# (8) Lesson 29: When Can We Reverse a Transformation? 

## Student Outcomes

- Students understand that an inverse transformation, when represented by a $2 \times 2$ matrix, exists precisely when the determinant of that matrix is non-zero.


## Lesson Notes

Lesson 29 is the second of a three-day lesson sequence. In Lesson 28, students were introduced to inverse matrices and asked to find inverses of matrices with a determinant of 1 by solving a system of equations. Lesson 29 has students finding the inverse of any matrix and understanding when a matrix does not have an inverse.

## Classwork

The Opening Exercise can be done individually or in pairs. Students will use the skills learned in Lesson 28 to find an inverse matrix and then compare that inverse to inverses of other matrices determined in Lesson 28 . Students will see a pattern. Then, they will see that that pattern only works if the determinant is 1 . This will lead to a general formula for any matrix followed by the question, "Do all matrices have inverses?"

## Opening Exercise (5 minutes)

> Opening Exercise
> Find the inverse of $\left[\begin{array}{cc}-7 & -2 \\ 4 & 1\end{array}\right]$. Show your work. Confirm that the matrices are inverses.
> $\left[\begin{array}{ll}a & c \\ b & d\end{array}\right]\left[\begin{array}{cc}-7 & -2 \\ 4 & 1\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
> $-7 a+4 c=1,-2 a+c=0,-7 b+4 d=0,-2 b+d=1$
> $a=1, b=-4, c=2$, and $d=-7$
> $\left[\begin{array}{cc}1 & 2 \\ -4 & -7\end{array}\right]\left[\begin{array}{cc}-7 & -2 \\ 4 & 1\end{array}\right]=\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]$

## Exploratory Challenge ( 10 minutes)

In this Exploratory Challenge, students will look at the patterns of matrices and their inverses that they found in Exercises 1-3 of Lesson 28 and the Opening Exercise of Lesson 29. This will lead to the discovery of the general formula for the inverse of any matrix. Students should work in small groups.

- Do you think all matrices have inverses? Explain why or why not.
- Answers will vary. Allow students to state their opinion and explain. Do not add to the discussion; students will discover the correct answer in this Exploratory Challenge.

Post or project the following:

| Matrix | Inverse |
| :---: | :---: |
| $\left[\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right]$ | $\left[\begin{array}{cc}1 & 0 \\ -1 & 1\end{array}\right]$ |
| $\left[\begin{array}{ll}3 & 1 \\ 5 & 2\end{array}\right]$ | $\left[\begin{array}{cc}2 & -1 \\ -5 & 3\end{array}\right]$ |
| $\left[\begin{array}{cc}-2 & -5 \\ 1 & 2\end{array}\right]$ | $\left[\begin{array}{cc}2 & 5 \\ -1 & -2\end{array}\right]$ |
| $\left[\begin{array}{cc}-7 & -2 \\ 4 & 1\end{array}\right]$ | $\left[\begin{array}{cc}1 & 2 \\ -4 & -7\end{array}\right]$ |

- In Lesson 28, you found the inverses of the first three matrices, and in the Opening Exercise, you found the inverse of the last matrix. Do you see any patterns between the original matrix and its inverse?
- The numbers in the top left and bottom right corners seem to change places.
- The numbers in the top right and bottom left corners change signs.
- Do you think this is true for the inverse of all matrices?
- Answers will vary, but most students will think that yes, this is true.
- Let's see if we are right. Find the inverse of the matrix in Exercise 1 using the pattern we discovered, and confirm that it is indeed the inverse.


## Exercise 1 (3 minutes)

## Exercises

## Scaffolding:

- Some student pairs may need targeted one-to-one guidance on this challenge. Consider pairing groups and having a larger group that is teacher led.
- Give advanced students a single task: "Write a formula for an inverse matrix after studying the patterns, and verify your formula." Ask them to develop an answer without the questions shown.

1. Find the inverse of $\left[\begin{array}{ll}5 & 3 \\ 2 & 4\end{array}\right]$. Confirm your answer.

$$
\left[\begin{array}{cc}
4 & -3 \\
-2 & 5
\end{array}\right]\left[\begin{array}{cc}
5 & 3 \\
2 & 4
\end{array}\right]=\left[\begin{array}{cc}
14 & 0 \\
0 & 14
\end{array}\right]
$$

- Was the matrix that you found using the pattern the inverse? What was missing?
- No, where we needed 1's, we had 14's.
- Let's look at this a little further. Look at the matrices in the table. Find the determinant of the matrices.
(Assign different groups/pairs different matrices from above.)
- All of the determinants were 1.
- Do you think that makes a difference? What was the determinant of the matrix in Exercise 1?
- The determinant was 14 .
- How does that compare to the matrix that resulted from multiplying the matrices in Exercise 1?
- That was the number that was in the position that should have been a 1.
- How do you think this ties into the way we find an inverse matrix?
- We can still use our pattern, but we need to divide each term by the determinant. (Answers may vary, but let students try out their hypothesis to come up with the right answer.)
- Try it on the inverse matrix in Exercise 1. Write the inverse matrix.
- $\left[\begin{array}{cc}\frac{4}{14} & -\frac{3}{14} \\ -\frac{2}{14} & \frac{5}{14}\end{array}\right]$
- Verify that is the inverse. Were you correct?
- $\left[\begin{array}{cc}\frac{4}{14} & -\frac{3}{14} \\ -\frac{2}{14} & \frac{5}{14}\end{array}\right]\left[\begin{array}{ll}5 & 3 \\ 2 & 4\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
- Yes, we get the identify matrix again.
- Explain to your neighbor how to find the inverse of a matrix.
- Switch the numbers in the top left and bottom right. Change the signs of the numbers in the top right and bottom left. Divide all of the terms by the determinant of the original matrix.


## Exercises 2-4 (10 minutes)

In Exercises 2 and 3, students practice finding an inverse matrix and confirm their results. In Exercise 4, students find the inverse matrix of a general matrix. Choose exercises based on the needs of students; Exercises 2 and 3 are simpler while Exercise 4 is more complicated. Students should complete this exercise in small groups and then present their findings to the class.

## Find the inverse matrix and verify.

2. $\left[\begin{array}{cc}3 & -3 \\ 1 & 4\end{array}\right]$

$$
\text { Determinant }=(3)(4)-(-3)(1)=12+3=15
$$

$$
\left[\begin{array}{cc}
\frac{4}{15} & \frac{3}{15} \\
\frac{-1}{15} & \frac{3}{15}
\end{array}\right]\left[\begin{array}{cc}
3 & -3 \\
1 & 4
\end{array}\right]=\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]
$$

3. $\left[\begin{array}{ll}5 & -2 \\ 4 & -3\end{array}\right]$

$$
\text { Determinant }=(5)(-3)-(-2)(4)=-15+8=-7
$$

$$
\left[\begin{array}{rr}
\frac{3}{7} & -\frac{2}{7} \\
\frac{4}{7} & -\frac{5}{7}
\end{array}\right]\left[\begin{array}{ll}
5 & -2 \\
4 & -3
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
$$

$$
\begin{aligned}
& \text { 4. }\left[\begin{array}{ll}
a & c \\
b & d
\end{array}\right] \\
& \text { Determinant }=(a)(d)-(c)(b)=a d-c b \\
& {\left[\begin{array}{cc}
\frac{d}{a d-c b} & -\frac{c}{a d-c b} \\
-\frac{b}{a d-c b} & \frac{a}{a d-c b}
\end{array}\right]\left[\begin{array}{ll}
a & c \\
b & d
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]}
\end{aligned}
$$

## Example 1 (10 minutes)

In this example, students calculate the determinant of a matrix and find that it is 0 ; they then try to find the inverse of the matrix. They discover that there is no inverse, then explore what that means about the resulting image. Students conclude that matrices with a determinant of 0 do not have inverses. Students need graph paper.

## Example 1

Find the determinant of $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]$.
The determinant is $\mathbf{0}$.

- Now that we have calculated the determinant and found it to be 0 , let's examine the inverse of $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]$.
- Students may struggle, but they should see that you cannot divide by 0 , so there will be an issue finding the inverse.
- Let's try to solve for the inverse with a system of equations.
- $\left[\begin{array}{ll}a & c \\ b & d\end{array}\right]\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]=\left[\begin{array}{ll}a+2 c & 2 a+4 c \\ b+2 d & 2 b+4 d\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
- $\quad a+2 c=1,2 a+4 c=0, b+2 d=0,2 b+4 d=1$
- What did you discover?
- We get a system of equations with no solutions.
- What do you think this means about the inverse of this matrix?
- This matrix does not have an inverse.
- Let's explore this further using what we know about matrix transformations of the unit square.
- Perform the operation $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]$ on the unit square. What are the coordinates of the vertices of the unit square on the image?

- $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]\left[\begin{array}{l}0 \\ 0\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]$
- $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]\left[\begin{array}{l}1 \\ 0\end{array}\right]=\left[\begin{array}{l}1 \\ 2\end{array}\right]$
- $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]\left[\begin{array}{l}0 \\ 1\end{array}\right]=\left[\begin{array}{l}2 \\ 4\end{array}\right]$
- $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]\left[\begin{array}{l}1 \\ 1\end{array}\right]=\left[\begin{array}{l}3 \\ 6\end{array}\right]$
- Plot the unit square and the transformation. What do you notice?
- The image is a line.
- What is the area of the image?
- The image is a line, not a parallelogram, so the area is 0 .
- What does the determinant of the transformation represent?
- It represents the area of the image of the unit square after the transformation.
- Is the area confirmed?
- Yes, the determinant is 0 , so the area of the transformation is 0 .
- The points $(1,0)$ and $\left(0, \frac{1}{2}\right)$ are both on the unit square. Perform this transformation on each of these points.
- $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]\left[\begin{array}{l}1 \\ 0\end{array}\right]=\left[\begin{array}{l}1 \\ 2\end{array}\right]$
- $\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]\left[\begin{array}{l}0 \\ \frac{1}{2}\end{array}\right]=\left[\begin{array}{l}1 \\ 2\end{array}\right]$
- What does this mean?
- Both points map to the same location.
- When the unit square "collapses" to a straight line under a transformation, we will always have more than one point mapping to the same location. This means that we cannot "undo" this transformation because there is no clear way to reverse the transformation. Would $(1,2)$ map back to $(1,0)$ or $\left(0, \frac{1}{2}\right)$ ? We are not sure.
- When does a matrix not have an inverse?
- When the image of the unit square "collapses" to a figure of 0 area, we have distinct points mapping to the same location, so there is no inverse.
- When the determinant of the matrix is 0 .


## Closing ( 2 minutes)

Students should do a 30-second quick write, then share with the class the answer to the following:

- What is an inverse matrix?
- An inverse matrix is a matrix that when multiplied by a given matrix, the product is the identity matrix.
- An inverse matrix "undoes" a transformation.
- Explain how to find an inverse matrix.
- Multiply a general matrix $\left[\begin{array}{ll}a & c \\ b & d\end{array}\right]$ by a given matrix, and set it equal to the identify matrix. Solve the system of equations for $a, b, c$, and $d$.


## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 29: When Can We Reverse a Transformation?

## Exit Ticket

$A=\left[\begin{array}{cc}4 & -2 \\ -1 & 3\end{array}\right]$

1. Find the inverse of $A$. Show your work and confirm your answer.
2. Explain why the matrix $\left[\begin{array}{ll}6 & 3 \\ 4 & 2\end{array}\right]$ has no inverse.

## Exit Ticket Sample Solutions

$$
A=\left[\begin{array}{cc}
4 & -2 \\
-1 & 3
\end{array}\right]
$$

1. Find the inverse of $A$. Show your work and confirm your answer.

Determinant $=(4)(3)-(-2)(-1)=12-2=10$
$\left[\begin{array}{cc}\frac{4}{10} & \frac{-2}{10} \\ \frac{-1}{10} & \frac{3}{10}\end{array}\right]\left[\begin{array}{ll}3 & 2 \\ 1 & 4\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
2. Explain why the matrix $\left[\begin{array}{ll}6 & 3 \\ 4 & 2\end{array}\right]$ has no inverse.

Determinant $=(6)(2)-(3)(4)=0$
This means the area of the image is $\mathbf{0}$ because the image of the unit square maps to a straight line, which has no area. This also means that distinct points map to the same location, so the transformation cannot be reversed.

## Problem Set Sample Solutions

## Find the inverse matrix of the following.

a. $\quad\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$

Determinant $=1-0=1 \quad$ Inverse matrix: $\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
Verify: $\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
b. $\quad\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$

Determinant $=0-1=-1 \quad$ Inverse matrix: $\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$
Verify: $\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
c. $\quad\left[\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right]$

Determinant $=1-1=0 \quad$ No inverse matrix
d. $\left[\begin{array}{ll}1 & 0 \\ 1 & 0\end{array}\right]$

Determinant $=0 \quad$ No inverse matrix
e. $\left[\begin{array}{ll}0 & 1 \\ 0 & 1\end{array}\right]$

Determinant $=0 \quad$ No inverse matrix
f. $\left[\begin{array}{ll}-2 & 2 \\ -5 & 4\end{array}\right]$

Determinant $=-8+10=2 \quad$ Inverse matrix: $\left[\begin{array}{cc}2 & -1 \\ \frac{5}{2} & -1\end{array}\right]$
Verify:
$\left[\begin{array}{ll}-2 & 2 \\ -5 & 4\end{array}\right]\left[\begin{array}{ll}2 & -1 \\ \frac{5}{2} & -1\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
g. $\quad\left[\begin{array}{ll}4 & 6 \\ 5 & 8\end{array}\right]$

Determinant $=32-30=2$
Inverse matrix: $\left[\begin{array}{cc}4 & -3 \\ \frac{5}{2} & 2\end{array}\right]$
Verify:
$\left[\begin{array}{ll}4 & 6 \\ 5 & 8\end{array}\right]\left[\begin{array}{cc}4 & -3 \\ \frac{5}{2} & 2\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
h. $\left[\begin{array}{ll}6 & -9 \\ 5 & -7\end{array}\right]$

Determinant $=-42+45=3, \quad$ Inverse matrix: $\left[\begin{array}{rr}-\frac{7}{3} & 3 \\ -\frac{5}{3} & 2\end{array}\right]$
Verify:
$\left[\begin{array}{ll}6 & -9 \\ 5 & -7\end{array}\right]\left[\begin{array}{rr}-\frac{7}{3} & 3 \\ -\frac{5}{3} & 2\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
i. $\left[\begin{array}{cc}\frac{1}{2} & -\frac{2}{3} \\ -6 & 4\end{array}\right]$

$$
\text { Determinant }=2-4=-2 \quad \text { Inverse matrix: }\left[\begin{array}{cc}
-2 & -\frac{1}{3} \\
-3 & \frac{1}{4}
\end{array}\right]
$$

Verify:
$\left[\begin{array}{cc}\frac{1}{2} & -\frac{2}{3} \\ -6 & 4\end{array}\right]\left[\begin{array}{cc}-2 & -\frac{1}{3} \\ -3 & \frac{1}{4}\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
j. $\quad\left[\begin{array}{cc}0.8 & 0.4 \\ -0.75 & -0.5\end{array}\right]$

Determinant $=-0.4+0.3=-0.1 \quad$ Inverse matrix: $\left[\begin{array}{cc}5 & -4 \\ -7.5 & -8\end{array}\right]$
Verify: $\left[\begin{array}{cc}0.8 & 0.4 \\ -0.75 & -0.5\end{array}\right]\left[\begin{array}{cc}5 & -4 \\ -7.5 & -8\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$

