation matrix $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ produce an image point of $\begin{bmatrix}4\\1\end{bmatrix}$?
  - *It transforms all points to the y- axis.*
- What do you notice about the determinants of the examples that cannot produce the image point of $\begin{bmatrix}4\\1\end{bmatrix}$?
  - *They all have determinants of* 0.
- What have we learned about matrices that have a determinant of 0?
  - *They do not have inverses.*
- If matrix L represents a linear transformation, what does L^{-1} represent geometrically?
  - *It represents undoing the transformation.*
- So if L represents a rotation $90°$ clockwise about the origin, what would L^{-1} represent?
  - *Rotation of* $90°$ *counterclockwise or* $270°$ *clockwise about the origin.*

Scaffolding:

- Advanced students can find additional examples of linear transformations that are not invertible.
- Advanced students could be asked to make conjectures about the properties of linear transformations that can/cannot produce the image point without being prompted about the determinant.
- When reviewing Example 2, select several pre-image points and multiply them by the transformation matrices. Ask students if they notice any patterns in the image points and to make a conjecture about whether they could produce the image point $\begin{bmatrix}4\\1\end{bmatrix}$.

- Then if L does not have an inverse, what does that suggest geometrically?
  - *The transformation cannot be undone using another single transformation.*
- How would this apply to the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$?
  - *You cannot perform a single transformation to reverse the transformation.*
- Looking at part (c), what does the equation $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ represent geometrically?
  - *A linear transformation is applied to the point $\begin{bmatrix} x \\ y \end{bmatrix}$ to produce the image point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.*
- Explain geometrically how we could find the coordinates of $\begin{bmatrix} x \\ y \end{bmatrix}$ given the equation $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.
  - *You can undo the linear transformation represented by matrix L.*
- What would this look like algebraically?
  - $L^{-1}L\begin{bmatrix} x \\ y \end{bmatrix} = L^{-1}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.
- And this simplifies to? Explain.
  - $\begin{bmatrix} x \\ y \end{bmatrix} = L^{-1}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ because $L^{-1}L = I$*; it represents applying and then undoing the linear transformation.*
- Does L have an inverse? How can you tell?
  - *Yes. It has a determinant of 5, and all square matrices with nonzero determinants are invertible.*
- Recall the formula that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$where $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Given this, find the pre-image point $\begin{bmatrix} x \\ y \end{bmatrix}$.
  - $L^{-1}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{2(1)-(-1)(3)}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ -10 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- How else could you have found the coordinates of $\begin{bmatrix} x \\ y \end{bmatrix}$ without using inverse matrices?
  - *Multiply the matrices on the left side of the equation $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ to create the system of equations:*

$$2x - y = 4$$
$$3x + y = 1$$

- How can we verify that the solution we found using inverse matrices is correct?
  - *Substitute the ordered pair $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ into both equations, and verify that the resulting number sentences are true.*
- What does our solution mean geometrically?
  - *If you apply the transformation represented by the matrix $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ to the point $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, the image point will have coordinates of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.*

MP.7

- What do you notice about the relationship between the system of equations and the transformation matrix L? Explain your observations to your partner.
  - *The entries in the matrix represent the coefficients of the system when both equations are written in standard form.*

- Why might it be useful to learn the technique of using inverse matrix operations to solve systems rather than using algebraic methods like substitution or elimination? Discuss your ideas with your partner.
  - *Algebraic methods can become messy and cumbersome with increasing numbers of equations and variables.*

Example 1

a. **Describe a transformation not already discussed that results in an image point of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and represent the transformation using a 2×2.**

Answers will vary. An example of an appropriate response is as follows: A rotation of the point $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ $90°$ to the point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ can be represented with the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

MP.2

b. **Determine whether any of the matrices listed represent linear transformations that can produce the image point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Justify your answers by describing the transformations represented by the matrices.**

i. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

This matrix represents a collapse to the origin, so it cannot produce the image point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

ii. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

This matrix represents a transformation to the diagonal defined by $y = x$, so it cannot produce the image point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

iii. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

This matrix represents a transformation to the y-axis, so it cannot produce the image point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

c. **Suppose a linear transformation L is represented by the matrix $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$. Find a point $L\begin{bmatrix} x \\ y \end{bmatrix}$ so that $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.**

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Exercises 1–4 (11 minutes)

Allow students to complete the work in pairs. They should both complete the work and then compare answers. Some pairs may need more help or additional instruction. After a few minutes, discuss Exercise 1 to ensure that students are able to represent the systems of equations as linear transformations using matrix notation. Exercise 3 will be discussed in detail to summarize the main points of the lesson. Exercise 4 is a challenge exercise.

Exercises 1–4

1. **Given the system of equations**

$$2x + 5y = 4$$
$$3x - 8y = -25$$

a. **Show how this system can be written as a statement about a linear transformation of the form $Lx = b$, with $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -25 \end{bmatrix}$.**

$$\begin{bmatrix} 2 & 5 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -25 \end{bmatrix}$$

b. **Determine whether L has an inverse. If it does, compute $L^{-1}b$, and verify that the coordinates represent the solution to the system of equations.**

$$L^{-1}b = \frac{1}{2(-8) - (5)(3)}\begin{bmatrix} -8 & -5 \\ -3 & 2 \end{bmatrix}$$
$$L^{-1}b = \frac{1}{-31}\begin{bmatrix} -8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -25 \end{bmatrix} = \frac{1}{-31}\begin{bmatrix} 93 \\ -62 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Verification using back substitution:

$$2(-3) + 5(2) = 4$$
$$3(-3) - 8(2) = -25$$

2. **The path of a piece of paper carried by the wind into a tree can be modeled with a linear transformation, where $L = \begin{bmatrix} 3 & -4 \\ 5 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$.**

a. **Write an equation that represents the linear transformation of the piece of paper.**

$$\begin{bmatrix} 3 & -4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

b. **Solve the equation from part (a).**

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{29}\begin{bmatrix} 3 & 4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 10 \end{bmatrix} = \frac{1}{29}\begin{bmatrix} 58 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

c. **Use your solution to provide a reasonable interpretation of the path of the piece of paper under the transformation by the wind.**

Answers will vary. An example of an appropriate response would be that the piece of paper started on the ground 2 feet to the right of the location defined as the origin, and it was moved by the wind to a spot 6 feet to the right of the origin and 10 feet above the ground (in the tree).

3. **For each system of equations, write the system as a linear transformation represented by a matrix and apply inverse matrix operations to find the solution, or explain why this procedure cannot be performed.**

a. $6x + 2y = 1$
$y = 3x + 1$

$$\begin{bmatrix} 6 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} -1 \\ 9 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} \\ \frac{3}{4} \end{bmatrix}$$

b. $4x - 6y = 10$
$2x - 3y = 1$

This system cannot be represented as a linear transformation because the transformation matrix L has a determinant of 0. The system represents parallel lines, so there is no solution.

4. In a two-dimensional plane, A represents a rotation of $30°$ counterclockwise about the origin, B represents a reflection over the line $y = x$, and C represents a rotation of $60°$ counterclockwise about the origin.

a. Write matrices A, B, and C.

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} C = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b. Transformations A, B, and C are applied to point $\begin{bmatrix} x \\ y \end{bmatrix}$ successively and produce the image point $\begin{bmatrix} 1+2\sqrt{3} \\ 2-\sqrt{3} \end{bmatrix}$. Use inverse matrix operations to find $\begin{bmatrix} x \\ y \end{bmatrix}$.

We must apply the inverse transformations in the reverse order. The inverse of matrix C is

$$C^{-1} = 1 \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Applied to the image

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1+2\sqrt{3} \\ 2-\sqrt{3} \end{bmatrix} = \begin{bmatrix} -1+2\sqrt{3} \\ -2-\sqrt{3} \end{bmatrix}$$

The inverse of matrix B is $B^{-1} = -1\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Applied to the image $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1+2\sqrt{3} \\ -2-\sqrt{3} \end{bmatrix} = \begin{bmatrix} -2-\sqrt{3} \\ -1+2\sqrt{3} \end{bmatrix}$.

The inverse of matrix A is

$$A^{-1} = 1 \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Applied to the image

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -2-\sqrt{3} \\ -1+2\sqrt{3} \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Closing (6 minutes)

As a class, discuss the results of Exercises 1–3, focusing on how to represent systems of two linear equations as a linear transformation represented by matrix multiplication.

- How does the format of a linear system affect how it can be written as a linear transformation?
  - *The equations need to be written in standard form so the coefficients are represented accurately in the transformation matrix L.*
- What is the geometric interpretation of your conclusion to Exercise 3, part (b)?
  - *There are no points in the coordinate plane that, when the transformation represented by the matrix $\begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}$ is applied, will result in an image point of $\begin{bmatrix} 10 \\ 1 \end{bmatrix}$.*
- How are problems involving linear transformations solved in the coordinate plane?
  - *Systems of equations with a single solution can be represented as a linear transformation in the form of the equation $Lx = b$, where L is the transformation matrix, $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and b = the coordinates of the image.*
  - *The coordinates of the pre-image can be found by multiplying both sides of the transformation equation by L^{-1}, which effectively undoes the transformation.*
  - *When the determinant of L is 0, the linear transformation matrix does not have an inverse, and there is no solution to the system of equations whose coefficients are represented by L.*

Exit Ticket (5 minutes)

Name ____ Date _______________

Lesson 14: Solving Equations Involving Linear Transformations of the Coordinate Plane

Exit Ticket

In two-dimensional space, point x is rotated $180°$ to the point $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a. Represent the transformation of point x using an equation in the format $Lx = b$.

b. Use inverse matrix operations to find the coordinates of x.

c. Verify that this solution makes sense geometrically.

Exit Ticket Sample Solutions

In two-dimensional space, point x is rotated $180°$ to the point $\begin{bmatrix}3\\4\end{bmatrix}$.

a. Represent the transformation of point x using an equation in the format $Lx = b$.

$$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$$

b. Use inverse matrix operations to find the coordinates of x.

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{1}\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}-3\\-4\end{bmatrix}$$

c. Verify that this solution makes sense geometrically.

When a point is rotated $180°$ about the origin in the coordinate plane, the x- and y- coordinates of the image point are the opposite of those of the pre-image point.

Problem Set Sample Solutions

1. In a two-dimensional plane, a transformation represented by $L = \begin{bmatrix}1 & 5\\2 & -4\end{bmatrix}$ is applied to point x, resulting in an image point $\begin{bmatrix}0\\5\end{bmatrix}$. Find the location of the point before it was transformed.

a. Write an equation to represent the linear transformation of point x.

$$\begin{bmatrix}1 & 5\\2 & -4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\5\end{bmatrix}$$

b. Solve the equation to find the coordinates of the pre-image point.

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{-1}{14}\begin{bmatrix}-4 & -5\\-2 & 1\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{-1}{14}\begin{bmatrix}-25\\5\end{bmatrix} = \begin{bmatrix}\frac{25}{14}\\\frac{-5}{14}\end{bmatrix}$$

2. Find the location of the point $\begin{bmatrix}x\\y\end{bmatrix}$ before it was transformed when given:

a. The transformation $L = \begin{bmatrix}3 & 5\\1 & 2\end{bmatrix}$ and the resultant is $\begin{bmatrix}1\\2\end{bmatrix}$. Verify your answer.

$$\begin{bmatrix}3 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{1}\begin{bmatrix}2 & -5\\-1 & 3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-8\\5\end{bmatrix}$$

$$\begin{bmatrix}3 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}-8\\5\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

b. The transformation $L = \begin{bmatrix} 4 & 7 \\ -1 & -2 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Verify your answer.

$$\begin{bmatrix} 4 & 7 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -2 & -7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 7 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

c. The transformation $L = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Verify your answer.

$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

d. The transformation $L = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Verify your answer.

$$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

e. The transformation $L = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Verify your answer.

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{1}{5} \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{8}{5} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

3. **On a computer assembly line, a robot is placing a CPU onto a motherboard. The robot's arm is carried out by the transformation $L = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.**

   a. **If the CPU is attached to the motherboard at point $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, at what location does the robot pick up the CPU?**

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ 8 \end{bmatrix}$$

   b. **If the CPU is attached to the motherboard at point $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, at what location does the robot pick up the CPU?**

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   c. **Find the transformation $L = \begin{bmatrix} -1 & c \\ b & 3 \end{bmatrix}$ that will place the CPU starting at $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ onto the motherboard at the location $\begin{bmatrix} -8 \\ 3 \end{bmatrix}$.**

$$\begin{bmatrix} -1 & c \\ b & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

$$-2 - 3c = -8,\ c = 2$$

$$2b - 9 = 3,\ b = 6$$

$$\begin{bmatrix} -1 & 2 \\ 6 & 3 \end{bmatrix}$$

4. **On a construction site, a crane is moving steel beams from a truck bed to workers. The crane is programed to perform the transformation $L = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.**

   a. **If the workers are at location $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, where does the truck driver need to unload the steel beams so that the crane can pick them up and bring them to the workers?**

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   b. **If the workers move to another location $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where does the truck driver need to unload the steel beams so that the crane can pick them up and bring them to the workers?**

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}$$

5. A video game soccer player is positioned at $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, where he kicks the ball. The ball goes into the goal, which is at point $\begin{bmatrix} 10 \\ 0 \end{bmatrix}$. When the player moves to point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and kicks the ball, he misses the goal. The ball lands at point $\begin{bmatrix} 10 \\ -1 \end{bmatrix}$. What is the program/transformation $L = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that this video soccer player uses?

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, 2c = 10, c = 5.\ 2d = 0, d = 0$$

$$\begin{bmatrix} a & 5 \\ b & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}, a + 5 = 10, a = 5.\ b = -1$$

$$\begin{bmatrix} 5 & 5 \\ -1 & 0 \end{bmatrix}$$

6. Tim bought 5 shirts and 3 pair of pants, and it cost him $\$250$. Scott bought 3 shirts and 2 pair of pants, and it cost him $\$160$. All the shirts have the same cost, and all the pants have the same cost.

   a. Write a system of linear equations to find the cost of the shirts and pants.

$$\begin{cases} 5S + 3P = 250 \\ 3S + 2P = 160 \end{cases}$$

   b. Show how this system can be written as a statement about a linear transformation of the form $Lx = b$ with $x = \begin{bmatrix} S \\ P \end{bmatrix}$ and $b = \begin{bmatrix} 250 \\ 160 \end{bmatrix}$.

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} S \\ P \end{bmatrix} = \begin{bmatrix} 250 \\ 160 \end{bmatrix}$$

   c. Determine whether L has an inverse. If it does, compute $L^{-1}b$, and verify your answer to the system of equations.

   The determinant of $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ is 1. $L^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

$$\begin{bmatrix} S \\ P \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 250 \\ 160 \end{bmatrix} = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$$

   Verification using back substitution: $5(20) + 3(50) = 250, 3(20) + 2(50) = 160$

7. In a two-dimensional plane, A represents a reflection over the x-axis, B represents a reflection over the y-axis, and C represents a reflection over the line $y = x$.

   a. Write matrices A, B, and C.

   $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. **Write an equation for each linear transformation, assuming that each one produces an image point of $\begin{bmatrix}-2\\-3\end{bmatrix}$.**

For transformation A,

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\-3\end{bmatrix}$$

For transformation B,

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\-3\end{bmatrix}$$

For transformation C,

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\-3\end{bmatrix}$$

c. **Use inverse matrix operations to find the pre-image point for each equation. Explain how your solutions make sense based on your understanding of the effect of each geometric transformation on the coordinates of the pre-image points.**

For transformation A, $\begin{bmatrix}x\\y\end{bmatrix}=-1\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-2\\-3\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}$. When a point is reflected over the x-axis, the x-coordinate remains unchanged, and the y-coordinate changes signs.

For transformation B, $\begin{bmatrix}x\\y\end{bmatrix}=-1\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-2\\-3\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix}$. When a point is reflected over the y-axis, the y-coordinate remains unchanged, and the x-coordinate changes signs.

For transformation C, $\begin{bmatrix}x\\y\end{bmatrix}=-1\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}-2\\-3\end{bmatrix}=\begin{bmatrix}-3\\-2\end{bmatrix}$. When a point is reflected over the line $y=x$, the coordinates of the pre-image point are interchanged (x and y are switched).

8. **A system of equations is shown:**

$$2x+5y+z=3$$
$$4x+y-z=5$$
$$3x+2y+4z=1$$

a. **Represent this system as a linear transformation in three-dimensional space represented by a matrix equation in the form of $Lx=b$.**

$$\begin{bmatrix}2 & 5 & 1\\4 & 1 & -1\\3 & 2 & 4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\5\\1\end{bmatrix}$$

b. **What assumption(s) need to be made to solve the equation in part (a) for x.**

To use inverse operations, we need to assume that L has an inverse.

c. **Use algebraic methods to solve the system.**

Adding equations 1 and 2 gives $6x + 6y = 8$.

Adding 4 times equation 2 and equation 3 gives $19x + 6y = 21$.

Subtracting $(6x + 6y = 8)$ from $(19x + 6y = 21)$ gives $13x = 13$, so $x = 1$.

Back substituting into $6x + 6y = 8$ gives $6y = 2$, or $y = \frac{1}{3}$.

Back substituting for y and x into the first equation gives $2(1) + 5\left(\frac{1}{3}\right) + z = 3$, so $z = -\frac{2}{3}$.

$$x = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{-2}{3} \end{bmatrix}$$

9. **Assume**

$$L^{-1} = \frac{1}{78}\begin{bmatrix} -6 & 18 & 6 \\ 19 & -5 & -6 \\ -5 & -11 & 18 \end{bmatrix}$$

Use inverse matrix operations to solve the equation from Problem 8, part (a) for x. Verify that your solution is the same as the one you found in Problem 8, part (c).

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{78}\begin{bmatrix} -6 & 18 & 6 \\ 19 & -5 & -6 \\ -5 & -11 & 18 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{-2}{3} \end{bmatrix},$$

which is the same solution found in part (c).