Lesson 12: Matrix Multiplication Is Distributive and Associative

Student Outcomes

- Students discover and verify that matrix multiplication is distributive.
- Students discover and verify that matrix multiplication is associative.

Lesson Notes

In this lesson, students use specific matrix transformations on points to show that matrix multiplication is distributive and associative. They then revisit some of the properties of matrices to prove that these properties hold for all matrices under multiplication.

Classwork

Opening Exercise (10 minutes)

Students review transformations represented by matrices studied in previous lessons in preparation for their work in Lesson 13. Students will write the matrix that represents the given transformation. Each transformation should be written on the board, and as the class agrees on the correct matrix that represents that transformation, a student should write the matrix under the transformation heading. Students can do this on paper at their desks, or this can be completed as a Rapid White Board Exchange. This could also be done in pairs or groups, and each group could present one transformation and explain it. The exercise can be modified to a matching activity if students are struggling.

MP.2

Opening Exercise

Write the 3×3 matrix that would represent the transformation listed.

a. **No change when multiplying (the multiplicative identity matrix)**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b. **No change when adding (the additive identity matrix)**

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c. **A rotation about the x-axis of θ degrees**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

MP.2

d. A rotation about the y-axis of θ degrees

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

e. A rotation about the z-axis of θ degrees

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

f. A reflection over the xy-plane

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

g. A reflection over the yz-plane

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

h. A reflection over the xz-plane

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i. A reflection over $y = x$ in the xy-plane

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaffolding:

Students could create a graphic organizer listing the matrix in one column and transformation represented in the next column.

Example 1 (15 minutes)

This example should be completed by students in pairs for parts (a) through (g). Part (h) should be completed as a teacher-led discussion after debriefing together the results of parts (a) through (g). Students look at transformations in two-dimensional space on a point. They predict the transformation, write a matrix that would represent that transformation, and apply the transformation to the point. Through a series of steps, they see that matrix multiplication is distributive and associative, revisit properties of matrices, and prove that these properties hold for all matrices under matrix multiplication.

- What are the dimensions of a matrix that represents two-dimensional space?
  - 2×2
- Using your geometric intuition, can you tell me how the point $(1, 1)$ would transform if it was rotated $90°$ clockwise? What are the new coordinates?
  - *The x-coordinate would stay the same, and the y-coordinate would change signs.* $(1, -1)$.

Scaffolding:

- Students with spatial issues may not be able to see the transformations in three-dimensions. Consider using a graphing program to help them visualize the transformations or demonstrating each one visually in some other way.
- For advanced learners, have students work in groups to complete this example with no leading questions. They should write up a summary and present their finding during the class discussion.

- What would happen to the point $(1, 1)$ if it was rotated $180°$ counterclockwise? What are the new coordinates?
  - *The x- and y-coordinates would change signs.* $(-1, -1)$.
- If matrix A represents a rotation of $90°$ clockwise, write matrix A. Show your steps.
  - $A = \begin{bmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- If matrix B represents a reflection about the x-axis, write matrix B.

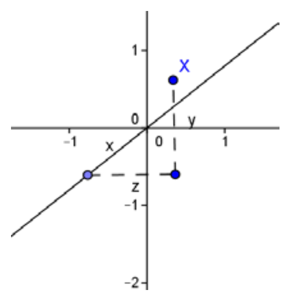
  $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- If matrix C represents a rotation of $180°$ about the y-axis, write matrix **C**. Show your steps.
  - $C = \begin{bmatrix} \cos(180°) & -\sin(180°) \\ \sin(180°) & \cos(180°) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
- $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; confirm your answers above by performing the transformations BX and CX. Were your instincts correct? If not, analyze your mistakes.
  - $BX = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $CX = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$; *answers will vary.*
- What is $BX + CX$? What did you notice about the results of part (e) and part (g)?
  - *The results are the same matrix.*
- Write out the mathematical statement.
  - $A(BX + CX) = (AB)X + (AC)X$
- Do you think matrix multiplication is distributive? If so, what would $A(BX + CX)$ be equivalent to?
  - $A(BX + CX) = A(BX) + A(CX)$
- What property would be necessary in order for us to prove matrix multiplication is distributive given what we have already proven?
  - *We have proven for this set of matrices that $A(BX + CX) = (AB)X + (AC)X$, but we need $A(BX + CX) = A(BX) + A(CX)$. That means $(AB)X + (AC)X = A(BX) + A(CX)$, which means the associative property must hold.*
- The sum of two matrices is the sum of corresponding elements. What would be the value of $(B + C)X$?
  - $BX + CX$.
- We also know that applying a matrix B and then matrix A is the same as applying the matrix product AB. So, $A(BX) = ?$ And $A(CX) = ?$
  - $(AB)X$
  - $(AC)X$
- Let's use what we have just stated to explain why $A(B + C)$ and $AB + AC$ have the same geometric effect on a point or points for any matrices A, B, and C.
- How do we know that $\big(A(B + C)\big)X = A((B + C)X)$?
  - *Applying the product of A and $(B + C)$ has the same effect as applying $(B + C)$ and then A.*
- Now $A\big((B + C)X\big) = A(BX + CX)$. Why?
  - *The sum of two matrices acts by summing the image.*

- We also know that $A(BX + CX) = A(BX) + A(CX)$. Why?
  - *Each matrix represents a linear transformation.*
- Continuing, $A(BX) + A(CX) = (AB)X + (AC)X$. Explain.
  - *Applying B then A is the same as applying AB.*
  - *Applying C then A is the same as applying AC.*
- And finally, why does $(AB)X + (AC)X = (AB + AC)X$?
  - *The sum of two matrices acts by summing the images.*
- Therefore, what must be true?
  - $A(B + C) = AB + AC$
- We have just shown that matrix multiplication is distributive and associative.
- Take a few minutes and discuss what we have just proven with your neighbor. Write a summary.

Example 1

In three-dimensional space, let A represent a rotation of $90°$ about the x-axis, B represent a reflection about the yz-plane, and C represent a rotation of $180°$ about the z-axis. Let $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a. **As best you can, sketch a three-dimensional set of axes and the location of the point X.**



b. **Using only your geometric intuition, what are the coordinates of BX? CX? Explain your thinking.**

$BX = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$; $CX = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$; *answers will vary but could include that when rotating about the x-axis $90°$, only the x-coordinate would change signs; however, when rotating about the z-axis $180°$, the x- and y-coordinates would change signs.*

c. **Write down matrices B and C, and verify or disprove your answers to part (b).**

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; BX = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; CX = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

d. **What is the sum of $BX + CX$?**

$$BX + CX = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

e. **Write down matrix A, and compute $A(BX + CX)$.**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A(BX + CX) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$$

f. **Compute AB and AC.**

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

g. **Compute $(AB)X$, $(AC)X$, and their sum. Compare your result to your answer to part (e). What do you notice?**

$$(AB)X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$(AC)X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$(AB)X + (AC)X = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$$

$$A(BX + CX) = (AB)X + (AC)X$$

h. **In general, must $A(B + C)$ and $AB + AC$ have the same geometric effect on point, no matter what matrices A, B, and C are? Explain.**

Yes. See full explanation in questions above.

Exercises 1–2 (10 minutes)

Allow students to complete the exercises in pairs, each doing the work individually and then comparing answers. Some groups may need more help or one-on-one instructions. Exercise 1, part (b) and Exercise 2, part (a) will be fully discussed as a class and are the focus of the lesson summary. Have early finishers write the results of Exercise 1, part (c) and Exercise 2, parts (a)–(b) on large paper, and prepare an explanation to present to the class as part of the lesson summary.

Exercises 1–2

1. Let $A = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$, $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and $C = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$.

   a. Write down the products AB, AC, and $A(B+C)$.

   $$AB = \begin{bmatrix} ax+bz & cx+dz \\ ay+bw & cy+dw \end{bmatrix};\ AC = \begin{bmatrix} ex+fz & gx+hz \\ ey+fw & gy+hw \end{bmatrix};$$

   $$A(B+C) = \begin{bmatrix} (a+e)x+(b+f)z & (c+g)x+(d+h)z \\ (a+e)y+(b+f)w & (c+g)y+(d+h)w \end{bmatrix}$$

   b. Verify that $A(B+C) = AB + AC$.

   $$A(B+C) = \begin{bmatrix} (a+e)x+(b+f)z & (c+g)x+(d+h)z \\ (a+e)y+(b+f)w & (c+g)y+(d+h)w \end{bmatrix} =$$
   $$\begin{bmatrix} ax+ex+bz+fz & cx+gx+dz+hz \\ ay+ey+bw+fw & cy+gy+dw+hw \end{bmatrix}$$

   $$AB + AC = \begin{bmatrix} ax+bz & cx+dz \\ ay+bw & cy+dw \end{bmatrix} + \begin{bmatrix} ex+fz & gx+hz \\ ey+fw & gy+hw \end{bmatrix} =$$
   $$\begin{bmatrix} ax+bz+ex+fz & cx+dz+gx+hz \\ ay+bw+ey+fw & cy+dw+gy+hw \end{bmatrix}$$

   Therefore, $A(B+C) = AB + AC$.

2. Suppose A, B, and C are 3×3 matrices, and X is a point in three-dimensional space.

   a. Explain why the point $\big(A(BC)\big)X$ must be the same point as $\big((AB)C\big)X$.

   $\big(A(BC)\big)X = (A)(B)(C)X$. Applying BC and then A is the same as applying C, then B, and then A.

   $(A)(B)(C)X = \big((AB)C\big)X$. Applying C, then B, and then A is the same as applying C and then AB.

   b. Explain why matrix multiplication must be associative.

   Matrix multiplication is associative because performing the transformation B and then A on a point X is the same as applying the product of AB to point X.

   c. Verify using the matrices from Exercise 1 that $A(BC) = (AB)C$.

   $$BC = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{bmatrix}$$

   $$A(BC) = \begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{bmatrix} = \begin{bmatrix} x(ae+cf)+z(be+df) & x(ag+ch)+z(bg+dh) \\ y(ae+cf)+w(be+df) & y(ag+ch)+w(bg+dh) \end{bmatrix}$$

   $$AB = \begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ax+bz & cx+dz \\ ay+bw & cy+dw \end{bmatrix}$$

   $$(AB)C = \begin{bmatrix} ax+bz & cx+dz \\ ay+bw & cy+dw \end{bmatrix}\begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} (ax+bz)e+(cx+dz)f & (ax+bz)g+(cx+dz)h \\ (ay+bw)e+(cy+dw)f & (ay+bw)g+(cy+dw)h \end{bmatrix}$$

   $$A(BC) = \begin{bmatrix} aex+cfx+bez+dfz & agx+chx+bgz+dhz \\ aey+dfy+bew+dfw & agy+chy+bgw+dhw \end{bmatrix} = (AB)C$$

Closing (5 minutes)

As a class, discuss the results of Exercises 1 and 2, focusing on the properties of matrix multiplication (distributive and associative) that were discovered/confirmed in this lesson and these exercises. Alternately, revisit the properties used in the example and the exercises of matrix multiplication.

- What matrix multiplication property did you prove in Exercise 1? Explain how you proved it in part (c).
  - *The distributive property. See exercises to ensure students understand properties.*
- In Exercise 2, parts (a) and (b), explain why you knew that matrix multiplication was associative.
  - *Review steps from exercises to ensure students understand properties*.
- Some properties of matrices:
  - Each matrix represents a linear transformation, so $A(BX + CX) = A(BX) + A(CX)$.
  - The sum of two matrices acts by summing the image points, $(B + C)X = BX + CX$.
  - Applying matrix B and then A is the same as applying the matrix product AB, meaning
$$A(BX) = (AB)X.$$

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 12: Matrix Multiplication Is Distributive and Associative

Exit Ticket

In three-dimensional space, matrix A represents a $180°$ rotation about the y-axis, matrix B represents a reflection about the xz-plane, and matrix C represents a reflection about xy-plane. Answer the following:

a. Write matrices A, B, and C.

b. If $X = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, compute $A(BX + CX)$.

c. What matrix operations are equivalent to $A(BX + CX)$? What property is shown?

d. Would $\big(A(BC)\big)X = \big((AB)C\big)X$? Why?

Exit Ticket Sample Solutions

In three-dimensional space, matrix A represents a $180°$ rotation about the y-axis, matrix B represents a reflection about the xz-plane, and matrix C represents a reflection about xy-plane. Answer the following:

a. Write matrices A, B, and C.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix};\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

b. If $X = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, compute $A(BX + CX)$.

$$BX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix};\ CX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

$$BX + CX = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix};\ A(BX + CX) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

c. What matrix operations are equivalent to $A(BX + CX)$? What property is shown?

$A(BX + CX) = (AB)X + (AC)X$; matrix multiplication is distributive.

d. Would $\big(A(BC)\big)X = \big((AB)C\big)X$? Why?

Yes; matrix multiplication is associative.

Problem Set Sample Solutions

1. Let matrix $A = \begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}$, matrix $B = \begin{pmatrix} 4 & 4 \\ 3 & 9 \end{pmatrix}$, and matrix $C = \begin{pmatrix} 8 & 2 \\ 7 & -5 \end{pmatrix}$. Calculate the following:

a. AB

$$\begin{pmatrix} 6 & -6 \\ -4 & -4 \end{pmatrix}$$

b. AC

$$\begin{pmatrix} 10 & 16 \\ -8 & -2 \end{pmatrix}$$

c. $A(B + C)$

$$B + C = \begin{pmatrix} 12 & 6 \\ 10 & 4 \end{pmatrix};\ A(B + C) = \begin{pmatrix} 16 & 10 \\ -12 & -6 \end{pmatrix};\ AB + AC = \begin{pmatrix} 16 & 10 \\ -12 & -6 \end{pmatrix}$$

We have that $A(B + C) = AB + AC$.

d. $AB + AC$

$$\begin{pmatrix} 16 & 10 \\ -12 & -6 \end{pmatrix}$$

e. $(A + B)C$

$(A + B)C = AC + BC$, *so* BC *has not been calculated yet. We get,*

$$BC = \begin{pmatrix} 60 & -12 \\ 87 & -39 \end{pmatrix}.$$

So,

$$(A + B)C = \begin{pmatrix} 70 & 4 \\ 79 & -41 \end{pmatrix}.$$

f. $A(BC)$

$$\begin{aligned} A(BC) &= A\begin{pmatrix} 60 & -12 \\ 87 & -39 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 42 \\ -60 & 12 \end{pmatrix} \end{aligned}$$

2. Apply each of the transformations you found in Problem 1 to the points $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, and $x + y$.

a. $(AB)x = \begin{pmatrix} 0 \\ -8 \end{pmatrix}$

$(AB)y = \begin{pmatrix} -30 \\ 4 \end{pmatrix}$

$(AB)(x + y) = \begin{pmatrix} -30 \\ -4 \end{pmatrix}$

b. $(AC)x = \begin{pmatrix} 26 \\ -10 \end{pmatrix}$

$(AC)y = \begin{pmatrix} 2 \\ 20 \end{pmatrix}$

$(AC)(x + y) = \begin{pmatrix} 28 \\ 10 \end{pmatrix}$

c. $\big(A(B + C)\big)x = \begin{pmatrix} 26 \\ -18 \end{pmatrix}$

$\big(A(B + C)\big)y = \begin{pmatrix} -28 \\ 24 \end{pmatrix}$

$\big(A(B + C)\big)(x + y) = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

d. *Same as part (c)*

$(AB + AC)x = \begin{pmatrix} 26 \\ -18 \end{pmatrix}$

$(AB + AC)y = \begin{pmatrix} -28 \\ 24 \end{pmatrix}$

$(AB + AC)(x + y) = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

e. $\big((A + B)C\big)x = \begin{pmatrix} 74 \\ 38 \end{pmatrix}$

$\big((A + B)C\big)y = \begin{pmatrix} -202 \\ -319 \end{pmatrix}$

$\big((A + B)C\big)(x + y) = \begin{pmatrix} -128 \\ -281 \end{pmatrix}$

f. $\left(A(BC)\right)x = \begin{pmatrix} 48 \\ -48 \end{pmatrix}$
$\left(A(BC)\right)y = \begin{pmatrix} 66 \\ 204 \end{pmatrix}$
$\left(A(BC)\right)(x+y) = \begin{pmatrix} 114 \\ 156 \end{pmatrix}$

3. Let A, B, C, and D be any four square matrices of the same dimensions. Use the distributive property to evaluate the following:

a. $(A+B)(C+D)$

$$(A+B)C + (A+B)D = AC + BC + AD + BD$$

b. $(A+B)(A+B)$

$$AA + AB + BA + BB$$

c. What conditions need to be true for part (b) to equal $AA + 2AB + BB$?

$AB = BA$ needs to be true.

4. Let A be a 2×2 matrix and B, C be the scalar matrices $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and $C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Answer the following questions.

a. Evaluate the following:

i. BC

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

ii. CB

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

iii. $B + C$

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

iv. $B - C$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

b. Are your answers to part (a) what you expected? Why or why not?

Answers may vary. Students should expect that the matrices will behave like real numbers since they represent scalars.

c. Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$; does $AB = BA$? Does $AC = CA$?

Yes. $AB = \begin{pmatrix} 2x & 2y \\ 2z & 2w \end{pmatrix} = BA$, and $AC = \begin{pmatrix} 3x & 3y \\ 3z & 3w \end{pmatrix}$.

d. **What is $(A+B)(A+C)$? Write the matrix A with the letter and not in matrix form. How does this compare to $(x+2)(x+3)$?**

$$\begin{aligned}(A+B)(A+C) &= AA+BA+AC+BC\\ &= AA+2A+3A+\begin{pmatrix}6 & 0\\ 0 & 6\end{pmatrix}\\ &= AA+5A+\begin{pmatrix}6 & 0\\ 0 & 6\end{pmatrix}\end{aligned}$$

This is identical to $(x+2)(x+3)$, with $x=A$.

e. **With B and C given as above, is it possible to factor $AA-A-BC$?**

Yes. We need factors of $-BC$ that add to -1. It looks like B and $-C$ work, so we get,

$$(A+B)(A-C).$$

5. **Define the sum of any two functions with the same domain to be the function $f+g$ such that for each x in the domain of f and g, $(f+g)(x)=f(x)+g(x)$. Define the product of any two functions to be the function fg, such that for each x in the domain of f and g, $(fg)(x)=\big(f(x)\big)\big(g(x)\big)$.**

Let f, g, and h be real-valued functions defined by the equations $f(x)=3x+1$, $g(x)=-\frac{1}{2}x+2$, and $h(x)=x^2-4$.

a. **Does $f(g+h)=fg+fh$?**

Yes. If we can show that $\big(f(g+h)\big)(x)=(fg)(x)+(fh)(x)$, then we will have shown that the functions are equal to each other.

$$\begin{aligned}\big(f(g+h)\big)(x) &= \big(f(x)\big)\big((g+h)(x)\big)\\ &= (3x+1)\left(-\frac{1}{2}x+2+x^2-4\right)\\ &= (3x+1)\left(\left(-\frac{1}{2}x+2\right)+(x^2-4)\right)\\ &= (3x+1)\left(-\frac{1}{2}x+2\right)+(3x+1)(x^2-4)\\ &= \big(f(x)\big)\big(g(x)\big)+\big(f(x)\big)\big(h(x)\big)\\ &= (fg)(x)+(fh)(x)\end{aligned}$$

b. **Show that this is true for any three functions with the same domains.**

$$\begin{aligned}\big(f(g+h)\big)(x) &= \big(f(x)\big)\big((g+h)(x)\big)\\ &= \big(f(x)\big)\big(g(x)+h(x)\big)\\ &= \big(f(x)\big)\big(g(x)\big)+\big(f(x)\big)\big(h(x)\big)\\ &= (fg)(x)+(fh)(x)\end{aligned}$$

c. Does $f \circ (g + h) = f \circ g + f \circ h$ for the functions described above?

No.

$$\begin{aligned}(f \circ (g + h))(x) &= f\left(-\frac{1}{2}x + 2 + x^2 - 4\right)\\ &= 3 \cdot \left(-\frac{1}{2}x + 2 + x^2 - 4\right) + 1\\ &= 3 \cdot \left(-\frac{1}{2}x + 2\right) + 3 \cdot (x^2 - 4) + 1\end{aligned}$$

The addition by 1 prevented it from working. If $f(x)$ would have been a proportion, then the composition would have worked.