Lesson 12: Matrix Multiplication Is Distributive and Associative

Classwork

Opening Exercise

Write the 3×3 matrix that would represent the transformation listed.

a. No change when multiplying (the multiplicative identity matrix)

b. No change when adding (the additive identity matrix)

c. A rotation about the x-axis of θ degrees

d. A rotation about the y-axis of θ degrees

e. A rotation about the z-axis of θ degrees

f. A reflection over the xy-plane

g. A reflection over the yz-plane

h. A reflection over the xz-plane

i. A reflection over $y = x$ in the xy-plane

Example 1

In three-dimensional space, let A represent a rotation of $90°$ about the x-axis, B represent a reflection about the yz-plane, and C represent a rotation of $180°$ about the z-axis. Let $\mathrm{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a. As best you can, sketch a three-dimensional set of axes and the location of the point X.

b. Using only your geometric intuition, what are the coordinates of BX? CX? Explain your thinking.

c. Write down matrices B and C, and verify or disprove your answers to part (b).

d. What is the sum of $BX + CX$?

e. Write down matrix A, and compute $A(BX + CX)$.

f. Compute AB and AC.

g. Compute $(AB)X$, $(AC)X$, and their sum. Compare your result to your answer to part (e). What do you notice?

h. In general, must $A(B + C)$ and $AB + AC$ have the same geometric effect on point, no matter what matrices A, B, and C are? Explain.

Exercises 1–2

1. Let $A = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$, $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and $C = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$.

   a. Write down the products AB, AC, and $A(B + C)$.

b. Verify that $A(B+C) = AB + AC$.

2. Suppose A, B, and C are 3×3 matrices, and X is a point in three-dimensional space.

a. Explain why the point $\big(A(BC)\big)X$ must be the same point as $\big((AB)C\big)X$.

b. Explain why matrix multiplication must be associative.

c. Verify using the matrices from Exercise 1 that $A(BC) = (AB)C$.

Problem Set

1. Let matrix $A = \begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}$, matrix $B = \begin{pmatrix} 4 & 4 \\ 3 & 9 \end{pmatrix}$, and matrix $C = \begin{pmatrix} 8 & 2 \\ 7 & -5 \end{pmatrix}$. Calculate the following:
   a. AB
   b. AC
   c. $A(B + C)$
   d. $AB + AC$
   e. $(A + B)C$
   f. $A(BC)$

2. Apply each of the transformations you found in Problem 1 to the points $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, and $x + y$.

3. Let A, B, C, and D be any four square matrices of the same dimensions. Use the distributive property to evaluate the following:
   a. $(A + B)(C + D)$
   b. $(A + B)(A + B)$
   c. What conditions need to be true for part (b) to equal $AA + 2AB + BB$?

4. Let A be a 2×2 matrix and B, C be the scalar matrices $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and $C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Answer the following questions.
   a. Evaluate the following:
      i. BC
      ii. CB
      iii. $B + C$
      iv. $B - C$
   b. Are your answers to part (a) what you expected? Why or why not?
   c. Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$; does $AB = BA$? Does $AC = CA$?
   d. What is $(A + B)(A + C)$? Write the matrix A with the letter and not in matrix form. How does this compare to $(x + 2)(x + 3)$?
   e. With B and C given as above, is it possible to factor $AA - A - BC$?

5. Define the sum of any two functions with the same domain to be the function $f+g$ such that for each x in the domain of f and g, $(f+g)(x) = f(x) + g(x)$. Define the product of any two functions to be the function fg, such that for each x in the domain of f and g, $(fg)(x) = \big(f(x)\big)\big(g(x)\big)$.
   Let f, g, and h be real-valued functions defined by the equations $f(x) = 3x + 1$, $g(x) = -\frac{1}{2}x + 2$, and $h(x) = x^2 - 4$.
   a. Does $f(g+h) = fg + fh$?
   b. Show that this is true for any three functions with the same domains.
   c. Does $f \circ (g+h) = f \circ g + f \circ h$ for the functions described above?