Lesson 6: Linear Transformations as Matrices

Classwork

Opening Exercise

Let $A = \begin{pmatrix} 7 & -2 \\ 5 & -3 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Does this represent a linear transformation? Explain how you know.

Exploratory Challenge 1: The Geometry of 3D Matrix Transformations

a. What matrix in $\mathbb{R}^2$ serves the role of 1 in the real number system? What is that role?

b. What matrix in $\mathbb{R}^2$ serves the role of 0 in the real number system? What is that role?

c. What is the result of scalar multiplication in $\mathbb{R}^2$?

d. Given a complex number $a + bi$, what represents the transformation of that point across the real axis?

Exploratory Challenge 2: Properties of Vector Arithmetic

a. Is vector addition commutative? That is, does $x + y = y + x$ for each pair of points in $\mathbb{R}^2$? What about points in $\mathbb{R}^3$?

b. Is vector addition associative? That is, does $(x+y)+r = x+(y+r)$ for any three points in $\mathbb{R}^2$? What about points in $\mathbb{R}^3$?

c. Does the distributive property apply to vector arithmetic? That is, does $k \cdot (x + y) = kx + ky$ for each pair of points in $\mathbb{R}^2$? What about points in $\mathbb{R}^3$?

d. Is there an identity element for vector addition? That is, can you find a point a in $\mathbb{R}^2$ such that $x + a = x$ for every point x in $\mathbb{R}^2$? What about for$\mathbb{R}^3$?

e. Does each element in $\mathbb{R}^2$ have an additive inverse? That is, if you take a point a in $\mathbb{R}^2$, can you find a second point b such that $a + b = 0$?

Problem Set

1. Show that the associative property, $x + (y + z) = (x + y) + z$, holds for the following.
   a. $x = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, y = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, z = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
   b. $x = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}, z = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

2. Show that the distributive property, $k\,(x + y) = kx + ky$, holds for the following.
   a. $x = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, y = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, k = -2$
   b. $x = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, y = \begin{pmatrix} -4 \\ 6 \\ -7 \end{pmatrix}, k = -3$

3. Compute the following.
   a. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
   b. $\begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$
   c. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

4. Let $x = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Compute $L(x) = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \cdot x$, plot the points, and describe the geometric effect to x.
   a. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
   c. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   d. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

5. Let $x = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Compute $L(x) = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \cdot x$. Describe the geometric effect to x.

   a. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   b. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

   c. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

   d. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   f. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

   g. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   h. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

   i. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

6. Find the matrix that will transform the point $x = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ to the following point:

   a. $\begin{pmatrix} -4 \\ -12 \\ -8 \end{pmatrix}$

   b. $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

7. Find the matrix/matrices that will transform the point $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ to the following point:

   a. $\mathbf{x}' = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$

   b. $\mathbf{x}' = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$