Lesson 9: Composition of Linear Transformations

Student Outcomes

- Students use technology to perform compositions of linear transformations in $\mathbb{R}^3$.

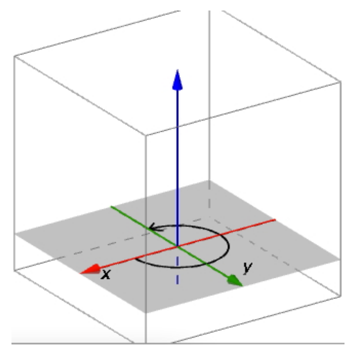
Lesson Notes

In Lesson 8, students discovered that if they compose two linear transformations in the plane, L_A and L_B, represented by matrices A and B respectively, then the resulting transformation can be produced using a single matrix AB. In this lesson, we extend this result from transformations in the plane to transformations in three-dimensional space, $\mathbb{R}^3$. This lesson was designed to be implemented using the GeoGebra applet TransformCubes (http://eureka-math.org/G12M2L9/geogebra-TransformCubes), which allows students to visualize the transformation of the cube.

Classwork

Opening Exercise (10 minutes)

The Opening Exercise reviews the discovery from the Problem Set in Lesson 7 that the matrix of a transformation in $\mathbb{R}^3$ is determined by the images of the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. This fact will be needed to construct matrices of transformations in this lesson before we compose them. Students may question what is meant by counterclockwise rotation in space; if this happens, let them know that we consider a rotation about the z-axis to be a counterclockwise rotation if it rotates the xy-plane counterclockwise when placed in its standard orientation as shown to the right.



Students should work these exercises with pencil and paper.

Opening Exercise

Recall from Problem 1, part (d) of the Problem Set of Lesson 7 that if you know what a linear transformation does to the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, you can find the matrix of the transformation. How do the images of these three points lead to the matrix of the transformation?

a. Suppose that a linear transformation L_1 rotates the unit cube by $90°$ counterclockwise about the z-axis. Find the matrix A_1 of the transformation L_1.

Since this transformation rotates by $90°$ counterclockwise in the xy-plane, a vector along the positive x-axis will be transformed to lie along the positive y-axis, a vector along the positive y-axis will be transformed to lie along the negative y-axis, and a vector along the z-axis will be left alone. Thus,

$$L_1\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix},\ L_1\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\0\end{bmatrix},\ \text{and } L_1\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

so the matrix of the transformation is $A_1=\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$.

b. Suppose that a linear transformation L_2 rotates the unit cube by $90°$ counterclockwise about the y-axis. Find

Scaffolding:
- Encourage struggling students to draw the image of a cube before and after the transformation to find the images of the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in space.

the matrix A_2 of the transformation L_2.

Since this transformation rotates by $90°$ counterclockwise in the xz-plane, a vector along the positive x-axis will be transformed to lie along the positive z-axis, a vector along the y-axis will be left alone, and a vector along the positive z-axis will be transformed to lie along the negative x-axis. Thus,

$$L_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix},\ L_2\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix},\text{ and } L_2\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\0\end{bmatrix},$$

so the matrix of the transformation is $A_2=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$.

c. Suppose that a linear transformation L_3 scales by 2 in the x-direction, scales by 3 in the y-direction, and scales by 4 in the z-direction. Find the matrix A_3 of the transformation L_3.

Since this transformation scales by 2 in the x-direction, by 3 in the y-direction, and by 4 in the z-direction , a vector along the x-axis will be multiplied by 2, a vector along the y-axis will be multiplied by 3, and a vector along the z-axis will be multiplied by 4. Thus,

$$L_3\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\0\\0\end{bmatrix},\ L_3\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\3\\0\end{bmatrix},\text{ and } L_3\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\4\end{bmatrix},$$

so the matrix of the transformation is $A_3=\begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}$.

d. Suppose that a linear transformation L_4 projects onto the xy-plane. Find the matrix A_4 of the transformation L_4.

Since this transformation projects onto the xy-plane, a vector along the x-axis will be left alone, a vector along the y-axis will be left alone, and a vector along the z-axis will be transformed into the zero vector. Thus,

$$L_4\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\ L_4\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix},\text{ and } L_4\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

so the matrix of the transformation is $A_4=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

e. Suppose that a linear transformation L_5 projects onto the xz-plane. Find the matrix A_5 of the transformation L_5.

Since this transformation projects onto the xz-plane, a vector along the x-axis will left alone, a vector along the y-axis will be transformed into the zero vector, and a vector along the z-axis will be left alone. Thus,

$$L_5\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\ L_5\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix},\text{ and } L_5\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

so the matrix of the transformation is $A_5=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$.

f. Suppose that a linear transformation L_6 reflects across the plane with equation $y=x$. Find the matrix A_6 of the transformation L_6.

Since this transformation reflects across the plane $y = x$, a vector along the positive x-axis will be transformed into a vector along the positive y-axis with the same length, a vector along the positive y-axis will be transformed into a vector along the positive x-axis, and a vector along the z-axis will be left alone. Thus,

$$L_6\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}, L_6\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}, \text{ and } L_6\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

so the matrix of the transformation is $A_6 = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$.

g. Suppose that a linear transformation L_7 satisfies $L_7\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\end{bmatrix}$, $L_7\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $L_7\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\\frac{1}{2}\end{bmatrix}$. Find the matrix A_7 of the transformation L_7. What is the geometric effect of this transformation?

The matrix of this transformation is $A_7 = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{2}\end{bmatrix}$. *The geometric effect of L_7 is to stretch by a factor of 2 in the x-direction and scale by a factor of $\frac{1}{2}$ in the z-direction.*

h. Suppose that a linear transformation L_8 satisfies $L_8\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $L_8\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\0\end{bmatrix}$, and $L_8\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Find the matrix of the transformation L_8. What is the geometric effect of this transformation?

The matrix of this transformation is $A_8 = \begin{bmatrix}1 & -1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$. *The geometric effect of L_8 is to rotate by $45°$ in the xy-plane, scale by $\sqrt{2}$ in both the x and y directions, and to not change in the z-direction.*

Once students have completed the Opening Exercise with pencil and paper, allow them to use the GeoGebra applet TransformCubes.ggb to check their work before continuing. The remaining exercises rely on the matrices A_1-A_8, so ensure that students have found the correct matrices before proceeding with the lesson.

Discussion (2 minutes)

- In Lesson 8, we saw that for linear transformations in the plane, if L_A is a linear transformation represented by a 2×2 matrix A and L_B is a linear transformation represented by a 2×2 matrix B, then the 2×2 matrix AB is the matrix of the composition of L_B followed by L_A. Today we will explore composition of linear transformations in $\mathbb{R}^3$ to see whether the same result extends to the case when A, B, and AB are 3×3 matrices.

Exploratory Challenge 1 (25 minutes)

In the Exploratory Challenge, students predict the geometric effect of composing pairs of transformations from the Opening Exercise and then check their predictions with the GeoGebra applet TransformCubes.ggb.

Exploratory Challenge 1

MP.3

Transformations L_1-L_8 refer to the linear transformations from the Opening Exercise. For each pair,

i. **Make a conjecture to predict the geometric effect of performing the two transformations in the order specified.**

ii. **Find the product of the corresponding matrices, in the order that corresponds to the indicated order of composition. Remember that if we perform a transformation L_B with matrix B and then L_A with matrix A, the matrix that corresponds to the composition $L_A \circ L_B$ is AB. That is, L_B is applied first, but matrix B is written second.**

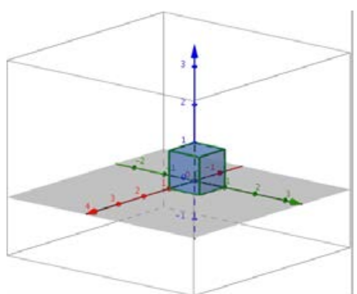
iii. **Use the GeoGebra applet TransformCubes.ggb to draw the image of the unit cube under the transformation induced by the matrix product in part (ii). Was your conjecture in part (i) correct?**

a. **Perform L_6 and then L_6.**

i. Since L_6 reflects across the plane through $y = x$ that is perpendicular to the xy-plane, performing L_6 twice in succession will result in the identity transformation.

ii. $A_6 \cdot A_6 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$*. Since $A_6 \cdot A_6$ is the identity matrix, we know that $L_6 \circ L_6$ is the identity transformation.*

iii. The conjecture was correct.

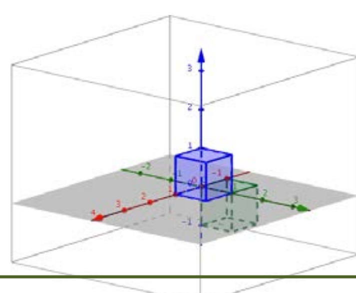


b. **Perform L_1 and then L_2.**

i. Sample student response: Since L_1 rotates $90°$ about the z-axis and L_2 rotates $90°$ about the y-axis, the composition $L_2 \circ L_1$ should rotate $180°$ about the line $y = -x$ in the xy-plane.

ii. $A_2 \cdot A_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.

iii. The conjecture in part (i) is not correct. While it appears that the composition $L_2 \circ L_1$ is a rotation by $180°$ about the y-axis, it is not because, for example, point $(0, 1, 0)$ is transformed to point $(-1, 0, 0)$ and does not remain on the y-axis after the transformation. Thus, this cannot be a rotation around the y-axis.

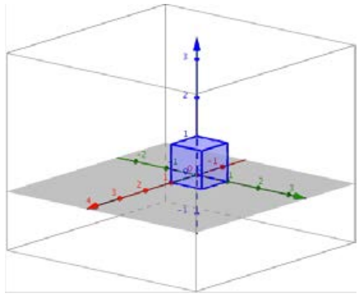


c. **Perform L_4 and then L_5.**

i. ***Since L_4 projects onto the xy-plane and L_5 projects onto the xz-plane, the composition $L_5 \circ L_4$ will project onto the x-axis.***

ii. $A_5 \cdot A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

iii. ***The conjecture in part (i) is correct. The cube is first transformed to a square in the xy-plane and then transformed onto a segment on the x-axis.***

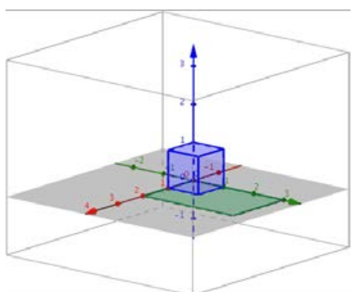


d. **Perform L_4 and then L_3.**

i. ***Since L_4 projects onto the xy-plane and L_3 scales in the x, y, and z directions, the composition $L_3 \circ L_4$ will project onto the xy-plane and scale in the x and y directions.***

ii. $A_3 \cdot A_4 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

iii. ***The conjecture in part (i) is correct.***

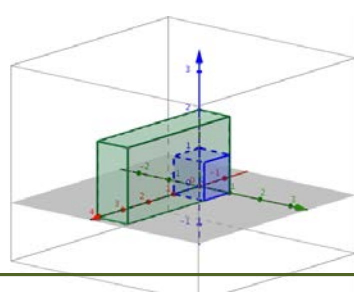


e. **Perform L_3 and then L_7.**

i. ***Since L_3 scales in the x, y, and z directions and so does L_7, the composition will be a transformation that also scales in all three directions but with different scale factors.***

ii. $A_7 \cdot A_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$

iii. ***The conjecture from (i) is correct.***

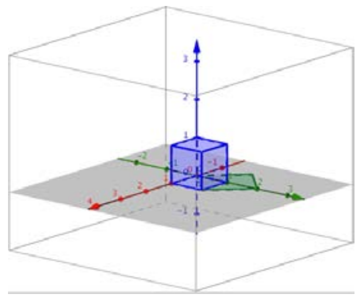


f. **Perform L_8 and then L_4.**

i. *Transformation L_8 rotates the unit cube by $45°$ about the z-axis and stretches by a factor of $\sqrt{2}$ in both the x and y directions, while L_4 projects the image onto the xy-plane. The composition will transform the unit cube into a larger square that has been rotated $45°$ in the xy-plane.*

ii. $A_4 \cdot A_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

iii. *The conjecture from part (i) is correct.*

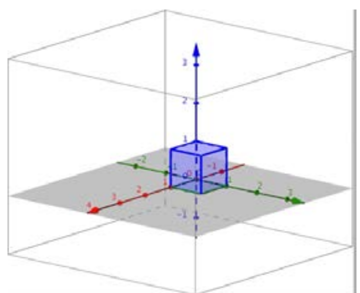


g. **Perform L_4 and then L_6.**

i. *Transformation L_4 projects onto the xy-plane, and transformation L_6 reflects across the plane through the line $y = x$ in the xy-plane and is perpendicular to the xy-plane, so the composition $L_6 \circ L_4$ will appear to be the reflection in the xy-plane across the line $y = x$.*

ii. $A_6 \cdot A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

iii. *The conjecture in part (i) is correct.*

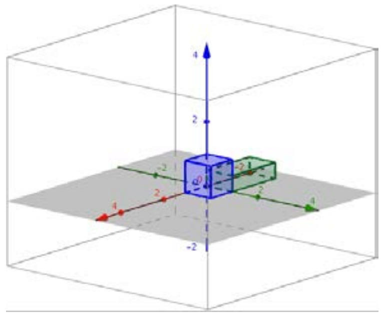


h. **Perform L_2 and then L_7.**

i. Since L_2 rotates $90°$ around the y-axis and L_7 scales in the x and z directions, the composition $L_7 \circ L_2$ will rotate and scale simultaneously.

ii. $A_7 \cdot A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$.

iii. The conjecture in part (i) is correct.

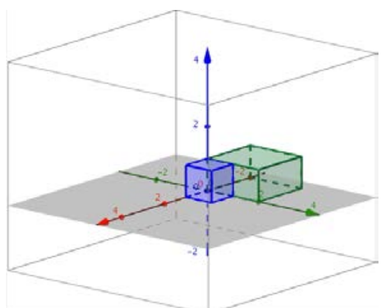


i. Perform L_8 and then L_8.

i. Since L_8 rotates by $45°$ about the z-axis and scales by $\sqrt{2}$ in the x and y directions, performing this transformation twice will rotate by $90°$ about the z-axis and scale by 2 in the x and y directions.

ii. $A_8 \cdot A_8 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

iii. The conjecture in part (i) is correct.



- Do a 30-second Quick Write on what we have discovered in Exploratory Challenge 1, and share with your neighbor.

Exploratory Challenge 2 (Optional)

This optional challenge is for students who finished Exploratory Challenge 1 early. The challenge below is designed to prompt the question of whether or not order matters when composing two linear transformations, a question that is definitively answered in the next lesson and demonstrates that matrix multiplication is in general not commutative. Students are asked to compose two linear transformations L_A and L_B, with matrices A and B respectively, and to compare $L_A \circ L_B$ with $L_B \circ L_A$. The directions for this challenge are left intentionally vague so that students may use either an algebraic or a geometric approach to answer the question.

Exploratory Challenge 2

Transformations L_1-L_8 refer to the transformations from the Opening Exercise. For each of the following pairs of matrices A and B below, compare the transformations $L_A \circ L_B$ and $L_B \circ L_A$.

a. L_4 and L_5

Transformation $L_4 \circ L_5$ has matrix representation $A_4 \cdot A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and transformation $L_5 \circ L_4$ has matrix representation $A_5 \circ A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Since the two transformations have the same matrix representation, they are the same transformation: $L_5 \circ L_4 = L_4 \circ L_5$.

b. L_2 and L_5

Transformation $L_2 \circ L_5$ has matrix representation $A_2 \cdot A_5 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and transformation $L_5 \circ L_2$ has matrix representation $A_5 \cdot A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Since the two transformations have the same matrix representation, they are the same transformation: $L_2 \circ L_5 = L_5 \circ L_2$.

c. L_3 and L_7

Transformation $L_3 \circ L_7$ has matrix representation $A_3 \cdot A_7 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and transformation $L_7 \circ L_3$ has matrix representation $A_7 \cdot A_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Since the two transformations have the same matrix representation, they are the same transformation: $L_3 \circ L_7 = L_7 \circ L_3$.

d. L_3 and L_6

Transformation $L_3 \circ L_6$ has matrix representation $A_3 \cdot A_6 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, and transformation $L_6 \circ L_3$ has matrix representation $A_6 \cdot A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Since the two transformations have different matrix representations, they are not the same transformation: $L_3 \circ L_6 \neq L_6 \circ L_3$.

e. L_7 and L_1

Transformation $L_7 \circ L_1$ has matrix representation $A_7 \cdot A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, and

transformation $L_1 \circ L_7$ has matrix representation $A_1 \cdot A_7 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

Since the two transformations have different matrix representations, they are not the same transformation: $L_1 \circ L_7 \neq L_7 \circ L_1$.

f. What can you conclude about the order in which you compose two linear transformations?

In some cases, the order of composition of two linear transformations matters: for two matrices A and B, the transformation $L_A \circ L_B$ is not always the same transformation as $L_B \circ L_A$.

Closing (4 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

Lesson Summary

- The linear transformation induced by a 3×3 matrix AB has the same geometric effect as the sequence of the linear transformation induced by the 3×3 matrix B followed by the linear transformation induced by the 3×3 matrix A.
- That is, if matrices A and B induce linear transformations L_A and L_B in $\mathbb{R}^3$, respectively, then the linear transformation L_{AB} induced by the matrix AB satisfies $L_{AB}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = L_A\left(L_B\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$.

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 9: Composition of Linear Transformations

Exit Ticket

Let A be the matrix representing a rotation about the z-axis of $45°$ and B be the matrix representing a dilation of 2.

a. Write down A and B.

b. Let $x = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Find the matrix representing a dilation of x by 2 followed by a rotation about the z-axis of $45°$.

c. Do your best to sketch a picture of x, x after the first transformation, and x after both transformations. You may use technology to help you.

Exit Ticket Sample Solutions

Let A be the matrix representing a rotation about the z-axis of $45°$ and B be the matrix representing a dilation of 2.

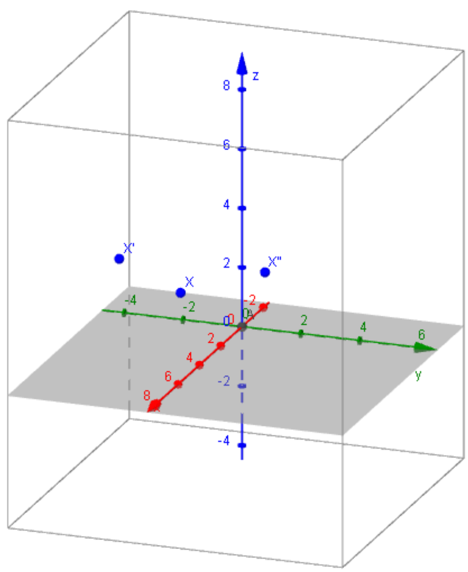
a. Write down A and B.

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

b. Let $x = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Find the matrix representing a dilation of x by 2 followed by a rotation about the z-axis of $45°$.

$$AB = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

c. Do your best to sketch a picture of x, x after the first transformation, and x after both transformations. You may use technology to help you.



Problem Set Sample Solutions

1. Let A be the matrix representing a dilation of $\frac{1}{2}$, and let B be the matrix representing a reflection across the yz-plane.

   a. Write A and B.

   $$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   b. Evaluate AB. What does this matrix represent?

   $$AB = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

   AB is a reflection across the yz-plane followed by a dilation of $\frac{1}{2}$.

   c. Let $x = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, and $z = \begin{bmatrix} 8 \\ -2 \\ -4 \end{bmatrix}$. Find $(AB)x$, $(AB)y$, and $(AB)z$.

   $$(AB)x = \begin{bmatrix} -\frac{5}{2} \\ 3 \\ 2 \end{bmatrix}, (AB)y = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}, (AB)z = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

2. Let A be the matrix representing a rotation of $30°$ about the x-axis, and let B be the matrix representing a dilation of 5.

   a. Write A and B.

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

   b. Evaluate AB. What does this matrix represent?

   $$AB = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{5\sqrt{3}}{2} & -\frac{5}{2} \\ 0 & \frac{5}{2} & \frac{5\sqrt{3}}{2} \end{bmatrix}$$

   AB is a dilation of 5 followed by a rotation of $30°$ about the x-axis.

c. Let $x = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $y = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $z = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Find $(AB)x$, $(AB)y$, and $(AB)z$.

$$(AB)x = \begin{bmatrix}5\\0\\0\end{bmatrix}$$

$$(AB)y = \begin{bmatrix}0\\ \frac{5\sqrt{3}}{2}\\ \frac{5}{2}\end{bmatrix}$$

$$(AB)z = \begin{bmatrix}0\\ \frac{5}{2}\\ \frac{5\sqrt{3}}{2}\end{bmatrix}$$

3. Let A be the matrix representing a dilation of 3, and let B be the matrix representing a reflection across the plane $y = x$.

a. Write A and B.

$$A = \begin{bmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}, B = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

b. Evaluate AB. What does this matrix represent?

$$AB = \begin{bmatrix}0 & 3 & 0\\3 & 0 & 0\\0 & 0 & 3\end{bmatrix}$$

AB is a reflection across the $y = x$ plane followed by a dilation of 3.

c. Let $x = \begin{bmatrix}-2\\7\\3\end{bmatrix}$. Find $(AB)x$.

$$(AB)x = \begin{bmatrix}21\\-6\\9\end{bmatrix}$$

4. Let $A = \begin{bmatrix}3 & 0 & 0\\3 & 3 & 0\\0 & 0 & 1\end{bmatrix}$, $B = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$.

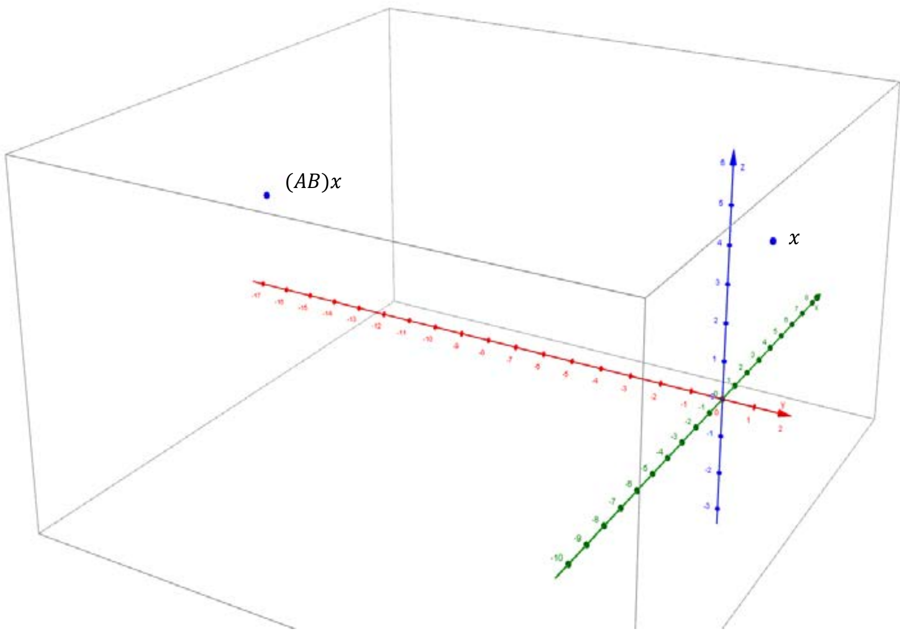
a. Evaluate AB.

$$\begin{bmatrix}0 & -3 & 0\\3 & -3 & 0\\0 & 0 & 1\end{bmatrix}$$

b. Let $x = \begin{bmatrix}-2\\2\\5\end{bmatrix}$. Find $(AB)x$.

$$\begin{bmatrix}-6\\-12\\5\end{bmatrix}$$

c. Graph x and $(AB)x$.



5. Let $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & \frac{1}{3} \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

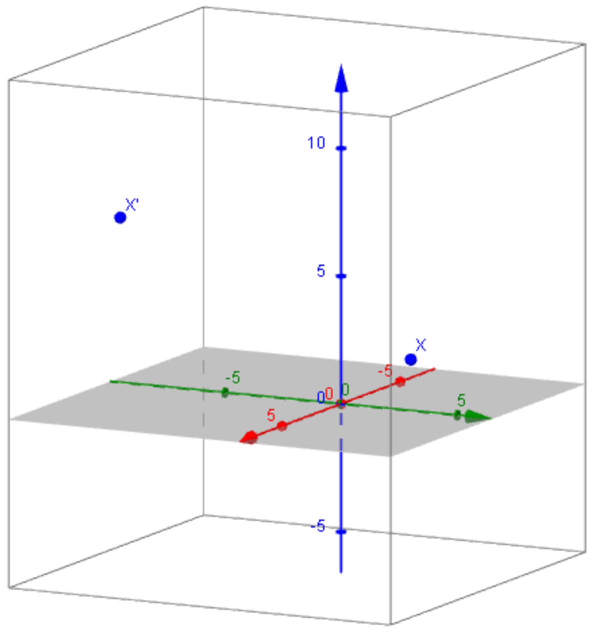
a. Evaluate AB.

$$\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 1 & -3 & 0 \\ 6 & 2 & \frac{1}{3} \end{bmatrix}$$

b. Let $x = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$. Find $(AB)x$.

$$\begin{bmatrix} 1 \\ -9 \\ \frac{20}{3} \end{bmatrix}$$

c. Graph x and $(AB)x$.



6. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

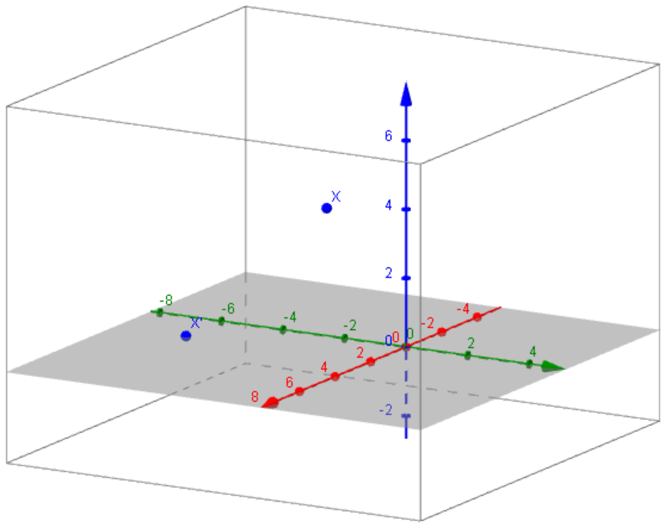
a. Evaluate AB.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. Let $x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. Find $(AB)x$.

$$\begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix}$$

c. Graph x and $(AB)x$.



d. What does AB represent geometrically?

AB represents a dilation of 2 in the x-direction, 3 in the y-direction, and a projection onto the xy-plane.

MP.7

7. Let A, B, C be 3×3 matrices representing linear transformations.

a. What does $A(BC)$ represent?

The linear transformation of applying the linear transformation that C represents followed by the transformation that B represents, followed by the transformation that A represents.

b. Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?

Yes, in general. When you multiply by a matrix on the left, you are applying a linear transformation after all linear transformations to the right have been applied.

c. Does $(AB)C$ represent something different from $A(BC)$? Explain.

No, it does not. This is the linear transformation obtained by applying C then AB, which is B followed by A.

8. Let AB represent any composition of linear transformations in $\mathbb{R}^3$. What is the value of $(AB)x$ where $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

Since a composition of linear transformations in $\mathbb{R}^3$ is also a linear transformation, we know that applying it to the origin will result in no change.