Lesson 1: Special Triangles and the Unit Circle

Classwork

Example 1

Find the following values for the rotation $\theta = \frac{\pi}{3}$ around the carousel. Create a sketch of the situation to help you. Interpret what each value means in terms of the position of the rider.

a. $\sin(\theta)$

b. $\cos(\theta)$

c. $\tan(\theta)$

Exercise 1

Assume that the carousel is being safety tested, and a safety mannequin is the rider. The ride is being stopped at different rotation values so technicians can check the carousel's parts. Find the sine, cosine, and tangent for each rotation indicated, and explain how these values relate to the position of the mannequin when the carousel stops at these rotation values. Use your carousel models to help you determine the values, and sketch your model in the space provided.

a. $\theta = \frac{\pi}{4}$

b. $\theta = \frac{\pi}{6}$

Example 2

Use your understanding of the unit circle and trigonometric functions to find the values requested.

a. $\sin\left(-\frac{\pi}{3}\right)$

b. $\tan\left(\frac{5\pi}{4}\right)$

Exercise 2

Use your understanding of the unit circle to determine the values of the functions shown.

a. $\sin\left(\frac{11\pi}{6}\right)$

b. $\cos\left(\frac{3\pi}{4}\right)$

c. $\tan(-\pi)$

Problem Set

1. Complete the chart below.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$			
$\cos(\theta)$			
$\tan(\theta)$			

2. Evaluate the following trigonometric expressions, and explain how you used the unit circle to determine your answer.
   a. $\cos\left(\pi + \frac{\pi}{3}\right)$
   b. $\sin\left(\pi - \frac{\pi}{4}\right)$
   c. $\sin\left(2\pi - \frac{\pi}{6}\right)$
   d. $\cos\left(\pi + \frac{\pi}{6}\right)$
   e. $\cos\left(\pi - \frac{\pi}{4}\right)$
   f. $\cos\left(2\pi - \frac{\pi}{3}\right)$
   g. $\tan\left(\pi + \frac{\pi}{4}\right)$
   h. $\tan\left(\pi - \frac{\pi}{6}\right)$
   i. $\tan\left(2\pi - \frac{\pi}{3}\right)$

3. Rewrite the following trigonometric expressions in an equivalent form using $\pi + \theta$, $\pi - \theta$, or $2\pi - \theta$ and evaluate.
   a. $\cos\left(\frac{\pi}{3}\right)$
   b. $\cos\left(\frac{-\pi}{4}\right)$
   c. $\sin\left(\frac{\pi}{6}\right)$
   d. $\sin\left(\frac{4\pi}{3}\right)$
   e. $\tan\left(\frac{-\pi}{6}\right)$
   f. $\tan\left(\frac{-5\pi}{6}\right)$

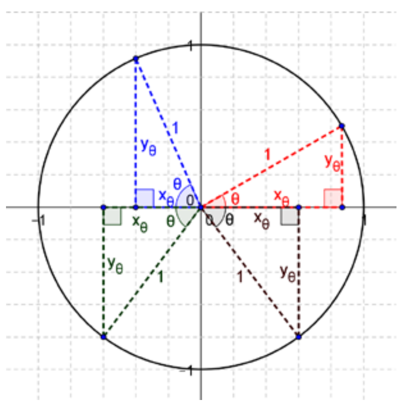
4. Identify the quadrant of the plane that contains the terminal ray of a rotation by θ if θ satisfies the given conditions.
   a. $\sin(\theta) > 0$ and $\cos(\theta) > 0$
   b. $\sin(\theta) < 0$ and $\cos(\theta) < 0$
   c. $\sin(\theta) < 0$ and $\tan(\theta) > 0$
   d. $\tan(\theta) > 0$ and $\sin(\theta) > 0$
   e. $\tan(\theta) < 0$ and $\sin(\theta) > 0$
   f. $\tan(\theta) < 0$ and $\cos(\theta) > 0$
   g. $\cos(\theta) < 0$ and $\tan(\theta) > 0$
   h. $\sin(\theta) > 0$ and $\cos(\theta) < 0$

5. Explain why $\sin^2(\theta) + \cos^2(\theta) = 1$.

6. Explain how it is possible to have $\sin(\theta) < 0$, $\cos(\theta) < 0$, and $\tan(\theta) > 0$. For which values of θ between 0 and 2π does this happen?

7. Duncan says that for any real number θ, $\tan(\theta) = \tan(\pi - \theta)$. Is he correct? Explain how you know.

8. Given the following trigonometric functions, identify the quadrant in which the terminal ray of θ lies in the unit circle shown below. Find the other two trigonometric functions of θ of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.



   a. $\sin(\theta) = \frac{1}{2}$ and $\cos(\theta) > 0$.
   b. $\cos(\theta) = -\frac{1}{2}$ and $\sin(\theta) > 0$.
   c. $\tan(\theta) = 1$ and $\cos(\theta) < 0$.
   d. $\sin(\theta) = -\frac{\sqrt{3}}{2}$ and $\cot(\theta) < 0$.
   e. $\tan(\theta) = -\sqrt{3}$ and $\cos(\theta) < 0$.
   f. $\sec(\theta) = -2$ and $\sin(\theta) < 0$.
   g. $\cot(\theta) = \sqrt{3}$ and $\csc(\theta) > 0$.

9. Toby thinks the following trigonometric equations are true. Use $\theta = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$ to develop a conjecture whether or not he is correct in each case below.
   a. $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$.
   b. $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$.

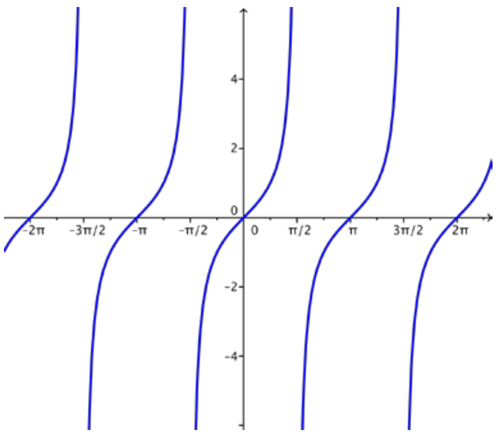
10. Toby also thinks the following trigonometric equations are true. Is he correct? Justify your answer.
   a. $\sin\left(\boldsymbol{\pi} - \frac{\pi}{3}\right) = \sin(\pi) - \sin\left(\frac{\pi}{3}\right)$
   b. $\cos\left(2\pi - \frac{\pi}{3}\right) = \cos(\pi) - \cos\left(\frac{\pi}{3}\right)$
   c. $\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right)$
   d. $\sin\left(\pi + \frac{\pi}{6}\right) = \sin(\pi) + \sin\left(\frac{\pi}{6}\right)$
   e. $\cos\left(\pi + \frac{\pi}{4}\right) = \cos(\pi) + \cos\left(\frac{\pi}{4}\right)$

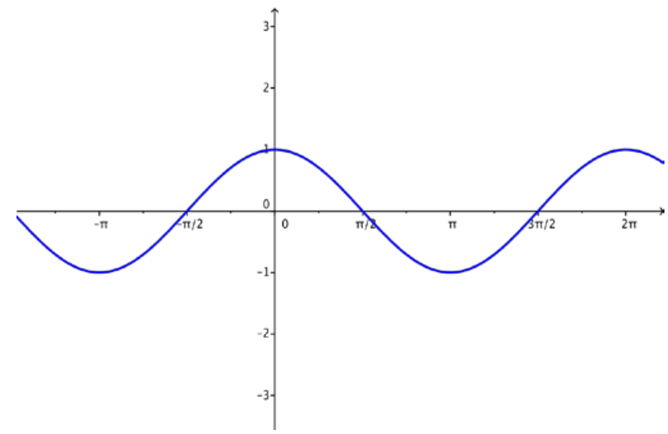
Lesson 2: Properties of Trigonometric Functions

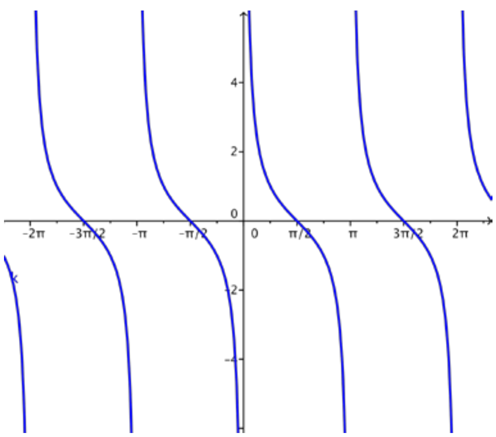
Classwork

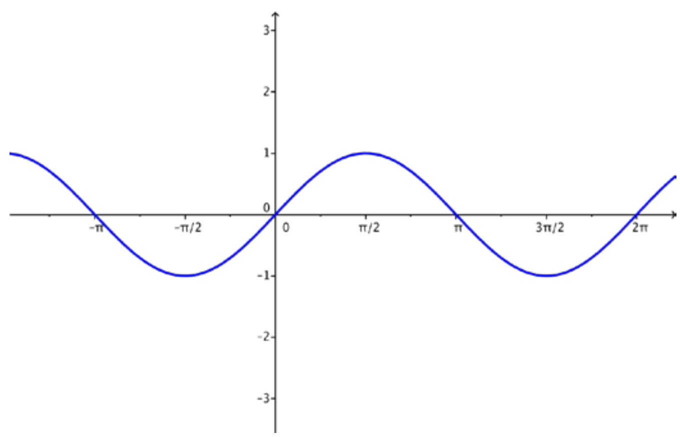
Opening Exercise

The graphs below depict four trigonometric functions. Identify which of the graphs are $f(x) = \sin(x)$, $g(x) = \cos(x)$, and $h(x) = \tan(x)$. Explain how you know.









Exercises 1–4

1. Use the unit circle to evaluate these expressions:

   a. $\sin\left(\frac{17\pi}{4}\right)$

   b. $\cos\left(\frac{19\pi}{6}\right)$

   c. $\tan(450\pi)$

2. Use the identity $\sin(\pi + \theta) = -\sin(\theta)$ for all real-numbered values of θ to verify the identity $\sin(2\pi + \theta) = \sin(\theta)$ for all real-numbered values of θ.

3. Use your understanding of the symmetry of the sine and cosine functions to evaluate these functions for the given values of θ.

   a. $\sin(-\frac{\pi}{2})$

   b. $\cos\left(-\frac{5\pi}{3}\right)$

4. Use your understanding of the symmetry of the sine and cosine functions to determine the value of $\tan(-\theta)$ for all real-numbered values of θ. Determine whether the tangent function is even, odd, or neither.

Exploratory Challenge/Exercises 5–6

5. Use your unit circle model to complete the table. Then use the completed table to answer the questions that follow.

$\boldsymbol{\theta}$	$\left(\frac{\pi}{2}+\theta\right)$	$\sin\left(\frac{\pi}{2}+\theta\right)$	$\cos\left(\frac{\pi}{2}+\theta\right)$
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			

a. What does the value $\left(\frac{\pi}{2}+\theta\right)$ represent with respect to the rotation of the carousel?

b. What pattern do you recognize in the values of $\sin\left(\frac{\pi}{2}+\theta\right)$ as θ increases from 0 to 2π?

c. What pattern do you recognize in the values of $\cos\left(\frac{\pi}{2}+\theta\right)$ as θ increases from 0 to 2π?

d. Fill in the blanks to formalize these relationships:

$\sin\left(\frac{\pi}{2}+\theta\right) =$

$\cos\left(\frac{\pi}{2}+\theta\right) =$

6. Use your unit circle model to complete the table. Then use the completed table to answer the questions that follow.

$\boldsymbol{\theta}$	$\left(\frac{\pi}{2}-\theta\right)$	$\sin\left(\frac{\pi}{2}-\theta\right)$	$\cos\left(\frac{\pi}{2}-\theta\right)$
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			

a. What does the value $\left(\frac{\pi}{2}-\theta\right)$ represent with respect to the rotation of a rider on the carousel?

b. What pattern do you recognize in the values of $\sin\left(\frac{\pi}{2}-\theta\right)$ as θ increases from 0 to 2π?

c. What pattern do you recognize in the values of $\cos\left(\frac{\pi}{2}-\theta\right)$ as θ increases from 0 to 2π?

d. Fill in the blanks to formalize these relationships:

$\sin\left(\frac{\pi}{2}-\theta\right) =$

$\cos\left(\frac{\pi}{2}-\theta\right) =$

Exercise 7

7. Use your understanding of the relationship between the sine and cosine functions to verify these statements.

a. $\cos\left(\frac{4\pi}{3}\right) = \sin\left(\frac{-\pi}{6}\right)$

b. $\cos\left(\frac{5\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right)$

Lesson Summary

For all real numbers θ for which the expressions are defined,

$\sin(\theta) = \sin(2\pi n + \theta)$ and $\cos(\theta) = \cos(2\pi n + \theta)$ for all integer values of n

$\tan(\theta) = \tan(\pi n + \theta)$ for all integer values of n

$\sin(-\theta) = -\sin(\theta)$, $\cos(-\theta) = \cos(\theta)$, and $\tan(-\theta) = -\tan(\theta)$

$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta)$ and $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$

Problem Set

1. Evaluate the following trigonometric expressions. Show how you used the unit circle to determine the solution.
   a. $\sin\left(\frac{13\pi}{6}\right)$
   b. $\cos\left(-\frac{5\pi}{3}\right)$
   c. $\tan\left(\frac{25\pi}{4}\right)$
   d. $\sin\left(-\frac{3\pi}{4}\right)$
   e. $\cos\left(-\frac{5\pi}{6}\right)$
   f. $\sin\left(\frac{17\pi}{3}\right)$
   g. $\cos\left(\frac{25\pi}{4}\right)$
   h. $\tan\left(\frac{29\pi}{6}\right)$
   i. $\sin\left(-\frac{31\pi}{6}\right)$
   j. $\cos\left(-\frac{32\pi}{6}\right)$
   k. $\tan\left(-\frac{18\pi}{3}\right)$

2. Given each value of β below, find a value of α with $0 \leq \alpha \leq 2\pi$ so that $\cos(\alpha) = \cos(\beta)$ and $\alpha \neq \beta$.
   a. $\beta = \frac{3\pi}{4}$
   b. $\beta = \frac{5\pi}{6}$
   c. $\beta = \frac{11\pi}{12}$
   d. $\beta = 2\pi$
   e. $\beta = \frac{7\pi}{5}$
   f. $\beta = \frac{17\pi}{30}$
   g. $\beta = \frac{8\pi}{11}$

3. Given each value of β below, find two values of α with $0 \leq \alpha \leq 2\pi$ so that $\cos(\alpha) = \sin(\beta)$.
   a. $\beta = \frac{\pi}{3}$
   b. $\beta = \frac{7\pi}{6}$
   c. $\beta = \frac{3\pi}{4}$
   d. $\beta = \frac{\pi}{8}$

4. Given each value of β below, find two values of α with $0 \leq \alpha \leq 2\pi$ so that $\sin(\alpha) = \cos(\beta)$.
   a. $\beta = \frac{\pi}{3}$
   b. $\beta = \frac{5\pi}{6}$
   c. $\beta = \frac{7\pi}{4}$
   d. $\beta = \frac{\pi}{12}$

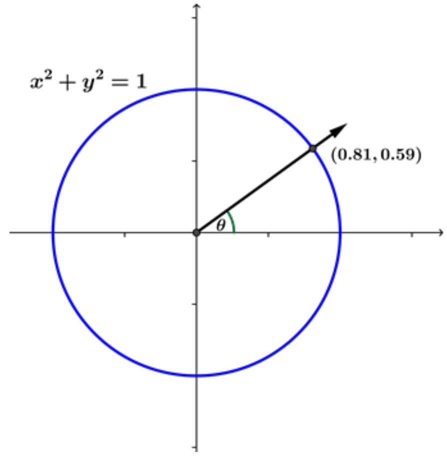
5. Jamal thinks that $\cos\left(\alpha - \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{4}\right)$ for any value of α. Is he correct? Explain how you know.

6. Shawna thinks that $\cos\left(\alpha - \frac{\pi}{3}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$ for any value of α. Is she correct? Explain how you know.

7. Rochelle looked at Jamal and Shawna's results from Problems 5 and 6 and came up with the conjecture below. Is she correct? Explain how you know.

   Conjecture: $\cos(\alpha - \beta) = \sin\left(\alpha + \left(\frac{\pi}{2} - \beta\right)\right)$.

8. A frog is sitting on the edge of a playground carousel with radius 1 meter. The ray through the frog's position and the center of the carousel makes an angle of measure θ with the horizontal, and his starting coordinates are approximately $(0.81, 0.59)$. Find his new coordinates after the carousel rotates by each of the following amounts.



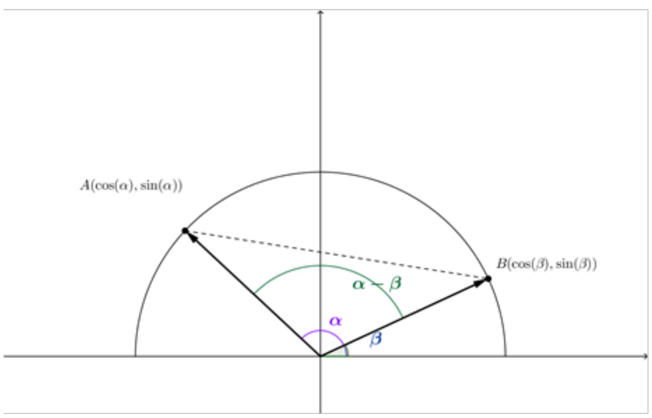
   a. $\frac{\pi}{2}$
   b. π
   c. 2π
   d. $-\frac{\pi}{2}$
   e. $-\pi$
   f. $\frac{\pi}{2}-\theta$
   g. $\pi-2\theta$
   h. -2θ

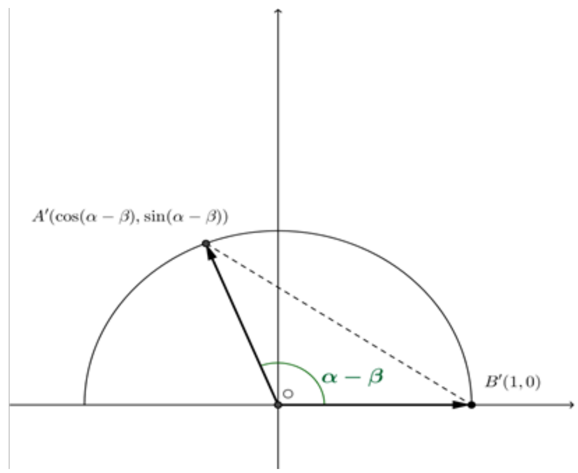
Lesson 3: Addition and Subtraction Formulas

Classwork

Example 1

Consider the figures below. The figure on the right is obtained from the figure on the left by rotating by $-\beta$ about the origin.





a. Calculate the length of $\overline{AB}$ in the figure on the left.

b. Calculate the length of $\overline{A'B'}$ in the figure on the right.

c. Set $\overline{AB}$ and $\overline{A'B'}$ equal to each other, and solve the equation for $\cos(\alpha - \beta)$.

Exercises 1–2

1. Use the fact that $\cos(-\theta) = \cos(\theta)$ to determine a formula for $\cos(\alpha + \beta)$.

2. Use the fact that $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$ to determine a formula for $\sin(\alpha - \beta)$.

Example 2

Use the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ to show that $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$.

Exercises 3–5

2. Verify the identity $\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$ for all $(\alpha-\beta)\neq\frac{\pi}{2}+\pi n$.

3. Use the addition and subtraction formulas to evaluate the expressions shown.

   a. $\cos\left(-\frac{5\pi}{12}\right)$

   b. $\sin\left(\frac{23\pi}{12}\right)$

   c. $\tan\left(\frac{5\pi}{12}\right)$

4. Use the addition and subtraction formulas to verify these identities for all real-number values of θ.

   a. $\sin(\pi-\theta)=\sin(\theta)$

   b. $\cos(\pi+\theta)=-\cos(\theta)$

Lesson Summary

The sum and difference formulas for sine, cosine, and tangent are summarized below.

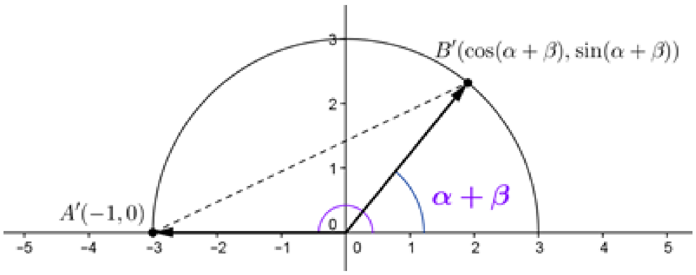
For all real numbers α and β for which the expressions are defined,

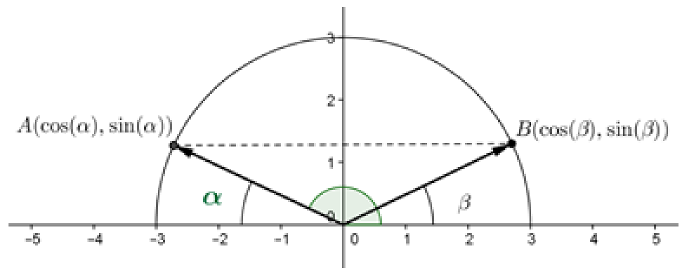
$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$
$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.$$

Problem Set

1. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.
   a. $\cos\left(\frac{\pi}{12}\right)$
   b. $\sin\left(\frac{\pi}{12}\right)$
   c. $\sin\left(\frac{5\pi}{12}\right)$
   d. $\cos\left(-\frac{\pi}{12}\right)$
   e. $\sin\left(\frac{7\pi}{12}\right)$
   f. $\cos\left(-\frac{7\pi}{12}\right)$
   g. $\sin\left(\frac{13\pi}{12}\right)$
   h. $\cos\left(-\frac{13\pi}{12}\right)$
   i. $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)$
   j. $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{6}\right)$
   k. $\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)$
   l. $\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)$
   m. $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{12}\right)$
   n. $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{12}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{12}\right)$

2. The figure below and to the right is obtained from the figure on the left by rotating the angle by α about the origin. Use the method shown in Example 1 to show that $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.





3. Use the sum formula for sine to show that $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$.

4. Evaluate $\tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$ to show $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$. Use the resulting formula to show that $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$.

5. Show an$(x - y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$.

6. Find the exact value of the following by using addition and subtraction formulas.
   a. $\tan\left(\frac{\pi}{12}\right)$
   b. $\tan\left(-\frac{\pi}{12}\right)$
   c. $\tan\left(\frac{7\pi}{12}\right)$
   d. $\tan\left(-\frac{13\pi}{12}\right)$
   e. $\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{12}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{12}\right)}$
   f. $\frac{\tan\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{12}\right)}{1+\tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{12}\right)}$
   g. $\frac{\tan\left(\frac{\pi}{12}\right)+\tan\left(\frac{\pi}{12}\right)}{1-\tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{12}\right)}$

Lesson 4: Addition and Subtraction Formulas

Classwork

Exercises

1. Derive formulas for the following:
   a. $\sin(2\theta)$

   b. $\cos(2\theta)$

2. Use the double-angle formulas for sine and cosine to verify these identities:
   a. $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

   b. $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

c. $\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta)$

3. Suppose that the position of a rider on the unit circle carousel is $(0.8, -0.6)$ for a rotation θ. What is the position of the rider after rotation by 2θ?

4. Use the double-angle formula for cosine to establish the identity $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta)+1}{2}}$.

5. Use the double-angle formulas to verify these identities:

a. $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$

b. $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

6. The position of a rider on the unit circle carousel is $(0.8, -0.6)$ after a rotation by θ where $0 \leq \theta < 2\pi$. What is the position of the rider after rotation by $\frac{\theta}{2}$?

7. Evaluate the following trigonometric expressions.

a. $\sin\left(\frac{3\pi}{8}\right)$

b. $\tan\left(\frac{\pi}{24}\right)$

Lesson Summary

The double-angle and half-angle formulas for sine, cosine, and tangent are summarized below.

For all real numbers θ for which the expressions are defined,

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 2\cos^2(\theta) - 1 \\
&= 1 - 2\sin^2(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)} \\
\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos(\theta)}{2}} \\
\cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{\cos(\theta) + 1}{2}} \\
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}
\end{aligned}$$

Problem Set

1. Evaluate the following trigonometric expressions.
   a. $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$
   b. $\frac{1}{2}\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$
   c. $4\sin\left(-\frac{5\pi}{12}\right)\cos\left(-\frac{5\pi}{12}\right)$
   d. $\cos^2\left(\frac{3\pi}{8}\right) - \sin^2\left(\frac{3\pi}{8}\right)$
   e. $2\cos^2\left(\frac{\pi}{12}\right) - 1$
   f. $1 - 2\sin^2\left(-\frac{\pi}{8}\right)$
   g. $\cos^2\left(-\frac{11\pi}{12}\right) - 2$
   h. $\frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$
   i. $\frac{2\tan\left(-\frac{5\pi}{12}\right)}{1-\tan^2\left(-\frac{5\pi}{12}\right)}$
   j. $\cos^2\left(\frac{\pi}{8}\right)$
   k. $\cos\left(\frac{\pi}{8}\right)$

l. $\cos\left(-\frac{9\pi}{8}\right)$
m. $\sin^2\left(\frac{\pi}{12}\right)$
n. $\sin\left(\frac{\pi}{12}\right)$
o. $\sin\left(-\frac{5\pi}{12}\right)$
p. $\tan\left(\frac{\pi}{8}\right)$
q. $\tan\left(\frac{\pi}{12}\right)$
r. $\tan\left(-\frac{3\pi}{8}\right)$

2. Show that $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$. (Hint: Use $\sin(2x) = 2\sin(x)\cos(x)$ and the sine sum formula.)

3. Show that $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$. (Hint: Use $\cos(2x) = \cos^2(x) - \sin^2(x)$ and the cosine sum formula.)

4. Use $\cos(2x) = \cos^2(x) - \sin^2(x)$ to establish the following formulas.
   a. $\cos^2(x) = \frac{1+\cos(2x)}{2}$
   b. $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

5. Jamia says that because sine is an odd function, $\sin\left(\frac{\theta}{2}\right)$ is always negative if θ is negative. That is, she says that for negative values of, $\sin\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos(\theta)}{2}}$. Is she correct? Explain how you know.

6. Ginger says that the only way to calculate $\sin\left(\frac{\pi}{12}\right)$ is using the difference formula for sine since $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. Fred says that there is another way to calculate $\sin\left(\frac{\pi}{12}\right)$. Who is correct, and why?

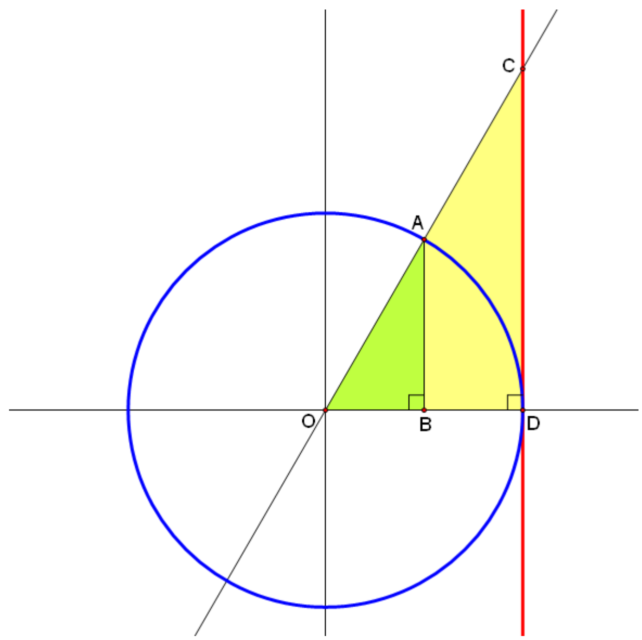
7. Henry says that by repeatedly applying the half-angle formula for sine we can create a formula for $\sin\left(\frac{\theta}{n}\right)$ for any positive integer n. Is he correct? Explain how you know.

Lesson 5: Tangent Lines and the Tangent Function

Classwork

Exercises

The circle shown to the right is a unit circle, and the length of $\widehat{DA}$ is $\frac{\pi}{3}$ radians.



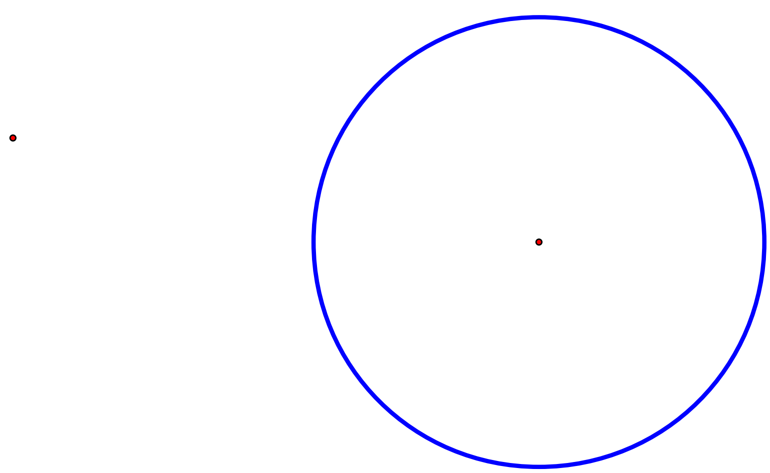
1. Which segment in the diagram has length $\sin\left(\frac{\pi}{3}\right)$?

2. Which segment in the diagram has length $\cos\left(\frac{\pi}{3}\right)$?

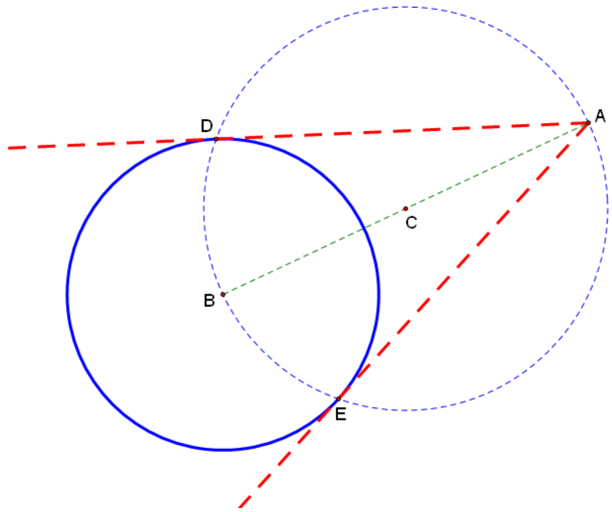
3. Which segment in the diagram has length $\tan\left(\frac{\pi}{3}\right)$?

4. Which segment in the diagram has length $\sec\left(\frac{\pi}{3}\right)$?

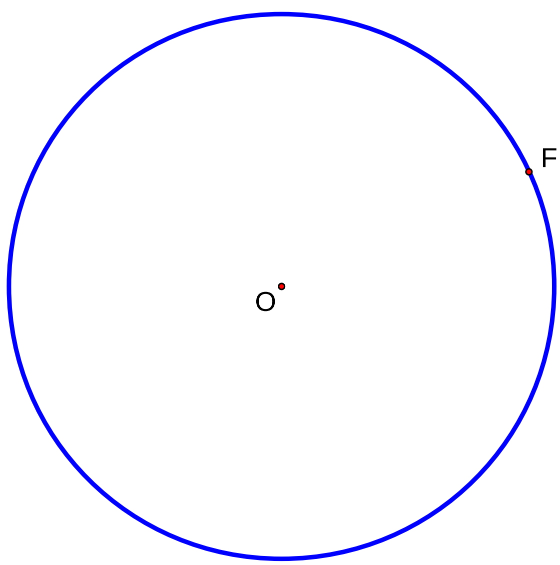
5. Use a compass to construct the tangent lines to the given circle that pass through the given point.



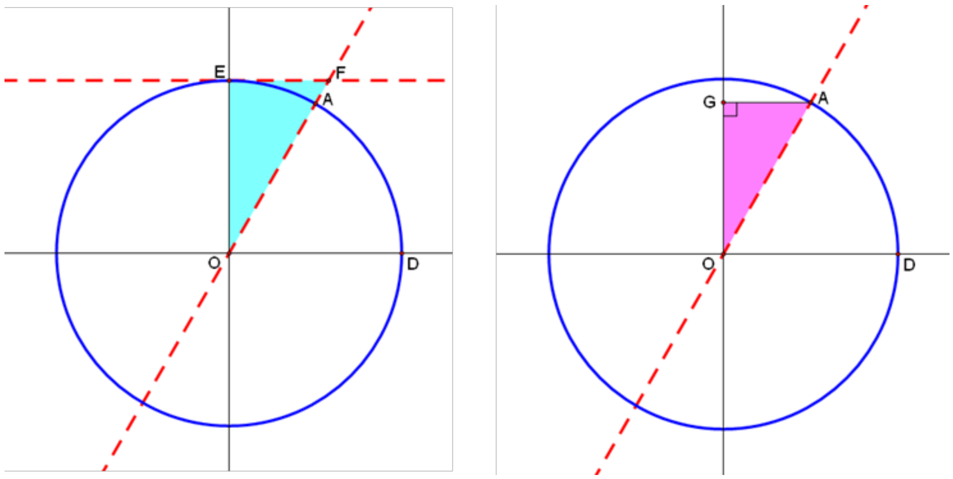
6. Analyze the construction shown below. Argue that the lines shown are tangent to the circle with center B.



7. Use a compass to construct a line that is tangent to the circle below at point F. Then choose a point G on the tangent line, and construct another tangent to the circle through G.



8. The circles shown below are unit circles, and the length of $\widehat{DA}$ is $\frac{\pi}{3}$ radians.



Which trigonometric function corresponds to the length of $\overline{EF}$?

9. Which trigonometric function corresponds to the length of $\overline{OF}$?

10. Which trigonometric identity gives the relationship between the lengths of the sides of ΔOEF?

11. Which trigonometric identities give the relationships between the corresponding sides of ΔOEF and ΔOGA?

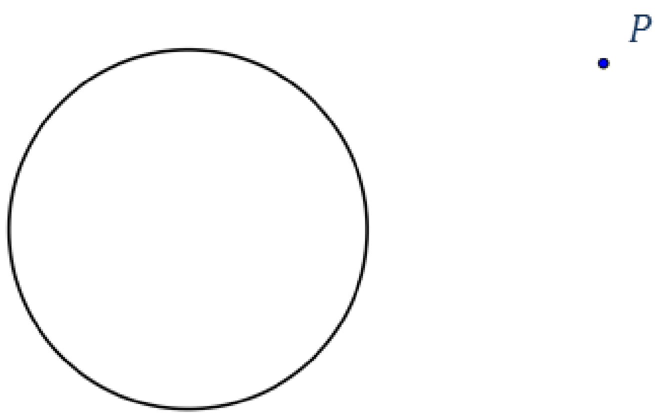
12. What is the value of $\csc\left(\frac{\pi}{3}\right)$? What is the value of $\cot\left(\frac{\pi}{3}\right)$? Use the Pythagorean theorem to support your answers.

Problem Set

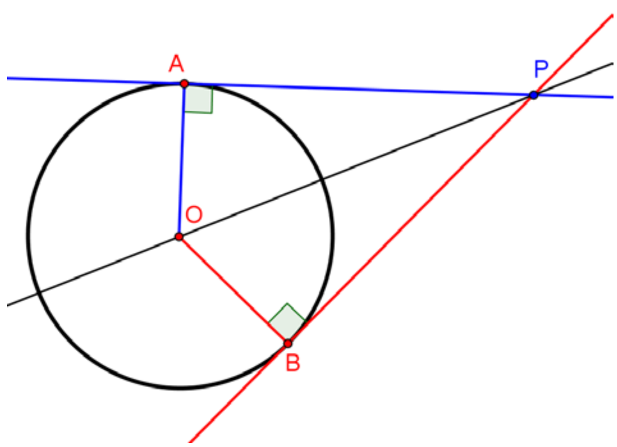
1. Prove Thales' theorem: If A, B, and P are points on a circle where $\overline{AB}$ is a diameter of the circle, then $\angle APB$ is a right angle.

2. Prove the converse of Thales' theorem: If $\overline{AB}$ is a diameter of a circle and P is a point so that $\angle APB$ is a right angle, then P lies on the circle for which $\overline{AB}$ is a diameter.

3. Construct the tangent lines from point P to the circle given below.



4. Prove that if segments from a point P are tangent to a circle at points A and B, then $\overline{PA} = \overline{PB}$.



5. Given points A, B, and C so that $AB = AC$, construct a circle so that $\overline{AB}$ is tangent to the circle at B and $\overline{AC}$ is tangent to the circle at C.

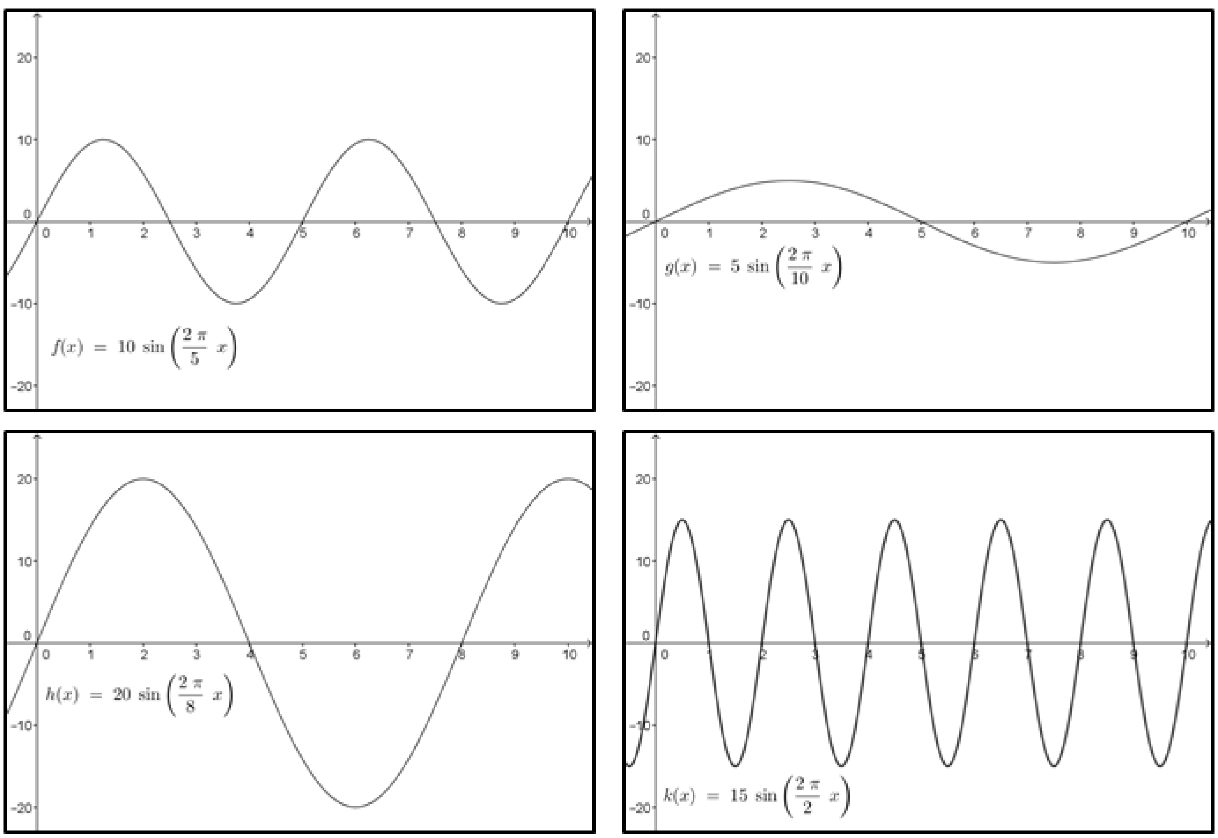
Lesson 6: Waves, Sinusoids, and Identities

Classwork

Opening Exercise

When you hear a musical note played on an instrument, the tones are caused by vibrations of the instrument. The vibrations can be represented graphically as a sinusoid. The amplitude is a measure of the loudness of the note, and the frequency is a measure of the pitch of the note. Recall that the frequency of a sinusoidal function is the reciprocal of its period. Louder notes have greater amplitude, and higher pitched notes have larger frequencies.

a. State the amplitude, period, and frequency of each sinusoidal function graphed below.



b. Order the graphs from quietest note to loudest note.

c. Order the graphs from lowest pitch note to highest pitch note.

Discussion

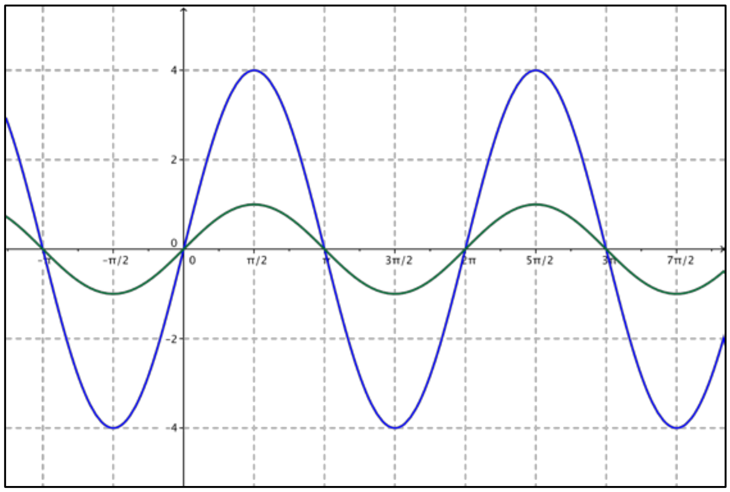
A wave is a disturbance moving through a medium that disrupts the particles that make up the medium. A medium can be any substance including solids, liquids, and gases. When a wave is present, the particles that make up the medium move about a fixed position. Energy is transferred between the particles, but the particles themselves always return to their fixed positions. This energy transfer phenomenon is a distinguishing feature of a wave.

One type of wave is a transverse wave where the particles oscillate perpendicular to the motion of the wave. Another type of wave is a longitudinal wave where the particles oscillate in the same direction as the motion of the wave. Sound waves are examples of longitudinal waves. The up and down motion of a buoy in the ocean as a wave passes is similar to a transverse wave.

Exercises 1–7

When two musical notes are played simultaneously, wave interference occurs. Wave interference is also responsible for the actual sound of the notes that you hear.

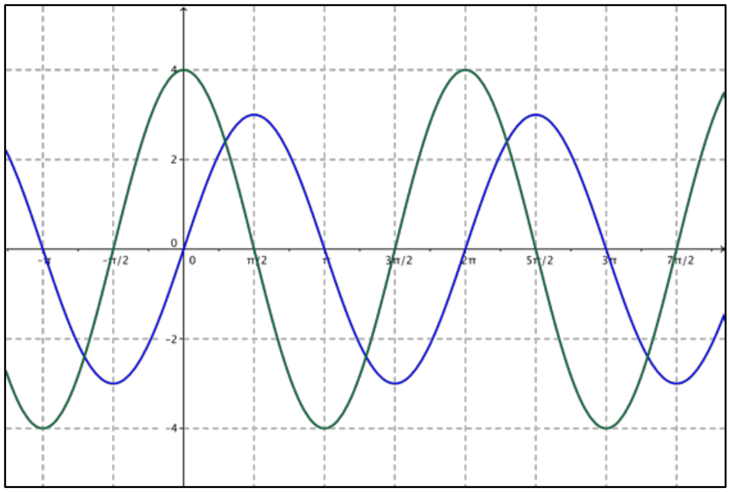
1. The graphs of two functions, f and g are shown below.



   a. Model wave interference by picking several points on the graphs of f and g and then using those points to create a graph of $h(x) = f(x) + g(x)$.

   b. What is a formula for h? Explain how you got your answer.

2. The graphs of f and g are shown below.

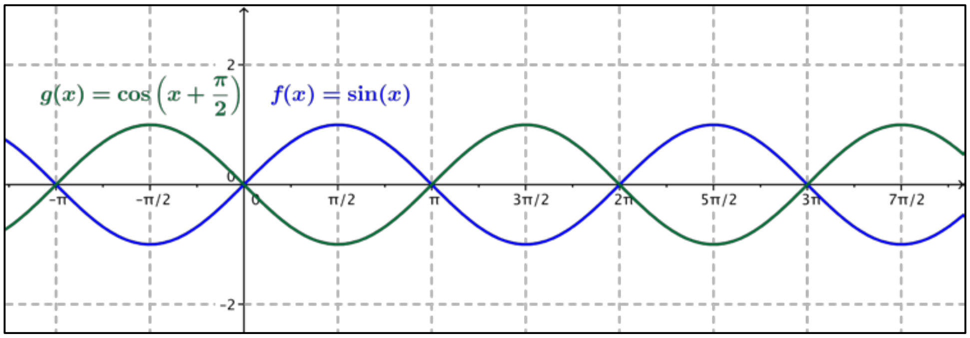


a. Model wave interference by picking several points on the graphs of f and g and then using those points to create a graph of $h(x) = f(x) + g(x)$.

b. What is an approximate formula for h? Explain how you got your answer.

3. Let $f(x) = \sin(x)$ and $g(x) = \cos\left(x + \frac{\pi}{2}\right)$.

   a. Predict what the graph of the wave interference function $h(x) = f(x) + g(x)$ would look like in this situation.



   b. Use an appropriate identity to confirm your prediction.

4. Show that in general, the function $h(x) = a\cos(bx - c)$ can be rewritten as the sum of a sine and cosine function with equal periods and different amplitudes.

5. Find an exact formula for $h(x) = 12\sin(x) + 5\cos(x)$ in the form $h(x) = a\cos(x - c)$. Graph $f(x) = 12\sin(x)$, $g(x) = 5\cos(x)$, and $h(x) = 12\sin(x) + 5\cos(x)$ together on the same axes.

6. Find an exact formula for $h(x) = 2\sin(x) - 3\cos(x)$ in the form $h(x) = a\cos(x - c)$. Graph $f(x) = 2\sin(x)$, $g(x) = -3\cos(x)$, and $h(x) = 2\sin(x) - 3\cos(x)$ together on the same axes.

7. Can you find an exact formula for $h(x) = 2\sin(2x) + 4\sin(x)$ in the form $h(x) = a\sin(x - c)$? If not, why not? Graph $f(x) = 2\sin(2x)$, $g(x) = 4\sin(x)$, and $h(x) = 2\sin(2x) + 4\sin(x)$ together on the same axes.

Lesson Summary

A wave is displacement that travels through a medium. Waves transfer energy, not matter. There are two types of waves: transverse and longitudinal. Sound waves are an example of longitudinal waves.

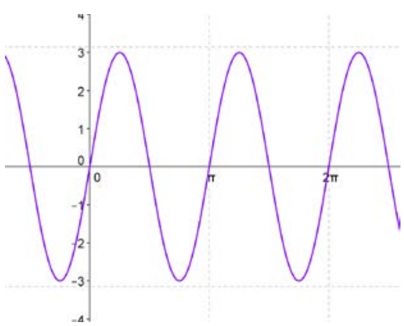
When two or more waves meet, interference occurs and can be represented mathematically as the sum of the individual waves.

The sum identity for sine is useful for analyzing the features of wave interference.

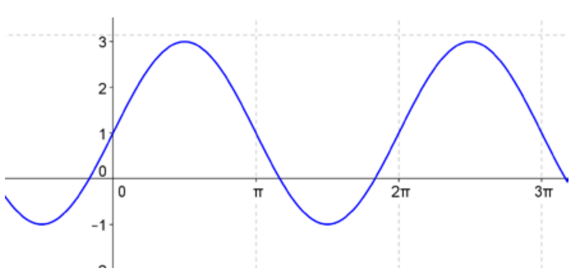
Problem Set

1. Rewrite the sum of the following functions in the form $(x) + g(x) = c\cos(x + k)$. Graph $y = f(x)$, $y = g(x)$, and $y = f(x) + g(x)$ on the same set of axes.
   a. $f(x) = 4\sin(x)$; $g(x) = 3\cos(x)$
   b. $f(x) = -6\sin(x)$; $g(x) = 8\cos(x)$
   c. $f(x) = \sqrt{3}\sin(x)$; $g(x) = 3\cos(x)$
   d. $f(x) = \sqrt{2}\sin(x)$; $g(x) = \sqrt{7}\cos(x)$
   e. $f(x) = 3\sin(x)$; $g(x) = -2\cos(x)$

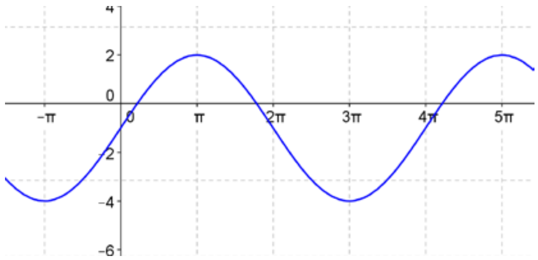
2. Find a sinusoidal function $f(x) = a\sin(bx + c) + d$ that fits each of the following graphs.
   a.



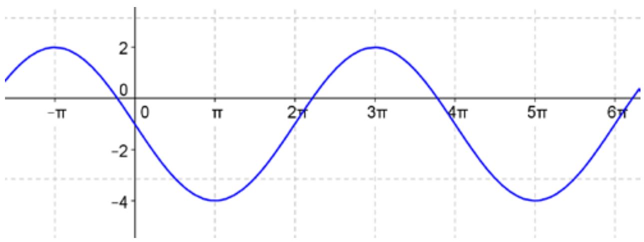
   b.



c.

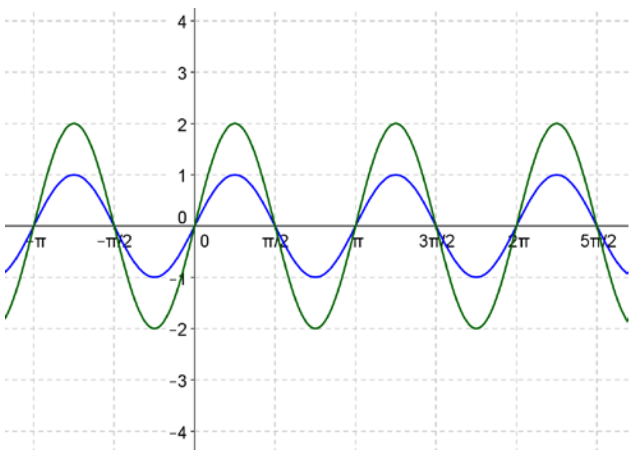


d.

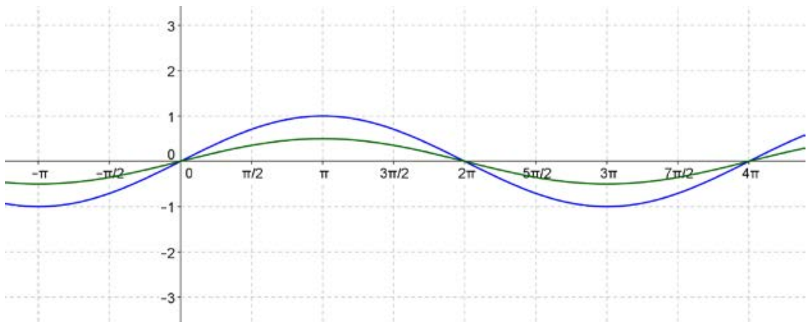


3. Two functions f and g are graphed below. Sketch the graph of the sum $f + g$.

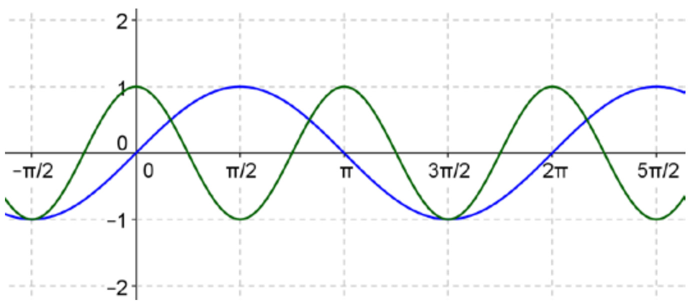
a.



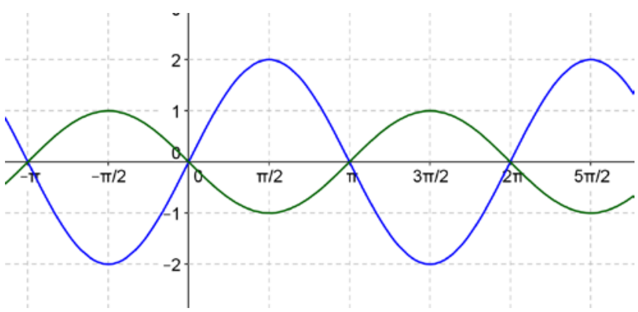
b.


3
2
1
0
-1
-2
-3
-π
-π/2
0
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

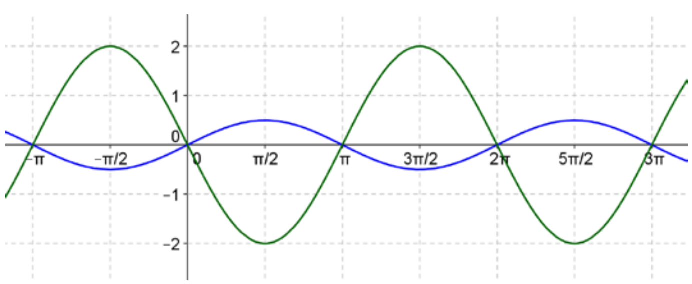
c.


2
1
0
-1
-2
-π/2
0
π/2
π
3π/2
2π
5π/2

d.


2
1
0
-1
-2
-π
-π/2
0
π/2
π
3π/2
2π
5π/2

e.


2
1
0
-1
-2
-π
-π/2
0
π/2
π
3π/2
2π
5π/2
3π

Lesson 7: An Area Formula for Triangles

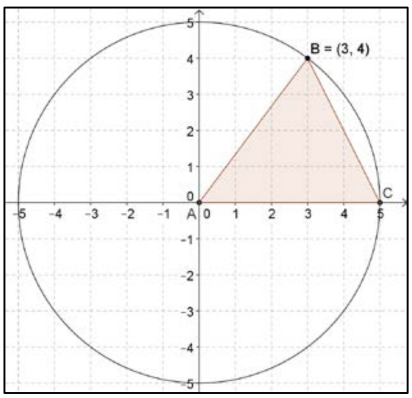
Classwork

Exploratory Challenge\Exercises 1–10: Triangles in Circles

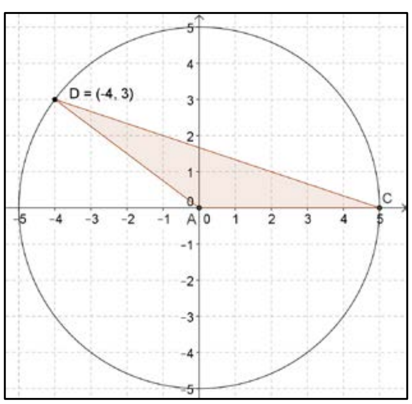
In this Exploratory Challenge, you will find the area of triangles with base along the positive x-axis and a third point on the graph of the circle $x^2 + y^2 = 25$.

1. Find the area of each triangle shown below. Show work to support your answer.

   a.

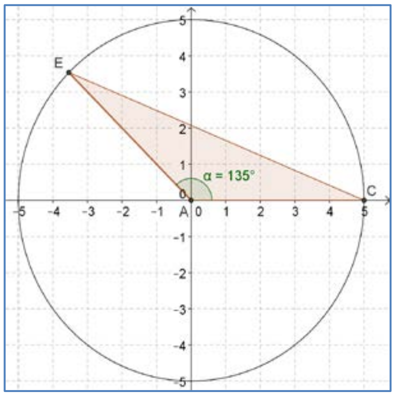


   b.

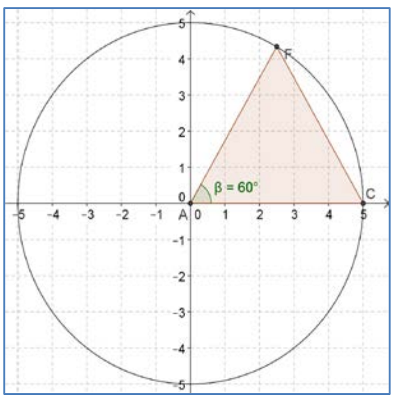


2. Find the area of each of the triangles shown below. Show work to support your answer.

   a.



   b.



3. Joni said that the area of triangle AFC in Exercise 2, part (b) can be found using the definition of the sine function.

   a. What are the coordinates of point F in terms of the cosine and sine functions? Explain how you know.

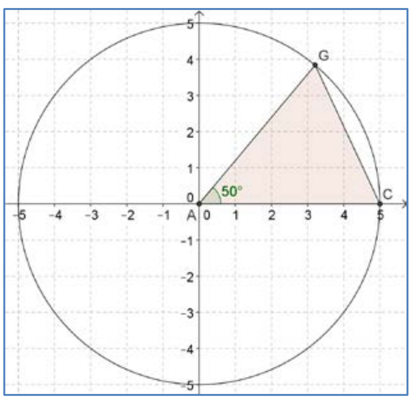
   b. Explain why the y-coordinate of point F is equal to the height of the triangle.

c. Write the area of triangle AFC in terms of the sine function.

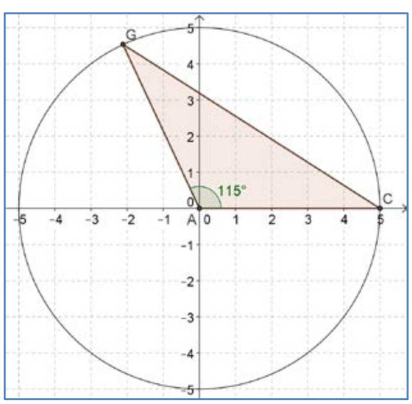
d. Does this method work for the area of triangle AEC?

4. Find the area of the following triangles.

a.



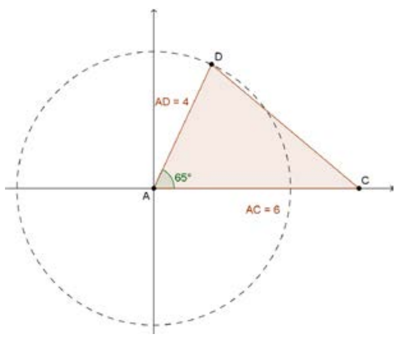
b.



5. Write a formula that will give the area of any triangle with vertices located at $A(0,0)$, $C(5,0)$ and $B(x, y)$ a point on the graph of $x^2 + y^2 = 25$ such that $y > 0$.

6. For what value of θ will this triangle have maximum area? Explain your reasoning.

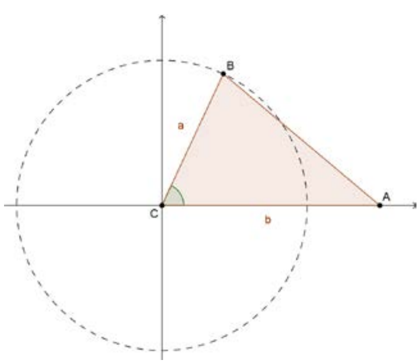
7. Find the area of the following triangle.



8. Prove that the area of any oblique triangle is given by the formula

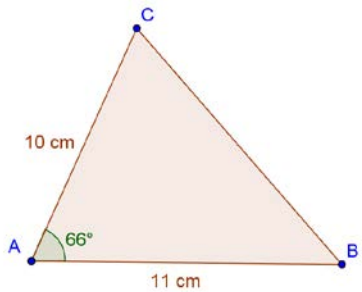
$$\text{Area} = \frac{1}{2}ab\sin(C)$$

where a and b are adjacent sides of ΔABC and C is the measure of the angle between them.

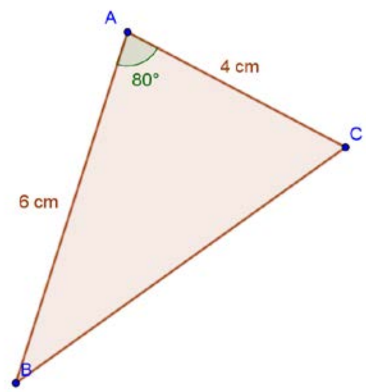


9. Use the area formula from Exercise 8 to calculate the area of the following triangles.

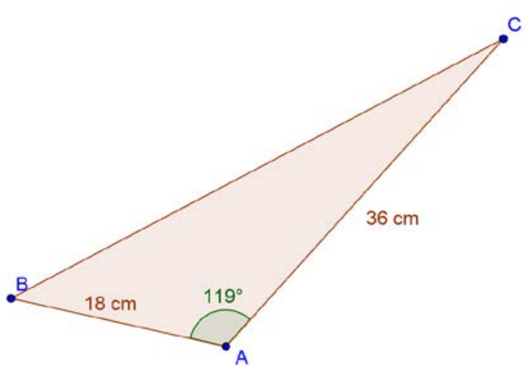
a.



b.

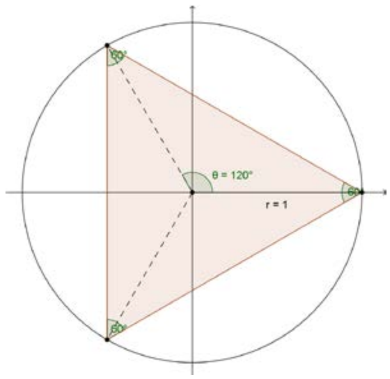


c. A quilter is making an applique design with triangular pieces like the one shown below. How much fabric is used in each piece?

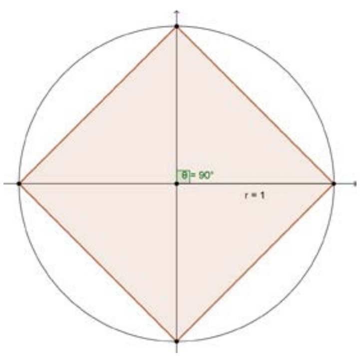


10. Calculate the area of the following regular polygons inscribed in a unit circle by dividing the polygon into congruent triangles where one of the triangles has a base along the positive x-axis.

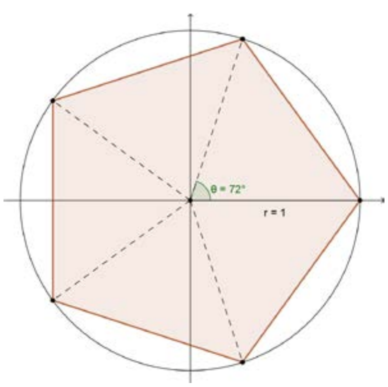
a.



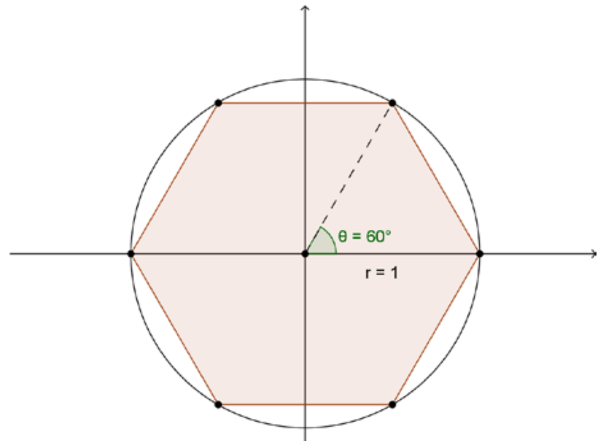
b.



c.



d. Sketch a regular hexagon inscribed in a unit circle with one vertex at $(1,0)$, and find the area of this hexagon.



e. Write a formula that will give the area of a regular polygon with n sides inscribed in a unit circle if one vertex is at $(1,0)$ and θ is the angle formed by the positive x-axis and the segment connecting the origin to the point on the polygon that lies in the first quadrant.

f. Use a calculator to explore the area of this regular polygon for large values of n. What does the area of this polygon appear to be approaching as the value of n increases?

Lesson Summary

The area of ΔABC is given by the formula:

$$\text{Area} = \frac{1}{2}ab\sin C$$

where A and B are the lengths of two sides of the triangle and C is the measure of angle between these sides.

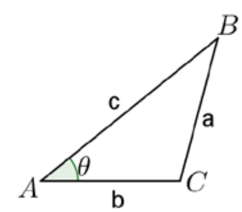
Problem Set

1. Find the area of the triangle ABC shown below, with the following data:

   a. $\theta = \frac{\pi}{6}$, $b = 3$, and $c = 6$.

   b. $\theta = \frac{\pi}{3}$, $b = 4$, and $c = 8$.

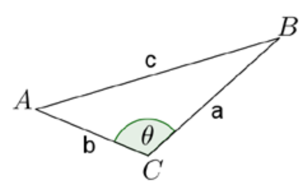
   c. $\theta = \frac{\pi}{4}$, $b = 5$, and $c = 10$.



2. Find the area of the triangle ABC shown below, with the following data:

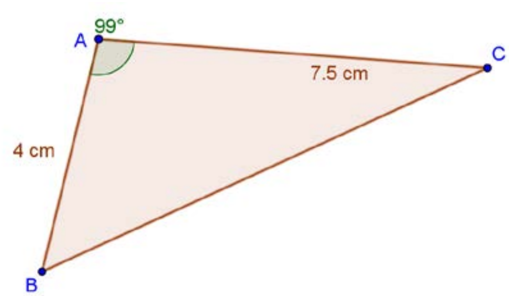
   a. $\theta = \frac{3\pi}{4}$, $a = 6$, and $b = 4$.

   b. $\theta = \frac{5\pi}{6}$, $a = 4$, and $b = 3$.

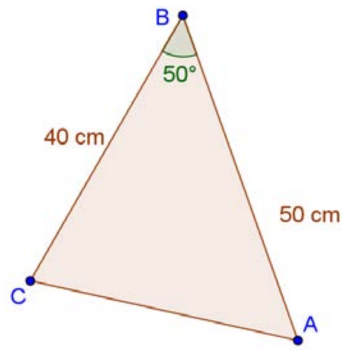


3. Find the area of each triangle shown below. State the area to the nearest tenth of a square centimeter.

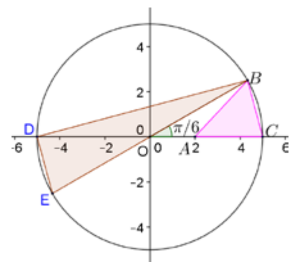
   a.



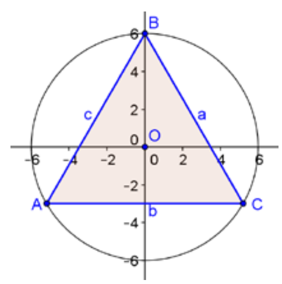
b.



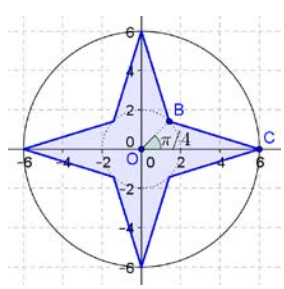
4. The diameter of the circle O in the figure shown below is $\overline{EB} = 10$.
   a. Find the area of the triangle OBA.
   b. Find the area of the triangle ABC.
   c. Find the area of the triangle DBO.
   d. Find the area of the triangle DBE.



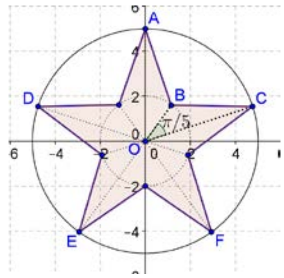
5. Find the area of the equilateral triangle ABC inscribed in a circle with a radius of 6.



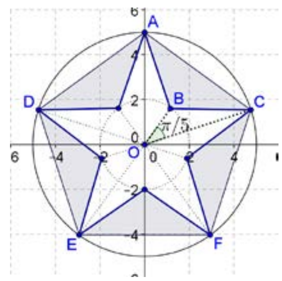
6. Find the shaded area in the diagram below.



7. Find the shaded area in the diagram below. The radius of the outer circle is 5; the length of the line segment $\overline{OB}$ is 2.



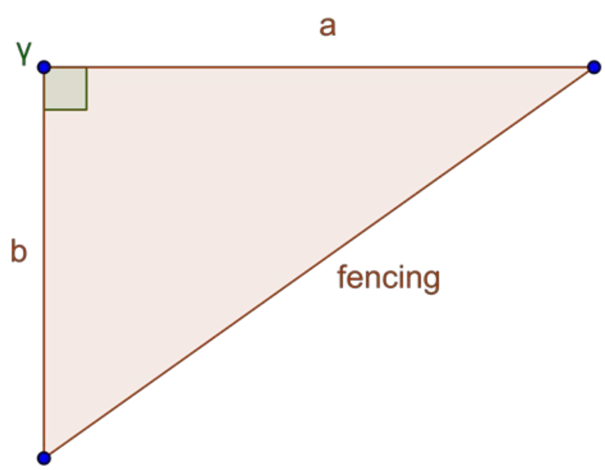
8. Find the shaded area in the diagram below. The radius of the outer circle is 5.



9. Find the area of the regular hexagon inscribed in a circle if one vertex is at $(2,0)$.

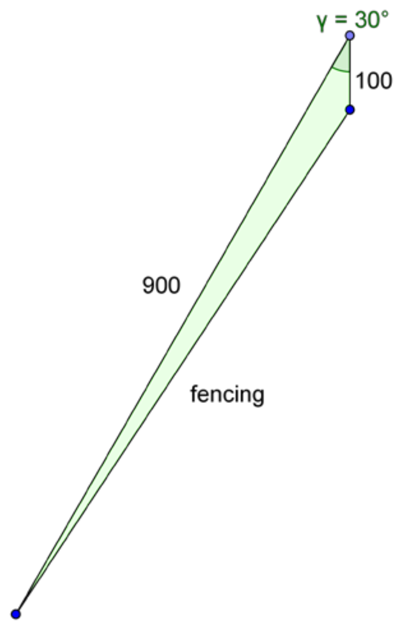
10. Find the area of the regular dodecagon inscribed in a circle if one vertex is at $(3,0)$.

11. A horse rancher wants to add on to existing fencing to create a triangular pasture for colts and fillies. She has 1000 feet of fence to construct the additional two sides of the pasture.
    a. What angle between the two new sides would produce the greatest area?



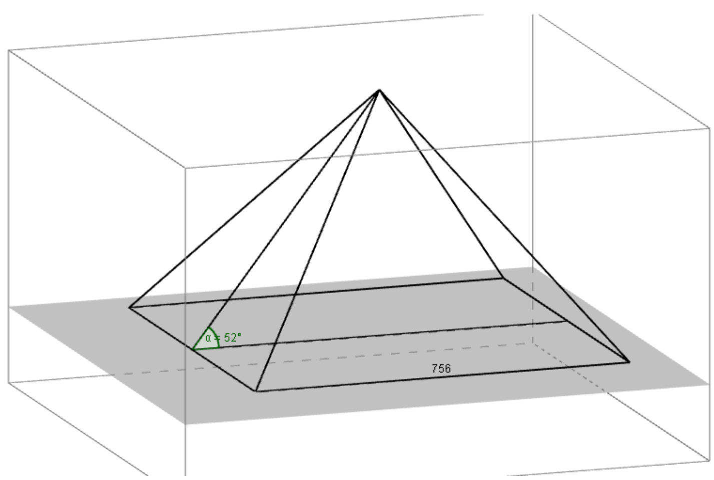
b. What is the area of her pasture if she decides to make two sides of 500 ft. each and uses the angle you found in part (a)?

c. Due to property constraints, she ends up using sides of 100 ft. and 900 ft. with an angle of $30°$ between them. What is the area of the new pasture?



12. An enthusiast of Egyptian history wants to make a life-size version of the Great Pyramid using modern building materials. The base of each side of the Great Pyramid was measured to be 756 ft. long, and the angle of elevation is about $52°$.

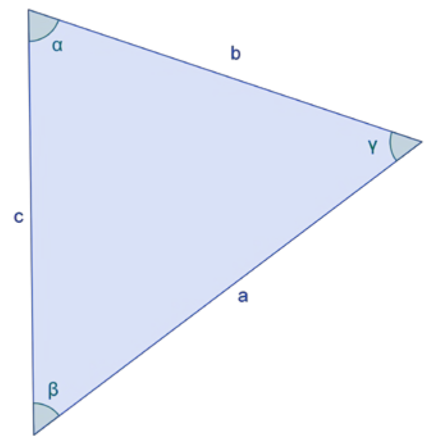
 a. How much material will go into the creation of the sides of the structure (the triangular faces of the pyramid)?



 b. If the price of plywood for the sides is $\$0.75$ per square foot, what is the cost of just the plywood for the sides?

13. Depending on which side you choose to be the "base," there are three possible ways to write the area of an oblique triangle, one being $A = \frac{1}{2}ab\sin(\gamma)$.

    a. Write the other two possibilities using $\sin(\alpha)$ and $\sin(\beta)$.



    b. Are all three equal?

    c. Find $\frac{2A}{abc}$ for all three possibilities.

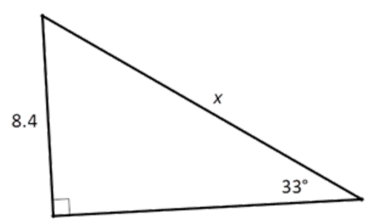
    d. Is the relationship you found in part (c) true for all triangles?

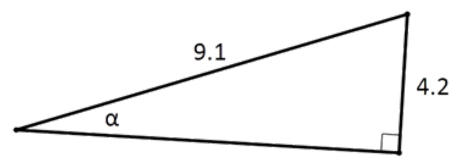
Lesson 8: Law of Sines

Classwork

Exercises

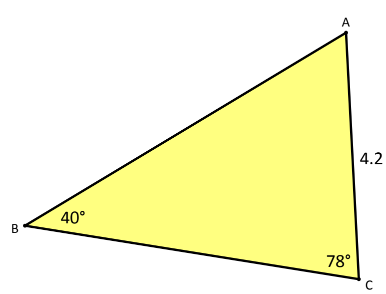
1. Find the value of x in the figure at the left.



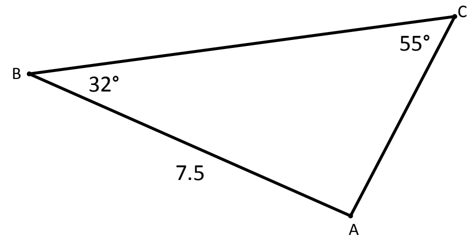


2. Find the value of α in the figure at the right.

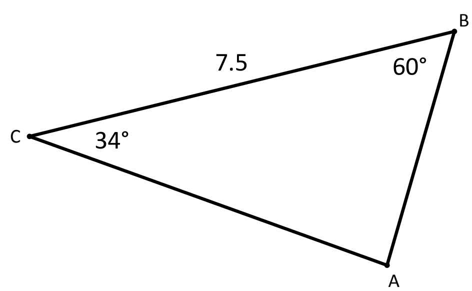
3. Find all of the measurements for the triangle below.



4. Find the length of side AC in the triangle below.



5. A hiker at point C is 7.5 kilometers from a hiker at point B; a third hiker is at point A. Use the angles shown in the diagram above to determine the distance between the hikers at points C and A.



6. Two sides of a triangle have lengths 10.4 and 6.4. The angle opposite 6.4 is $36°$. What could the angle opposite 10.4 be?

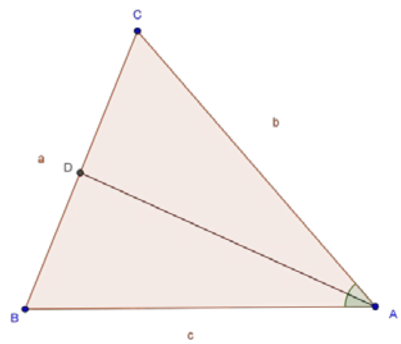
7. Two sides of a triangle have lengths 9.6 and 11.1. The angle opposite 9.6 is $59°$. What could the angle opposite 11.1 be?

Problem Set

1. Let ΔABC be the triangle with the given lengths and angle measurements. Find all possible missing measurements using the law of sines.
   a. $a = 5, m\angle A = 43, m\angle B = 80$.
   b. $a = 3.2, m\angle A = 110, m\angle B = 35$.
   c. $a = 9.1, m\angle A = 70, m\angle B = 95$.
   d. $a = 3.2, m\angle B = 30, m\angle C = 45$.
   e. $a = 12, m\angle B = 29, m\angle C = 31$.
   f. $a = 4.7, m\angle B = 18.8, m\angle C = 72$.
   g. $a = 6, b = 3, m\angle A = 91$.
   h. $a = 7.1, b = 7, m\angle A = 70$.
   i. $a = 8, b = 5, m\angle A = 45$.
   j. $a = 3.5, b = 3.6, m\angle A = 37$.
   k. $a = 9, b = 10.1, m\angle A = 61$.
   l. $a = 6, b = 8, m\angle A = 41.5$.

2. A surveyor is working at a river that flows north to south. From her starting point, she sees a location across the river that is $20°$ north of east from her current position, she labels the position S. She moves 110 feet north and measures the angle to S from her new position, seeing that it is $32°$ south of east.
   a. Draw a picture representing this situation.
   b. Find the distance from her starting position to S.
   c. Explain how you can use the procedure the surveyor used in this problem (called triangulation) to calculate the distance to another object.

3. Consider the triangle pictured below.



Use the law of sines to prove the generalized angle bisector theorem, that is, $\frac{\overline{BD}}{\overline{DC}} = \frac{c\sin(\angle BAD)}{b\sin(\angle CAD)}$. (Although this is called the generalized angle bisector theorem, we do not assume that the angle bisector of BAC intersects side $\overline{BC}$ at D. In the case that AD is an angle bisector, then the formula simplifies to $\frac{\overline{BD}}{\overline{DC}} = \frac{c}{b}$.)

a. Use the triangles ABD and ACD to express $\frac{c}{\overline{BD}}$ and $\frac{b}{\overline{DC}}$ as a ratio of sines.

b. Note that angles BDA and ADC form a linear pair. What does this tell you about the value of the sines of these angles?

c. Solve each equation in part (a) to be equal to the sine of either $\angle BDA$ or $\angle ADC$.

d. What do your answers to parts (b) and (c) tell you?

e. Prove the generalized angle bisector theorem.

4. As an experiment, Carrie wants to independently confirm the distance to Alpha Centauri. She knows that if she measures the angle of Alpha Centauri and waits 6 months and measures again, then she will have formed a massive triangle with two angles and the side between them being 2 AU long.

a. Carrie measures the first angle at $82°\,8'24.5''$ and the second at $97°\,51'34''$. How far away is Alpha Centauri according to Carrie's measurements?

b. Today, astronomers use the same triangulation method on a much larger scale by finding the distance between different spacecraft using radio signals, and then measuring the angles to stars. Voyager 1 is about 122 AU away from Earth. What fraction of the distance from Earth to Alpha Centauri is this? Do you think that measurements found in this manner are very precise?

5. A triangular room has sides of length 3.8, 5.1, and 5.1 m. What is the area of the room?

6. Sara and Paul are on opposite sides of a building that a telephone pole fell on. The pole is leaning away from Paul at an angle of $59°$ and towards Sara. Sara measures the angle of elevation to the top of the telephone pole to be $22°$, and Paul measures the angle of elevation to be $34°$. Knowing that the telephone pole is about 35 ft. tall, answer the following questions.

a. Draw a diagram of the situation.

b. How far apart are Sara and Paul?

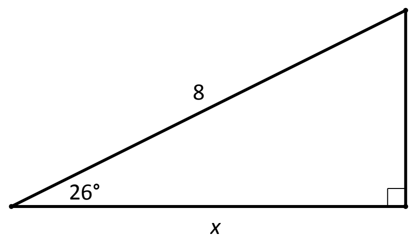
c. If we assume the building is still standing, how tall is the building?

Lesson 9: Law of Cosines

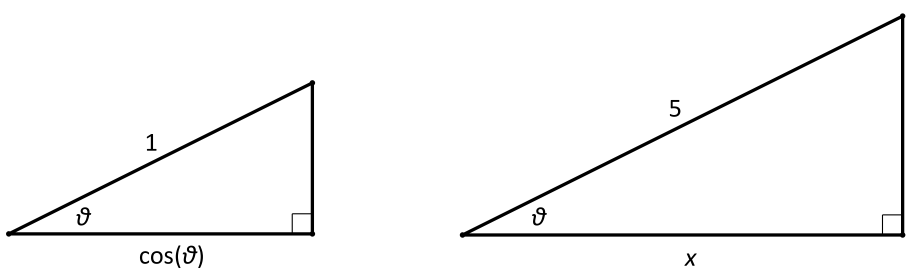
Classwork

Exercises

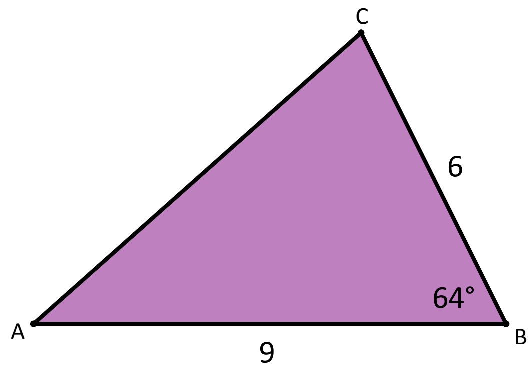
1. Find the value of x in the triangle below.



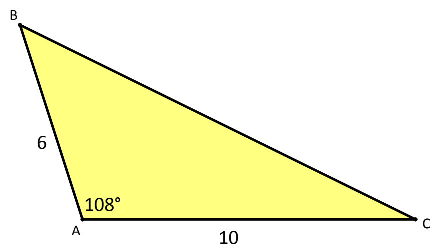
2. Explain how the figures below are related. Then, describe x in terms of ϑ.



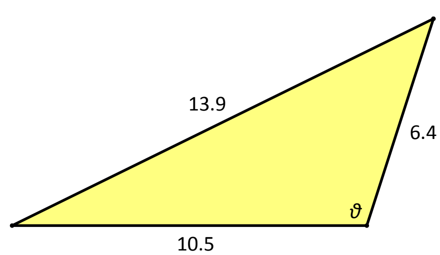
3. Find the length of side $\overline{AC}$ in the triangle below.



4. Points B and C are located at the edges of a large body of water. Point A is $6\ \mathrm{km}$ from point B and $10\ \mathrm{km}$ from point C. The angle formed between segments $\overline{BA}$ and $\overline{AC}$ is $108°$. How far apart are points B and C?



5. Use the law of cosines to find the value of ϑ in the triangle below.



Problem Set

1. Consider the case of a triangle with sides 5, 12, and the angle between them $90°$.
   a. What is the easiest method to find the missing side?
   b. What is the easiest method to find the missing angles?
   c. Can you use the law of cosines to find the missing side? If so, perform the calculations. If not, show why not.
   d. Can you use the law of cosines to find the missing angles? If so, perform the calculations. If not, show why not.
   e. Consider a triangle with sides a, b, and the angle between them $90°$. Use the law of cosines to prove a well-known theorem. State the theorem.
   f. Summarize what you have learned in parts (a) through (e).

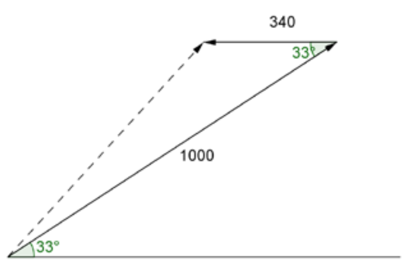
2. Consider the case of two line segments $\overline{CA}$ and $\overline{CB}$ of lengths 5 and 12, respectively, with $m\angle C = 180°$.
   a. Is ABC a triangle?
   b. What is the easiest method to find the distance between A and B?
   c. Can you use the law of cosines to find the distance between A and B? If so, perform the calculations. If not, show why not.
   d. Summarize what you have learned in parts (a) through (c).

3. Consider the case of two line segments $\overline{CA}$ and $\overline{CB}$ of lengths 5 and 12, respectively, with $m\angle C = 0°$.
   a. Is ABC a triangle?
   b. What is the easiest method to find the distance between A and B?
   c. Can you use the law of cosines to find the distance between A and B? If so, perform the calculations. If not, show why not.
   d. Summarize what you have learned in parts (a) through (c).

4. Consider the case of two line segments $\overline{CA}$ and $\overline{CB}$ of lengths 5 and 12, respectively, with $m\angle C > 180°$.
   a. Is the law of cosines consistent in being able to calculate the length of $\overline{AB}$ even using an angle this large? Try it for $m\angle C = 200°$, and compare your results to the triangle with $m\angle C = 160°$. Explain your findings.
   b. Consider what you have learned in Problems 1–4. If you were designing a computer program to be able to measure sides and angles of triangles created from different line segments and angles, would it make sense to use the law of cosines or several different techniques depending on the shape? Would a computer program created from the law of cosines have any errors based on different inputs for the line segments and angle between them?

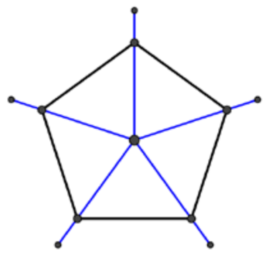
5. Consider triangles with the following measurements. If two sides are given, use the law of cosines to find the measure of the third side. If three sides are given, use the law of cosines to find the measure of the angle between a and b.
   a. $a = 4, b = 6, C = 35$.
   b. $a = 2, b = 3, C = 110$.
   c. $a = 5, b = 5, C = 36$.
   d. $a = 7.5, b = 10, C = 90$.
   e. $a = 4.4, b = 6.2, C = 9$.
   f. $a = 12, b = 5, C = 45$.
   g. $a = 3, b = 6, C = 60$.
   h. $a = 4, b = 5, c = 6$.
   i. $a = 1, b = 1, c = 1$.
   j. $a = 7, b = 8, c = 3$.
   k. $a = 6, b = 5.5, c = 6.5$.
   l. $a = 8, b = 5, c = 12$.
   m. $a = 4.6, b = 9, c = 11.9$.

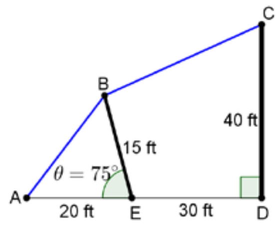
6. A trebuchet launches a boulder at an angle of elevation of $33°$ at a force of $1000\ N$. A strong gale wind is blowing against the boulder parallel to the ground at a force of $340\ N$. The figure is shown below.



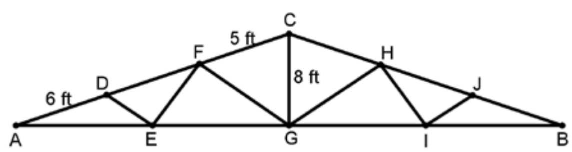
   a. What is the force in the direction of the boulder's path?
   b. What is the angle of elevation of the boulder after the wind has influenced its path?

7. Cliff wants to build a tent for his son's graduation party. The tent is a regular pentagon, as illustrated below. How much guide wire (show in blue) does Cliff need to purchase to build this tent? Round your answers to the nearest thousandths.





8. A roofing contractor needs to build roof trusses for a house. The side view of the truss is shown below. Given that G is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AG}$, I is the midpoint of $\overline{GB}$, $\overline{AB} = 32$ ft., $\overline{AD} = 6$ ft., $\overline{FC} = 5$ ft., and $\angle AGC = 90^\circ$. Find $\overline{DE}$, $\overline{EF}$, and $\overline{FG}$. Round your answers to the nearest thousandths.

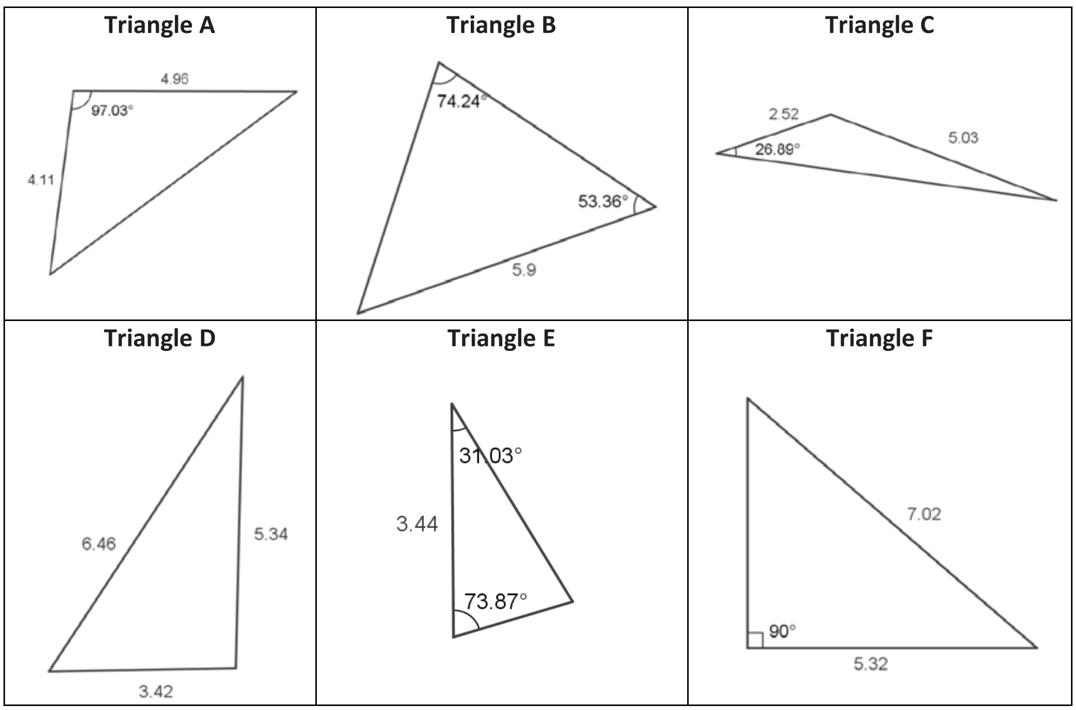


Lesson 10: Putting the Law of Cosines and the Law of Sines to Use

Classwork

Opening Exercise

a. For each triangle shown below, decide whether you should use the law of sines, the law of cosines, or neither to begin finding the missing measurements. Explain how you know.



b. What types of given information will help you to decide which formula to use to determine missing measurements? Summarize your ideas in the table shown below:

Determining Missing Measurements

Given Measurements	**Formulas to Use**
Right Triangle	Trigonometry Functions Pythagorean Theorem
Non-Right Triangle	Law of Sines
Non-Right Triangle	Law of Cosines

Exercises 1–7

1. A landscape architect is given a survey of a parcel of land that is shaped like a parallelogram. On the scale drawing the sides of the parcel of land are 8 in. and 10 in., and the angle between these sides measures $75°$. The architect is planning to build a fence along the longest diagonal. If the scale on the survey is $1 \text{ in.} = 120 \text{ ft.}$, how long will the fence be?

2. A regular pentagon is inscribed in a circle with a radius of 5 cm. What is the perimeter of the pentagon?

3. At the base of a pyramid, a surveyor determines that the angle of elevation to the top is $53°$. At a point 75 meters from the base, the angle of elevation to the top is $35°$. What is the distance from the base of the pyramid up the slanted face to the top?

4. A surveyor needs to determine the distance across a lake between an existing ferry dock at point A and a second dock across the lake at point B. He locates a point C along the shore from the dock at point A that is 750 meters away. He measures the angle at A between the sight lines to points B and C to be $65°$ and the angle at C between the sight lines to points A and B to be $82°$. How far is it from the dock at A and the dock at B?

5. Two people located 500 yards apart have spotted a hot air balloon. The angle of elevation from one person to the balloon is $67°$. From the second person to the balloon the angle of elevation is $46°$. How high is the balloon when it is spotted?

When applying mathematics to navigation, direction is often given as a bearing. The <u>bearing</u> of an object is the degrees rotated clockwise from north that indicates the direction of travel or motion. The next exercises apply the law of cosines and the law of sines to navigation problems.

6. Two fishing boats start from a port. One travels $15 \text{ nautical miles}$ per hour on a bearing of $25°$ and the other travels $18 \text{ nautical miles}$ per hour on a bearing of $100°$. Assuming each maintains its course and speed, how far apart will the fishing boats be after two hours?

7. An airplane travels on a bearing of $200°$ for 1500 miles and then changes to a bearing of $250°$ and travels an additional 500 miles. How far is the airplane from its starting point?

Example: Revisiting Vectors and Resultant Forces

The goalie on the soccer team kicks a ball with an initial force of 135 Newtons at a $40°$ angle with the ground. The mass of a soccer ball is 0.45 kg. Assume the acceleration due to gravity is $9.8\ \frac{\mathrm{m}}{\mathrm{s}^2}$.

a. Draw a picture representing the force vectors acting on the ball and the resultant force vector.

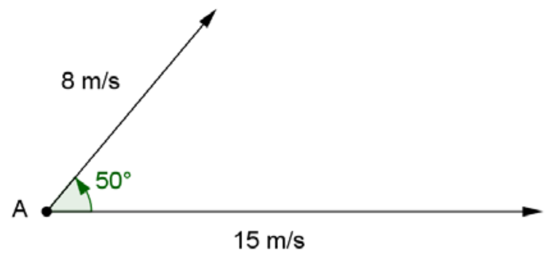
b. What is magnitude of the resultant force vector?

c. What are the horizontal and vertical components of this vector?

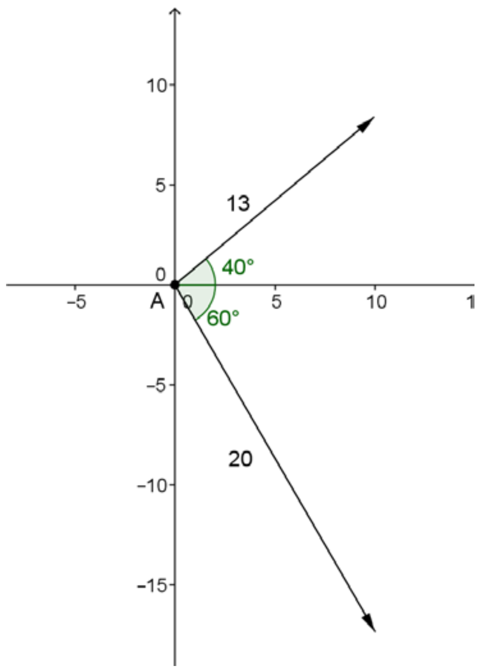
d. What is the angle of elevation of the resulting vector?

Exercises 8–10

8. Suppose a soccer player runs up to a moving soccer ball located at A and kicks the ball into the air. The diagram below shows the initial velocity of the ball along the ground and the initial velocity and direction of the kick. What is the resultant velocity and angle of elevation of the soccer ball immediately after it is kicked?



9. A 13 lb. force and a 20 lb. force are applied to an object located at A as shown in the diagram below. What is the resulting force and direction being applied to the object at A?



10. A motorboat is travels across a lake at a speed of 10 mph at a bearing of $25°$. The current of the lake due to the wind is a steady 2 mph at a bearing of $340°$.

    a. Draw a diagram that shows the two velocities that are affecting the boat's motion across the lake.

    b. What is the resulting speed and direction of the boat?

Lesson Summary

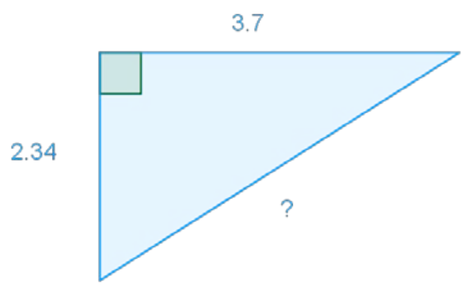
The law of sines and the law of cosines can be used to solve problems that can be represented with triangles with three known measurements.

The law of sines and the law of cosines can be used to find the magnitude and direction of the resultant sum of two vectors, which can represent velocities, distances, or forces.

Problem Set

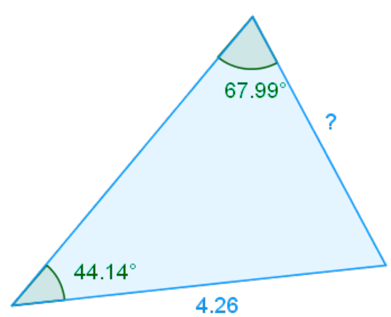
1. For each of the situations below, determine whether to use Pythagorean theorem, right-triangle trigonometry, law of sines, law of cosines, or some other method.

   a.



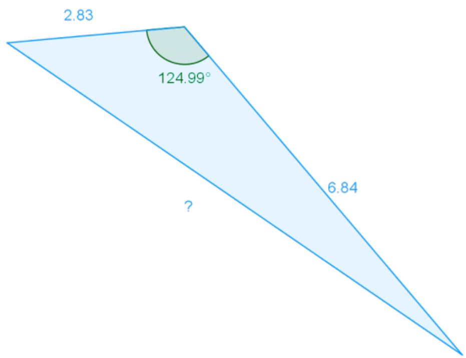
   b. Know one side and an angle of a right triangle, and want to find any other side.

   c.



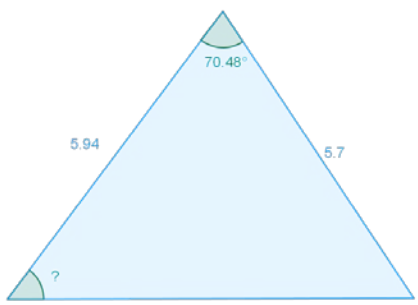
   d. Know two angles of a triangle and want to find the third.

e.



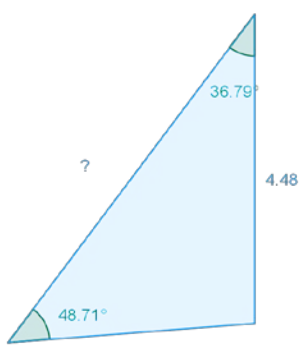
f. Know three sides of a triangle and want to find an angle.

g.



h. Know a side and two angles and want to find the third angle.

i.



2. Mrs. Lane's trigonometry class has been asked to judge the annual unmanned hot-air balloon contest, which has a prize for highest flying balloon.
   a. Sarah thinks that the class needs to set up two stations to sight each balloon as it passes between them. Construct a formula that Mrs. Lane's class can use to find the height of the balloon by plugging the two angles of elevation so that they can program their calculators to automatically output the height of the balloon. Use 500 ft. for the distance between the stations and α and β for the angles of elevation.
   b. The students expect the balloons to travel no higher than ft. What distance between the stations would you recommend? Explain.
   c. Find the heights of balloons sighted with the following angles of elevation to the nearest ten feet. Assume a distance of 500 ft. between stations.
      i. $5°, 15°$
      ii. $38°, 72°$
      iii. $45°, 45°$
      iv. $45°, 59°$
      v. $28°, 44°$
      vi. $50°, 66°$
      vii. $17°, 40°$
   d. Based on your results in part (c), which balloon won the contest?
   e. The balloons were released several hundred feet away, but directly in the middle of the two stations. If the first angle represents the West station and the second angle represents the East station, what can you say about the weather conditions during the contest?
   f. Are there any improvements to Mrs. Lane's class's methods that you would suggest? Explain.

3. Bearings on ships are often given as a clockwise angle from the direction the ship is heading ($0°$ represents something in the path of the boat and $180°$ represents something behind the boat). Two ships leave port at the same time. The first ship travels at a constant speed of 30 kn. After $2\frac{1}{2}$ hours, the ship sights the second at a bearing of $110°$ and 58 nautical miles away.
   a. How far is the second ship from the port where it started?
   b. How fast is the second ship traveling on average?

4. A paintball is fired from a gun with a force of 59 N at an angle of elevation of $1°$. If the force due to gravity on the paintball is 0.0294 N, then answer the following:
   a. Is this angle of elevation enough to overcome the initial force due to gravity and still have an angle of elevation greater than $0.5°$?
   b. What is the resultant magnitude of the vector in the direction of the paintball?

5. Valerie lives 2 miles west of her school and her friend Yuri lives 3 miles directly northeast of her.
   a. Draw a diagram representing this situation.
   b. How far does Yuri live from school?
   c. What is the bearing of the school to Yuri's house?

6. A 2.1-kg rocket is launched at an angle of $33°$ with an initial force of 50 N. Assume the acceleration due to gravity is $9.81 \frac{\text{m}}{\text{s}^2}$.
   a. Draw a picture representing the force vectors and their resultant vector.
   b. What is the magnitude of the resultant vector?
   c. What are the horizontal and vertical components of the resultant vector?
   d. What is the angle of elevation of the resultant vector?

7. Use the distance formula to find c, the distance between A and B for ΔABC, with $A = (b\cos(\gamma), b\sin(\gamma))$, $B = (a, 0)$, and $C = (0,0)$. After simplifying, what formula have you proven?

8. For isosceles triangles with $a = b$, show the law of cosines can be written as $\cos(\gamma) = 1 - \frac{c^2}{2a^2}$.

Lesson 11: Revisiting the Graphs of the Trigonometric Functions

Classwork

Opening Exercise

Graph each of the following functions on the interval $-2\pi \le x \le 4\pi$ by making a table of values. The graph should show all key features (intercepts, asymptotes, relative maxima and minima).

a. $f(x) = \sin(x)$

x	
$\sin(x)$	

b. $f(x) = \cos(x)$

x	
$\cos(x)$	

Exercises 1–7

1. Consider the trigonometric function $f(x) = \tan(x)$.

   a. Rewrite $\tan(x)$ as a quotient of trigonometric functions. Then, state the domain of the tangent function.

   b. Why is this the domain of the function?

   c. Complete the table.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\tan(x)$													

   d. What will happen on the graph of $f(x) = \tan(x)$ at the values of x for which the tangent function is undefined?

   e. Expand the table to include angles that have a reference angle of $\frac{\pi}{4}$.

x	$-\frac{7\pi}{2}$	$-\frac{5\pi}{2}$	$-\frac{3\pi}{2}$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$\frac{7\pi}{2}$	$\frac{9\pi}{2}$	$\frac{11\pi}{2}$	$\frac{13\pi}{2}$	$\frac{15\pi}{2}$
$\tan(x)$												

f. Sketch the graph of $f(x) = \tan(x)$ on the interval $-2\pi \leq x \leq 4\pi$. Verify by using a graphing utility.

2. Use the graphs of the sine, cosine, and tangent functions to answer each of the following.

   a. How do the graphs of the sine and cosine functions support the following identities for all real numbers x?

$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$

   b. Use the symmetry of the graph of the tangent function to write an identity. Explain your answer.

   c. How do the graphs of the sine and cosine functions support the following identities for all real numbers x?

$$\sin(x + 2\pi) = \sin(x)$$
$$\cos(x + 2\pi) = \cos(x)$$

d. Use the periodicity of the tangent function to write an identity. Explain your answer.

3. Consider the function $f(x) = \cos\left(x - \frac{\pi}{2}\right)$.

a. Graph $y = f(x)$ by using transformations of functions.

b. Based on your graph, write an identity.

4. Verify the identity $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$ for all real numbers x by using a graph.

5. Use a graphing utility to explore the graphs of the family of functions in the form $f(x) = A\sin\big(\omega(x-h)\big) + k$. Write a summary of the effect that changing each parameter has on the graph of the sine function.
   a. A

   b. ω

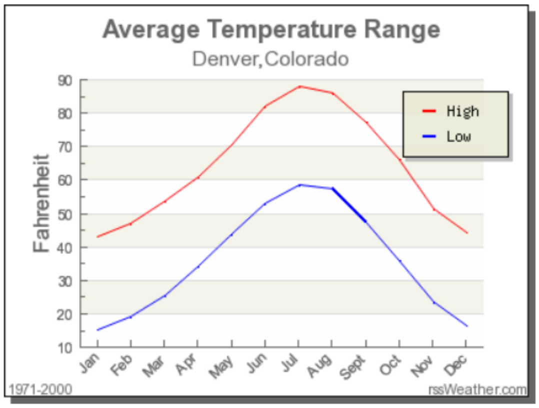
   c. h

   d. k

6. Graph at least one full period of the function $f(x) = 3\sin\left(\frac{1}{3}(x-\pi)\right) + 2$. Label the amplitude, period, and midline on the graph.

7. The graph and table below show the average monthly high and low temperature for Denver, Colorado. (source: http://www.rssweather.com/climate/Colorado/Denver/)

Average Temperatures for Denver



Month	Low	High
Jan	15.2°F	43.2°F
Feb	19.1°F	47.2°F
Mar	25.4°F	53.7°F
Apr	34.2°F	60.9°F
May	43.8°F	70.5°F
Jun	53.0°F	82.1°F
Jul	58.7°F	88.0°F
Aug	57.4°F	86.0°F
Sept	47.3°F	77.4°F
Oct	35.9°F	66.0°F
Nov	23.5°F	51.5°F
Dec	16.4°F	44.1°F

a. Why would a sinusoidal function be appropriate to model this data?

b. Write a function to model the average monthly high temperature as a function of the month.

c. What does the midline represent within the context of the problem?

d. What does the amplitude represent within the context of the problem?

e. Name a city whose temperature graphs would have a smaller amplitude. Explain your reasoning.

f. Name a city whose temperature graphs would have a larger vertical shift. Explain your reasoning.

Problem Set

1. Sketch the graph of $y = \sin(x)$ on the same set of axes as the function $f(x) = \sin(4x)$. Explain the similarities and differences between the two graphs.

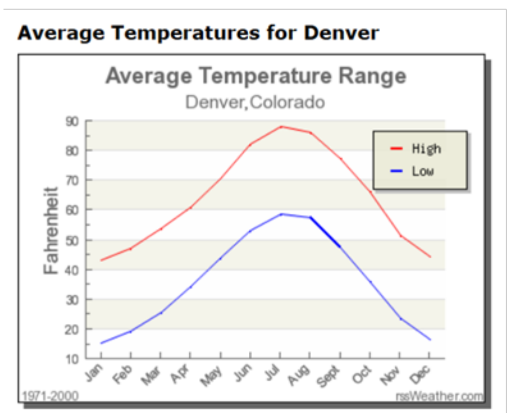
2. Sketch the graph of $y = \sin\left(\frac{x}{2}\right)$ on the same set of axes as the function $g(x) = 3\sin\left(\frac{x}{2}\right)$. Explain the similarities and differences between the two graphs.

3. Indicate the amplitude, frequency, period, phase shift, horizontal and vertical translations, and equation of the midline. Graph the function on the same axes as the graph of the cosine function $f(x) = \cos(x)$. Graph at least one full period of each function.

$$g(x) = \cos\left(x - \frac{3\pi}{4}\right).$$

4. Sketch the graph of the pairs of functions on the same set of axes: $f(x) = \sin(4x)$, $g(x) = \sin(4x) + 2$.

5. The graph and table below show the average monthly high and low temperature for Denver, Colorado. (source: http://www.rssweather.com/climate/Colorado/Denver/)



Month	Low	High
Jan	15.2°F	43.2°F
Feb	19.1°F	47.2°F
Mar	25.4°F	53.7°F
Apr	34.2°F	60.9°F
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Jun	53.0°F	82.1°F
Jul	58.7°F	88.0°F
Aug	57.4°F	86.0°F
Sept	47.3°F	77.4°F
Oct	35.9°F	66.0°F
Nov	23.5°F	51.5°F
Dec	16.4°F	44.1°F

Write a function to model the average monthly low temperature as a function of the month.

Extension:

6. Consider the cosecant function.
   a. Use technology to help you sketch $y = \csc(x)$ for $0 \leq x \leq 4\pi$, $-4 \leq y \leq 4$.
   b. What do you notice about the graph of the function? Compare this to your knowledge of the graph of $y = \sin(x)$.

7. Consider the secant function.
   a. Use technology to help you sketch $y = \sec(x)$ for $0 \leq x \leq 4\pi$, $-4 \leq y \leq 4$.
   b. What do you notice about the graph of the function? Compare this to your knowledge of the graph of $y = \cos(x)$.

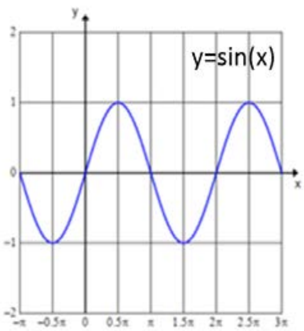
8. Consider the cotangent function.
   a. Use technology to help you sketch $y = \cot(x)$ for $0 \leq x \leq 2\pi$, $-4 \leq y \leq 4$.
   b. What do you notice about the graph of the function? Compare this to your knowledge of the graph of $y = \tan(x)$.

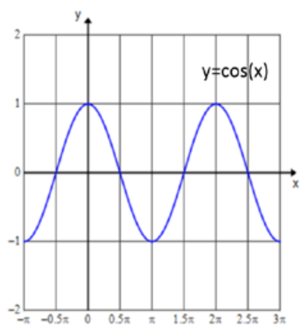
Lesson 12: Inverse Trigonometric Functions

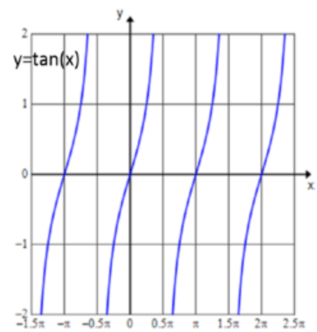
Classwork

Opening Exercise

Use the graphs of the sine, cosine, and tangent functions to answer each of the following questions.







a. State the domain of each function.

b. Would the inverse of the sine, cosine, or tangent functions also be functions? Explain.

c. For each function, select a suitable domain that will make the function invertible.

Example 1

Consider the function $f(x) = \sin(x), -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

a. State the domain and range of this function.

b. Find the equation of the inverse function.

c. State the domain and range of the inverse.

Exercises 1–3

1. Write an equation for the inverse cosine function, and state its domain and range.

2. Write an equation for the inverse tangent function, and state its domain and range.

3. Evaluate each of the following expressions without using a calculator. Use radian measures.

a. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

c. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

d. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

e. $\sin^{-1}(1)$

f. $\sin^{-1}(-1)$

g. $\cos^{-1}(1)$

h. $\cos^{-1}(-1)$

i. $\tan^{-1}(1)$

j. $\tan^{-1}(-1)$

Example 2

Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Round to three decimal places when necessary.

a. $2\cos(x) - 1 = 0$

b. $3\sin(x) + 2 = 0$

Exercises 4–8

4. Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Give answers in exact form.

   a. $\sqrt{2}\cos(x) + 1 = 0$

   b. $\tan(x) - \sqrt{3} = 0$

   c. $\sin^2(x) - 1 = 0$

5. Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Round answers to three decimal places.

   a. $5\cos(x) - 3 = 0$

   b. $3\cos(x) + 5 = 0$

c. $3\sin(x) - 1 = 0$

d. $\tan(x) = -0.115$

6. A particle is moving along a straight line for $0 \leq t \leq 18$. The velocity of the particle at time t is given by the function $v(t) = \cos\left(\frac{\pi}{5}t\right)$. Find the time(s) on the interval $0 \leq t \leq 18$ where the particle is at rest ($v(t) = 0$).

7. In an amusement park, there is a small Ferris wheel, called a kiddie wheel, for toddlers. The formula $H(t) = 10\sin\left(2\pi\left(t - \frac{1}{4}\right)\right) + 15$ models the height H (in feet) of the bottom-most car t minutes after the wheel begins to rotate. Once the ride starts, it lasts 4 minutes.

   a. What is the initial height of the car?

   b. How long does it take for the wheel to make one full rotation?

   c. What is the maximum height of the car?

d. Find the time(s) on the interval $0 \leq t \leq 4$ when the car is at its maximum height.

8. Many animal populations fluctuate periodically. Suppose that a wolf population over an 8-year period is given by the function $W(t) = 800\sin\left(\frac{\pi}{4}t\right) + 2200$, where t represents the number of years since the initial population counts were made.

a. Find the time(s) on the interval $0 \leq t \leq 8$ such that the wolf population equals 2500.

b. On what time interval during the 8-year period is the population below 2000?

c. Why would an animal population be an example of a periodic phenomenon?

Problem Set

1. Solve the following equations. Approximate values of the inverse trigonometric functions to the thousandths place, where x refers to an angle measured in radians.
   a. $5 = 6\cos(x)$
   b. $-\frac{1}{2} = 2\cos\left(x - \frac{\pi}{4}\right) + 1$
   c. $1 = \cos\big(3(x-1)\big)$
   d. $1.2 = -0.5\cos(\pi x) + 0.9$
   e. $7 = -9\cos(x) - 4$
   f. $2 = 3\sin(x)$
   g. $-1 = \sin\left(\frac{\pi(x-1)}{4}\right) - 1$
   h. $\pi = 3\sin(5x+2) + 2$
   i. $\frac{1}{9} = \frac{\sin(x)}{4}$
   j. $\cos(x) = \sin(x)$
   k. $\sin^{-1}(\cos(x)) = \frac{\pi}{3}$
   l. $\tan(x) = 3$
   m. $-1 = 2\tan(5x+2) - 3$
   n. $5 = -1.5\tan(-x) - 3$

2. Fill out the following tables.

x	$\sin^{-1}(x)$	$\cos^{-1}(x)$
-1		
$-\frac{\sqrt{3}}{2}$		
$-\frac{\sqrt{2}}{2}$		
$-\frac{1}{2}$		

x	$\sin^{-1}(x)$	$\cos^{-1}(x)$
0		
$\frac{1}{2}$		
$\frac{\sqrt{2}}{2}$		
$\frac{\sqrt{3}}{2}$		
1		

3. Let the velocity v in miles per second of a particle in a particle accelerator after t seconds be modeled by the function $v = \tan\left(\frac{\pi t}{6000} - \frac{\pi}{2}\right)$ on an unknown domain.
   a. What is the t-value of the first vertical asymptote to the right of the y-axis?
   b. If the particle accelerates to 99% of the speed of light before stopping, then what is the domain? Note: $c \approx 186000$. Round your solution to the ten-thousandths place.
   c. How close does the domain get to the vertical asymptote of the function?
   d. How long does it take for the particle to reach the velocity of Earth around the sun (about 18.5 miles per second)?
   e. What does it imply that v is negative up until $t = 3000$?

Lesson 13: Modeling with Inverse Trigonometric Functions

Classwork

Example

The Statue of Liberty is 151 feet tall and sits on a pedestal that is 154 feet above the ground. An observer who is 6 feet tall wants to stand at the ideal viewing distance in front of the statue.

a. Sketch the statue and observer. Label all appropriate measurements on the sketch, and define them in context.

b. How far back from the statue should the observer stand so that his or her viewing angle (from the feet of the statue to the tip of the torch) is largest? What is the value of the largest viewing angle?

c. What would be your best viewing distance from the statue?

d. If there are 66 meters of dry land in front of the statue, is the viewer still on dry land at the best viewing distance?

Exercise

Hanging on a museum wall is a picture with base a inches above a viewer's eye level and top b inches above the viewer's eye level.

a. Model the situation with a diagram.

b. Determine an expression that could be used to find the ideal viewing distance x that maximizes the viewing angle y.

c. Find the ideal viewing distance, given the a and b values assigned to you. Calculate the maximum viewing angle in degrees.

d. Complete the table using class data, which indicates the ideal values for x given different assigned values of a and b. Note any patterns you see in the data.

a (inches)	**b (inches)**	**x at max (inches)**	**y max (degrees)**

Problem Set

1. Consider the situation of sitting down with eye level at 46 in.. Find the missing distances and heights for the following:
   a. The bottom of the picture is at 50 in. and the top is at 74 in. What is the optimal viewing distance?
   b. The bottom of the picture is at 52 in. and the top is at 60 in. What is the optimal viewing distance?
   c. The bottom of the picture is at 48 in. and the top is at 64 in. What is the optimal viewing distance?
   d. What is the height of the picture if the optimal viewing distance is 1 ft. and the bottom of the picture is hung at 47 in.?

2. Consider the situation where you are looking at a painting a inches above your line of sight and b inches below your line of sight.
   a. Find the optimal viewing distance if it exists.
   b. If the average standing eye height of Americans is 61.4 in., at what height should paintings and other works of art be hung?

3. The amount of daylight per day is periodic with respect to the day of the year. The function $y = -3.016\cos\left(\frac{2\pi x}{365}\right) + 12.25$ gives the number of hours of daylight in New York, y, as a function of the number of days since the winter solstice (December 22), which is represented by x.
   a. On what days will the following hours of sunlight occur?
      i. 15 hours, 15 minutes.
      ii. 12 hours.
      iii. 9 hours, 15 minutes.
      iv. 10 hours.
      v. 9 hours.
   b. Give a function that will give the day of the year from the solstice as a function of the hours of daylight.
   c. What is the domain of the function you gave in part (b)?
   d. What does the domain tell you in the context of the problem?
   e. What is the range of the function? Does this make sense in the context of the problem? Explain.

4. Ocean tides are an example of periodic behavior. At a particular harbor, data was collected over the course of 24 hours to create the following model: $y = 1.236\sin\left(\frac{\pi}{3}x\right) + 1.798$, which gives the water level, y, in feet above the MLLW (mean lower low water) as a function of the time, x, in hours.
   a. How many periods are there each day?
   b. Write a function that gives the time in hours as a function of the water level. How many other times per day will have the same water levels as those given by the function?

Lesson 14: Modeling with Inverse Trigonometric Functions

Classwork

Example 1

A designer wants to test the safety of a wheelchair ramp she has designed for a building before constructing it, so she creates a scale model. To meet the city's safety requirements, an object that starts at a standstill from the top of the ramp and rolls down it should not experience an acceleration exceeding $2.4\,\frac{m}{s^2}$.

a. A ball of mass 0.1 kg is used to represent an object that rolls down the ramp. As it is placed at the top of the ramp, the ball experiences a downward force due to gravity, which causes it to accelerate down the ramp. Knowing that the force applied to an object is the product of its mass and acceleration, create a sketch to model the ball as it accelerates down the ramp.

b. If the ball rolls at the maximum allowable acceleration of $2.4\ \frac{\mathrm{m}}{\mathrm{s}^2}$, what is the angle of elevation for the ramp?

c. If the designer wants to exceed the safety standards by ensuring the acceleration of the object does not exceed $2.0\,\frac{m}{s^2}$, by how much will the maximum angle of elevation decrease?

d. How does the mass of the object used in the scale model affect the value of θ? Explain your response.

Exercise 1

A vehicle with a mass of 1000 kg rolls down a slanted road with an acceleration of $0.07\,\frac{m}{s^2}$. The frictional force between the wheels of the vehicle and the wet concrete road is 2800 Newtons.

a. Sketch the situation.

b. What is the angle of elevation of the road?

c. What is the maximum angle of elevation the road could have so that the vehicle described would not slide down the road?

Example 2

The declination of the sun is the path the sun takes overhead the earth throughout the year. When the sun passes directly overhead, the declination is defined as $0°$, while a positive declination angle represents a northward deviation and a negative declination angle represents a southward deviation. Solar declination is periodic and can be roughly estimated using the equation$\delta = -23.44°(\cos\left(\frac{360}{365}\right)(N + 10))$, where N represents a calendar date, e.g., $N = 1$ is January 1, and δ is the declination angle of the sun measured in degrees.

a. Describe the domain and range of the function.

b. Write an equation that represents N as a function of δ.

c. Determine the calendar date(s) for the given angles of declination:

i. $10°$

ii. $-5.2°$

iii. $25°$

d. When will the sun trace a direct path above the equator?

Exercises 2–3

2. The average monthly temperature in a coastal city in the United States is periodic and can be modeled with the equation $y = -8\cos\left((x-1)\left(\frac{\pi}{6}\right)\right) + 17.5$, where y represents the average temperature in degrees Celsius and x represents the month, with $x = 1$ representing January.

a. Write an equation that represents x as a function of y.

b. A tourist wants to visit the city when the average temperature is closest to $25°$ Celsius. What recommendations would you make regarding when the tourist should travel? Justify your response.

3. The estimated size for a population of rabbits and a population of coyotes in a desert habitat are shown in the table. The estimated population sizes were recorded as part of a long-term study related to the effect of commercial development on native animal species.

Years since initial count (n)	0	3	6	9	12	15	18	21	24
Estimated number of rabbits (r)	$14{,}989$	$10{,}055$	$5{,}002$	$10{,}033$	$15{,}002$	$10{,}204$	$4{,}999$	$10{,}002$	$14{,}985$
Estimated number of coyotes (c)	$1{,}995$	$2{,}201$	$2{,}003$	$1{,}795$	$1{,}999$	$2{,}208$	$2{,}010$	$1{,}804$	$2{,}001$

a. Describe the relationship between sizes of the rabbit and coyote populations throughout the study.

b. Plot the relationship between the number of years since the initial count and the number of rabbits. Fit a curve to the data.

c. Repeat the procedure described in part (b) for the estimated number of coyotes over the course of the study.

d. During the study, how many times was the rabbit population approximately $12{,}000$? When were these times?

e. During the study, when was the coyote population estimate below $2{,}100$?

Problem Set

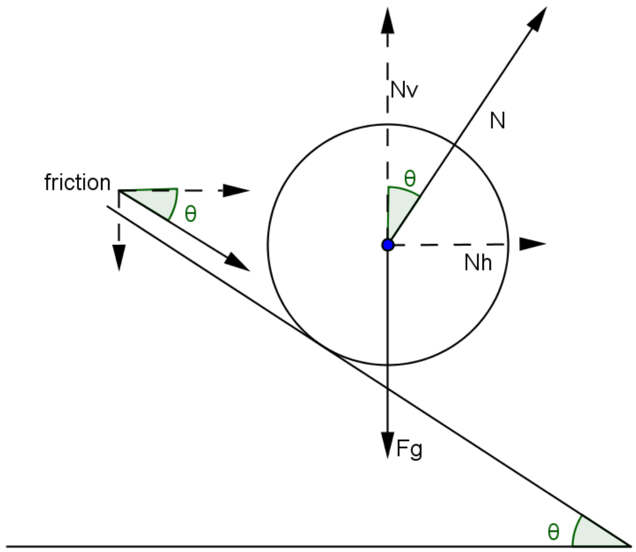
1. A particle is moving along a line at a velocity of $y = 3\sin\left(\frac{2\pi x}{5}\right) + 2\ \frac{\text{m}}{\text{s}}$ at location x meters from the starting point on the line for $0 \leq x \leq 20$.
   a. Find a formula that represents the location of the particle given its velocity.
   b. What is the domain and range of the function you found in part (a)?
   c. Use your answer to part (a) to find where the particle is when it is traveling $5\ \frac{\text{m}}{\text{s}}$ for the first time.
   d. How can you find the other locations the particle is traveling at this speed?

2. In general, since the cosine function is merely the sine function under a phase shift, mathematicians and scientists regularly choose to use the sine function to model periodic phenomena instead of a mixture of the two. What behavior in data would prompt a scientist to use a tangent function instead of a sine function?

3. A vehicle with a mass of 500 kg rolls down a slanted road with an acceleration of $0.04\ \frac{\text{m}}{\text{s}^2}$. The frictional force between the wheels of the vehicle and the road is 1800 Newtons.
   a. Sketch the situation.
   b. What is the angle of elevation of the road?
   c. The steepness of a road is frequently measured as grade, which expresses the slope of a hill as a percent that the change in height is of the change in horizontal distance. What is the grade of the hill described in this problem?

4. Canton Avenue in Pittsburgh, PA is considered to be one of the steepest roads in the world with a grade of 37%.
   a. Assuming no friction on a particularly icy day, what would be the acceleration of a 1000 kg car with only gravity acting on it?
   b. The force due to friction is equal to the product of the force perpendicular to the road and the coefficient of friction μ. For icy roads of a non-moving vehicle, assume the coefficient of friction is $\mu = 0.3$. Find the force due to friction for the car above. If the car is in park, will it begin sliding down Canton Avenue if the road is this icy?
   c. Assume the coefficient of friction for moving cars on icy roads is $\mu = 0.2$. What is the maximum angle of road that the car will be able to stop on?

5. Talladega Superspeedway has some of the steepest turns in all of NASCAR. The main turns have a radius of about 305 m and are pitched at $33°$. Let N be the perpendicular force on the car, and N_v and N_h be the vertical and horizontal components of this force, respectively. See the diagram below.



a. Let μ represent the coefficient of friction; recall that μN gives the force due to friction. To maintain the position of a vehicle traveling around the bank, the centripetal force must equal the horizontal force in the direction of the center of the track. Add the horizontal component of friction to the horizontal component of the perpendicular force on the car to find the centripetal force. Set your expression equal to $\frac{mv^2}{r}$, the centripetal force.

b. Add the vertical component of friction to the force due to gravity, and set this equal to the vertical component of the perpendicular force.

c. Solve one of your equations in part (a) or part (b) for m, and use this with the other equation to solve for v.

d. Assume $\mu = 0.75$, the standard coefficient of friction for rubber on asphalt. For the Talladega Superspeedway, what is the maximum velocity on the main turns? Is this about how fast you might expect NASCAR stock cars to travel? Explain why you think NASCAR takes steps to limit the maximum speeds of the stock cars.

e. Does the friction component allow the cars to travel faster on the curve or force them to drive slower? What is the maximum velocity if the friction coefficient is zero on the Talladega roadway?

f. Do cars need to travel slower on a flat roadway making a turn than on a banked roadway? What is the maximum velocity of a car traveling on a 305 m turn with no bank?

6. At a particular harbor over the course of 24 hours, the following data on peak water levels was collected (measurements are in feet above the MLLW):

Time	$1:30$	$7:30$	$14:30$	$20:30$
Water Level	-0.211	8.21	-0.619	7.518

   a. What appears to be the average period of the water level?
   b. What appears to be the average amplitude of the water level?
   c. What appears to be the average midline for the water level?
   d. Fit a curve of the form $y = A\sin\big(\omega(x-h)\big) + k$ or $y = A\cos\big(\omega(x-h)\big) + k$ modeling the water level in feet as a function of the time.
   e. According to your function, how many times per day will the water level reach its maximum?
   f. How can you find other time values for a particular water level after finding one value from your function?
   g. Find the inverse function associated with the function in part (d). What is the domain and range of this function? What type of values does this function output?