# Lesson 10: Matrix Multiplication Is Not Commutative

## Student Outcomes

- Students understand that, unlike multiplication of numbers, matrix multiplication is not commutative.

## Lesson Notes

In this lesson, students first demonstrate that linear transformations in the coordinate plane do not commute. Since each linear transformation corresponds to a $2 \times 2$ matrix, students see that the corresponding matrix multiplication must also fail to commute (**N-VM.C.9**). Students verify this fact algebraically by multiplying matrices in both orders. Work is then extended to coordinates in 3-D space to prove that multiplication of $3 \times 3$ matrices is also not commutative.

## Classwork

### Opening Exercise (7 minutes)

Allow student time to work on the Opening Exercise independently before discussing as a class. Solicit volunteers to demonstrate each transformation in order to illustrate that the result is not the same.

*Scaffolding:*

- Consider providing students who struggle with transformations a coordinate grid, a transparency, and a fine-point dry erase marker to perform each transformation.
- Students place the transparency on the grid and draw vector $(0, 1)$; then, they perform the rotation with the transparency to help determine the location of the transformed vector.

**Consider the vector $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.**

**If $\mathbf{v}$ is rotated $45^\circ$ counterclockwise and then reflected across the $y$-axis, what is the resulting vector?**

$$\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

**If $\mathbf{v}$ is reflected across the $y$-axis and then rotated $45^\circ$ counterclockwise about the origin, what is the resulting vector?**

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

**Did these linear transformations commute? Explain.**

*No.* $R_{0,45^\circ} \circ R_y \neq R_y \circ R_{0,45^\circ}$

MP.3

- Do you think linear transformations, in general, commute? Explain why or why not.
  - *No. The resulting vector varies depending on the order in which the transformations are applied.*

Write this on the board: $R_{0,45^\circ} \circ R_y \neq R_y \circ R_{0,45^\circ}$

- What matrix corresponds to a $45^\circ$ counterclockwise rotation?
  - $\begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix}$
- What matrix corresponds to a reflection across the $y$-axis?
  - $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- What does the fact that the linear transformations failed to commute tell us about the corresponding matrix multiplication?
  - *The multiplication of the two transformation matrices must also fail to commute.*

### Exercise 1 (5 minutes)

Allow students time to verify algebraically that the matrix multiplication fails to commute. Share results as a class.

**Exercises 1–4**

1. **Let $A$ equal the matrix that corresponds to a $45^\circ$ rotation counterclockwise and $B$ equal the matrix that corresponds to a reflection across the $y$-axis. Verify that matrix multiplication does not commute by finding the products $AB$ and $BA$.**

$$\boldsymbol{AB} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\boldsymbol{AB \neq BA}$$

- What can be said about multiplication of two $2 \times 2$ matrices?
  - *Unlike multiplication of two numbers, it is not commutative.*
- What do you think about multiplication of two $3 \times 3$ matrices? Why?
  - *I do not think the multiplication would be commutative either. Each $3 \times 3$ matrix represents a transformation in 3-D space. These transformations will not be commutative; therefore, the corresponding multiplication will not be commutative.*

### Exercise 2 (7 minutes)

Allow students time to verify algebraically that the matrix multiplication fails to commute. Share results as a class.

> **2. Let $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Verify that matrix multiplication does not commute by finding the products $AB$ and $BA$.**
>
> $$AB = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
>
> $$BA = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$
>
> $$AB \neq BA$$

- Summarize to a partner what can be said about linear transformations and, therefore, matrix multiplication.
  - *Neither of them commutes. The order in which transformations are applied affects the outcome; therefore, the order in which square matrices are multiplied affects the product.*

### Exercise 3 (7 minutes)

In this exercise, students will find that the products are equal. We want students to realize that although there are special cases when matrix multiplication *appears* to be commutative, matrix multiplication is *not* commutative because as we have seen in previous examples, there are many cases where $AB$ does not equal $BA$ (for square matrices $A$ and $B$). Allow students time to work with a partner. Share results as a class.

> **3. Consider the vector v = $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.**
>
> **a. If v is rotated $45°$ counterclockwise about the $z$-axis and then reflected across the $xy$-plane, what is the resulting vector?**
>
> $$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$
>
> **b. If v is reflected across the $xy$-plane and then rotated $45°$ counterclockwise about the $z$-axis, what is the resulting vector?**
>
> $$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

*Scaffolding:*

- Exercise 4 could be varied by providing each student with a card that has a matrix in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Students could then pair up with another student and work the exercise.
- Advanced students may try working with the general matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$.
- Show the matrix $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ and ask students, "Explain how you know this is in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$."

c. **Verify algebraically that the product of the two corresponding matrices is the same regardless of the order in which they are multiplied.**

*Let* $A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *(the matrix representing* $45°$ *rotation about the* $z$*-axis)*

*Let* $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. *(the matrix representing a reflection across the* $xy$*-plane)*

$$AB = BA = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- What can be said about the order in which the transformations are applied in this particular example?
  - *The resulting point is the same regardless of the order in which the transformations are applied.*
- What can be said about the corresponding matrix multiplication?
  - *In this case,* $AB = BA$*, but this is a particular case.*
- Summarize with a partner what can be said about matrix multiplication.
  - *Matrix multiplication is not commutative. However, there are instances where the products are the same. This would correspond geometrically to cases in which the order that transformations are applied does not affect the location of the point.*

## Exercise 4 (9 minutes)

Allow students time to work with a partner. Have several groups show their results from part (a) in order to establish that multiplication of any two matrices in this form will have the same product. Stress that this is a special case and does not mean that matrix multiplication is commutative.

**Exercise 4**

4. **Write two matrices in the form** $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

   a. **Verify algebraically that the products of these two matrices are equal.**

   *Let* $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$.

   $AB = \begin{pmatrix} 4 & -7 \\ 7 & 4 \end{pmatrix}$ *and* $BA = \begin{pmatrix} 4 & -7 \\ 7 & 4 \end{pmatrix}$.

   $AB = BA$

   b. **Write each of your matrices as a complex number. Find the product of the two complex numbers.**

   $(2 + i)(3 + 2i) = 4 + 7i$

c. **Why is it the case that any two matrices in the form $\begin{pmatrix} \boldsymbol{a} & -\boldsymbol{b} \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix}$ have products that are equal regardless of the order in which they are multiplied?**

*Matrices in this form represent the geometric effect of complex multiplication. Multiplying a complex number $z$ by a complex number $\alpha$ and then by a complex number $\beta$ gives the same answer as multiplying by $\beta$ and then $\alpha$; that is, $\beta(\alpha z) = \alpha(\beta z)$; thus, the corresponding matrix multiplication yields the same product.*

MP.8

- What did you discover about the matrices above? (Allow several groups to share their work.)
  - $AB = BA$
- Does this mean matrix multiplication is commutative? Explain.
  - *No. This is a special case because the matrices are in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.*
- What is the relationship between these matrices and complex numbers?
  - *Matrices in this form can be used to represent a corresponding complex number. Multiplying these matrices is the same as multiplying two complex numbers.*
- Is the multiplication of two complex numbers commutative?
  - *Yes. Therefore, two matrices in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ have the same product, but this does not mean that matrix multiplication is commutative.*

## Closing (5 minutes)

Ask students to summarize in writing what can be said about multiplication of two square matrices and then share with a partner. Discuss key points as a class.

- Is matrix multiplication commutative?
  - *No.*
- How does what we know about linear transformations support this notion?
  - *The product of the two matrices represents two linear transformations. Since linear transformations do not commute, matrix multiplication must also fail to commute.*
- Are there instances when, for two matrices $A$ and $B$, $AB = BA$? Explain.
  - *Yes, particularly when the two matrices represent complex numbers. The product of the two matrices is also a complex number. Since complex multiplication is commutative, matrix multiplication appears to be commutative in this instance.*

## Exit Ticket (5 minutes)

Name ___________________________________________ Date ________________

# Lesson 10: Matrix Multiplication Is Not Commutative

## Exit Ticket

1. Let $A$ be the matrix representing a rotation about the origin by $60°$ and $B$ be the matrix representing a reflection across the $x$-axis.
   a. Give two reasons why $AB \neq BA$.

   b. Let $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Evaluate $A(Bx)$ and $B(Ax)$.

2. Write two matrices, $A, B$, that represent linear transformations where $A(Bx)$ and $B(Ax)$ where $x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Explain why the products are the same.

## Exit Ticket Sample Solutions

1. Let $A$ be the matrix representing a rotation about the origin by $60°$ and $B$ be the matrix representing a reflection across the $x$-axis.

   a. Give two reasons why $AB \neq BA$.

   *The linear transformations these matrices represent do not commute, and the matrices themselves do not commute.*

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}; B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; AB = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}; BA = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

   b. Let $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Evaluate $A(Bx)$ and $B(Ax)$.

$$A(Bx) = (AB)x = \begin{pmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{pmatrix}$$

$$B(Ax) = (BA)x = \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{pmatrix}$$

2. Write two matrices, $A, B$, that represent linear transformations where $A(Bx)$ and $B(Ax)$ where $x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Explain why the products are the same.

   *Answers may vary. Examples include any two rotations about the origin, any two matrices representing complex numbers, and a dilation with any other matrix (since scalars commute with all matrices). For example,*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$AB = BA = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

$$A(Bx) = B(Ax) = \begin{pmatrix} 2 \\ 10 \end{pmatrix}.$$

   *Matrices in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ represent the geometric effect of complex multiplication. Multiplying a complex number $z$ by a complex number $\alpha$ and then by a complex number $\beta$ gives the same answer as multiplying by $\beta$ and then $\alpha$; that is, $\beta(\alpha z) = \alpha(\beta z)$; thus, the corresponding matrix multiplication yields the same product.*

## Problem Set Sample Solutions

1. Let $A$ be the matrix representing a dilation of $2$, $B$ the matrix representing a rotation of $30°$, and $x = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

   a. Evaluate $AB$.

$$\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$$

b. Evaluate $BA$.

$$\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$$

c. Find $ABx$.

$$\begin{pmatrix} -2\sqrt{3}-3 \\ -2+3\sqrt{3} \end{pmatrix}$$

d. Find $BAx$.

$$\begin{pmatrix} -2\sqrt{3}-3 \\ -2+3\sqrt{3} \end{pmatrix}$$

2. Let $A$ be the matrix representing a reflection across the line $y = x$, $B$ the matrix representing a rotation of $90°$, and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

a. Evaluate $AB$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b. Evaluate $BA$.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

c. Find $Ax$.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

d. Find $Bx$.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e. Find $ABx$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

f. Find $BAx$.

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

g. Describe the linear transformation represented by $AB$.

*$AB$ represents a reflection across the $x$-axis.*

h. **Describe the linear transformation represented by $BA$.**

*$BA$ represents a reflection across the $y$-axis.*

3. **Let matrices $A, B$ represent scalars. Answer the following questions.**

a. **Would you expect $AB = BA$? Explain why or why not.**

*Yes. Scalars are real numbers, and real numbers commute.*

b. **Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Show $AB = BA$ through matrix multiplication and explain why.**

$$AB = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} ba & 0 \\ 0 & ba \end{pmatrix} = BA$$

*These matrices represent complex numbers and are both of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, so their products will be the same regardless of the order.*

4. **Let matrices $A, B$ represent complex numbers. Answer the following questions.**

a. **Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$. Show $AB = BA$ through matrix multiplication.**

$$AB = \begin{pmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{pmatrix}$$

$$BA = \begin{pmatrix} ca - db & -cb - da \\ da + cb & -db + ca \end{pmatrix}$$

They are equivalent.

b. **Would you expect $AC = CA$ for any matrix $C$? Explain.**

*No. Matrix multiplication is not commutative.*

c. **Let $C$ be any $2 \times 2$ matrix, $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Show $AC \neq CA$ through matrix multiplication.**

$$AC = \begin{pmatrix} ax - bz & ay - bw \\ bx + az & by + aw \end{pmatrix}$$

$$CA = \begin{pmatrix} ax + bz & ay + bw \\ az + bx & -bz + aw \end{pmatrix}$$

d. **Summarize your results from Problems 4 and 5.**

*Matrices representing scalars commute with any other matrix, and matrices representing complex number multiplication only commute when multiplying by other complex numbers (or if $b = 0$).*

5. Quaternions are a number system that extends to complex numbers discovered by William Hamilton in 1843. Multiplication of quaternions is not commutative and is defined as the quotient of two vectors. They are useful in 3-dimensional rotation calculations for computer graphics. Quaternions are formed the following way:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Multiplication by $-1$ and $1$ works normally. We can represent all possible multiplications of quaternions through a table:

| $\times$ | $i$ | $j$ | $k$ |
|---|---|---|---|
| $i$ | $-1$ | $k$ | $-j$ |
| $j$ | $-k$ | $-1$ | $i$ |
| $k$ | $j$ | $-i$ | $-1$ |

a. Is multiplication of quaternions commutative? Explain why or why not.

*No. For example, $ij = k$, but $ji = -k$.*

5.

b. Is multiplication of quaternions associative? Explain why or why not.

*Yes. $i(jk) = i(-i) = -1$, and $(ij)k = (-k)k = -1$. $1$ and $-1$ work normally.*