

Lesson 8: Composition of Linear Transformations

Classwork

Opening Exercise

Compute the product AB for the following pairs of matrices.

a. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

e. $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Exploratory Challenge

1. For each pair of matrices A and B given below:

- i. Describe the geometric effect of the transformation $L_B \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = B \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
- ii. Describe the geometric effect of the transformation $L_A \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
- iii. Draw the image of the unit square under the transformation $L_B \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = B \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
- iv. Draw the image of the transformed square under the transformation $L_A \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
- v. Describe the geometric effect on the unit square of performing first L_B then L_A .
- vi. Compute the matrix product AB .
- vii. Describe the geometric effect of the transformation $L_{AB} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = AB \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.

a. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

e. $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

2. Make a conjecture about the geometric effect of the linear transformation produced by the matrix AB . Justify your answer.

Lesson Summary

The linear transformation produced by matrix AB has the same geometric effect as the sequence of the linear transformation produced by matrix B followed by the linear transformation produced by matrix A .

That is, if matrices A and B produce linear transformations L_A and L_B in the plane, respectively, then the linear transformation L_{AB} produced by the matrix AB satisfies

$$L_{AB} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = L_A \left(L_B \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) \right).$$

Problem Set

- Let A be the matrix representing a dilation of $\frac{1}{2}$, and let B be the matrix representing a reflection across the y -axis.
 - Write A and B .
 - Evaluate AB . What does this matrix represent?
 - Let $x = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and $z = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$. Find $(AB)x$, $(AB)y$, and $(AB)z$.
- Let A be the matrix representing a rotation of 30° , and let B be the matrix representing a dilation of 5.
 - Write A and B .
 - Evaluate AB . What does this matrix represent?
 - Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $(AB)x$.
- Let A be the matrix representing a dilation of 3, and let B be the matrix representing a reflection across the line $y = x$.
 - Write A and B .
 - Evaluate AB . What does this matrix represent?
 - Let $x = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$. Find $(AB)x$.
- Let $A = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
 - Evaluate AB .
 - Let $x = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Find $(AB)x$.
 - Graph x and $(AB)x$.

5. Let $A = \begin{bmatrix} \frac{1}{3} & 0 \\ 2 & \frac{1}{3} \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$.
- Evaluate AB .
 - Let $x = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Find $(AB)x$.
 - Graph x and $(AB)x$.
6. Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$.
- Evaluate AB .
 - Let $x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Find $(AB)x$.
 - Graph x and $(AB)x$.
7. Let A, B, C be 2×2 matrices representing linear transformations.
- What does $A(BC)$ represent?
 - Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?
 - Does $(AB)C$ represent something different from $A(BC)$? Explain.
8. Let AB represent any composition of linear transformations in $\mathbb{R}^2$. What is the value of $(AB)x$ where $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?