Lesson 28: When Can We Reverse a Transformation?

Student Outcomes

- Students determine inverse matrices using linear systems.

Lesson Notes

In the final three lessons of this module, we will discover how to reverse a transformation by discovering the inverse matrix. In Lesson 28, students are introduced to inverse matrices and find inverses of matrices with a determinant of 1 by solving a system of equations. Lesson 29 expands this idea to include inverses of matrices with a determinant other than 1 and finding a general formula for an inverse matrix. In Lesson 30, students discover matrices with determinants of zero do not have an inverse.

Classwork

The Opening Exercise can be done individually or in pairs. It allows students to practice a 2×2 matrix transformation on a unit square. Students need graph paper.

Opening Exercise (8 minutes)

Opening Exercise

Perform the operation $\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ on the unit square.

a. **State the vertices of the transformation.**

$(0,0)$, $(3,1)$, $(-2,1)$, *and* $(1,2)$

b. **Explain the transformation in words.**

$(0,0)$ *stays at the origin, the vertex* $(1,0)$ *moves to* $(3,1)$, $(0,1)$ *moves to* $(-2,1)$, *and* $(1,1)$ *moves to* $(1,2)$.

c. **Find the area of the transformed figure.**

$|(3)(1)-(-2)(1)| = 5$ square units

d. **If the original square was 2×2 instead of a unit square, how would the transformation change?**

The coordinates of the vertices of the image would all double. The vertices would be $(0,0)$, $(6,2)$, $(-4,2)$, *and* $(2,4)$.

e. **What is the area of the image? Explain how you know.**

The area of the image is 20 *square units. The area of the original square was* 4 *square units, multiply that by the determinant which is* 5*, and the area of the new figure is* $4 \times 5 = 20$ *square units.*

Discussion (10 minutes)

This discussion is a whole class discussion that wraps up the Opening Exercise and gets students to think about reversing transformations.

> *Scaffolding:*
> - Advanced learners can do the discussion in small homogenous groups and do Example 1 with no guiding questions.
> - Remind students how to multiply 2×2 matrices by displaying this graphic:
> $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} ab + bt & as + bu \\ cr + dt & cs + du \end{bmatrix}$$
> - Remind students of the trigonometry Pythagorean identities by displaying the following:
> $$\sin^2\theta + \cos^2\theta = 1$$

- What are some differences between the unit square and a 2×2 square?
  - *The coordinates of the vertices of the* 2×2 *square were double the coordinates of the vertices of the unit square.*
  - *The area of the* 2×2 *square is four times the area of the unit square.*
- Was the same thing true when the matrix transformation was applied?
  - *Yes.*
- How can the determinant of the transformation matrix be used to find the area of a transformed image if the original image was not a unit square?
  - *Find the area of the original square, then multiply that area by the value of the determinant.*
- What matrix have we studied that produces only a counter-clockwise rotation through an angle θ about the origin? Call it R_θ.
  - $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

MP.3

- What transformation would "undo" this transformation? Describe it in words or symbols.
  - *We can undo this transformation by rotating in the opposite direction, or through an angle of* $-\theta$.
  - $\sin(-\theta) = -\sin\theta$ *because it is an odd function and is symmetric about the origin.* $f(-x) = -f(x)$
  - $\cos(-\theta) = \cos\theta$ *because it is an even function and is symmetric about the* y*-axis.* $f(-x) = f(x)$
- Write the matrix that represents the rotation through $-\theta$. Call it $R_{-\theta}$.
  - $R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
- What do you think will happen if we apply R_θ and then $R_{-\theta}$?
  - *We should end up with what we started with.*
- Let's confirm this. What matrix do you expect to see when you compute the product $R_{-\theta}R_\theta$?
  - *The identity matrix* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- Perform this operation. Were you correct?
  - $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - *Yes, we got the identity matrix.*
- What about $R_\theta R_{-\theta}$?
  - $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - *Yes, we get the identify matrix again.*
- Explain to your neighbor what we have just discovered.
  - $R_{-\theta}R_\theta = R_\theta R_{-\theta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *the identify matrix.*
  - *If the transformation R_θ is performed, it can be reversed by performing the transformation $R_{-\theta}$.*

Example 1 (10 minutes)

In this example, students solve a system of equations to find the transformation that reverses the pure dilation matrix with a scale factor of k. This example concludes with students writing their own definition of an inverse matrix, then comparing it to the formal definition. Students should work in small homogenous groups or pairs. Some groups can work through without guided questions while others may need targeted teacher support.

Example 1

What transformation reverses a pure dilation from the origin with a scale factor of k?

a. **Write the pure dilation matrix and multiply it by $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.**

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} ak & ck \\ bk & dk \end{bmatrix}$$

b. **What values of a, b, c, and d would produce the identity matrix? (Hint: Write and solve a system of equations.)**

$$\begin{bmatrix} ak & ck \\ bk & dk \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$ak = 1, \quad ck = 0, \quad bk = 0, \quad dk = 1$

$a = \frac{1}{k}, \quad c = 0, \quad b = 0, \quad d = \frac{1}{k}$

c. **Write the matrix and confirm that it reverses the pure dilation with a scale factor of k.**

$$\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- What is the pure dilation matrix with a scale factor of k?
  - $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
- Multiply by a general matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. What is the resulting matrix?
  - $\begin{bmatrix} ak & ck \\ bk & dk \end{bmatrix}$
- What matrix would this have to be equal to if the transformation had been reversed? Write that matrix.
  - *The identity matrix,* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Equate the two matrices and write a system of equations that would have to be true for the matrices to be equal.
  - $\begin{bmatrix} ak & ck \\ bk & dk \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - $ak = 1, bk = 0, ck = 0,$ *and* $dk = 1$
- Solve this system for $a, b, c,$ and d in terms of k.
  - $a = \frac{1}{k}, b = 0, c = 0,$ *and* $d = \frac{1}{k}$
- Write the matrix that reverses the pure dilation transformation with a scale factor of k.
  - $\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$
- Confirm that this is the matrix that reverses the transformation. Explain.
  - $\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - *When the two matrices are multiplied, you get the identity matrix, which means that the transformation has been reversed.*
- If the transformations were done in the reverse order, would they still "undo" each other? Show your work.
  - *Yes,* $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Let's call original matrix A, the matrix that reverses the transformation B, and the identity matrix I. Write a statement that is true that relates the three matrices.
  - $AB = I, BA = I$
- We call matrix B an inverse matrix of matrix A. Write a definition of the inverse matrix.
  - *Matrix B is an inverse matrix to matrix A if $AB = I$ and $BA = I$.*

Exercises 1–3 (10 minutes)

Students find the inverse matrices of each matrix given. Exercises 1 and 2 will require students to solve a system of four equations and four variables. This is not as difficult as it may seem, since two of the equations are equal to zero. In Lesson 29, we will develop a general formula for the inverse of any matrix; in this exercise, we want students to start seeing patterns relating the inverse matrix and the original matrix. These problems were all chosen because their determinant is zero, so students can focus on the movement of terms and changing of signs. All students should do Exercises 1 and 2. Early finishers can also do Exercise 3. We will use the results of this exercise in the Opening Exercise of Lesson 29, asking if students see a pattern.

Exercises 1–3

Find the inverse matrix and verify.

1. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a+c & c \\ b+d & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 3a+5c & a+2c \\ 3b+5d & b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

3. $\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -2a+c & -5a+2c \\ -2b+d & -5b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$

Closing (2 minutes)

Students should do a 30-second quick write, then share with the class the answer to the following:

- What is an inverse matrix?
  - *An inverse matrix is a matrix that when multiplied by a given matrix, the product is the identity matrix.*
  - *An inverse matrix "undoes" a transformation.*
- Explain how to find an inverse matrix.
  - *Multiply a general matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ by a given matrix and set it equal to the identify matrix. Solve the system of equations for $a, b, c,$ and d.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 28: When Can We Reverse a Transformation?

Exit Ticket

$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

1. Is matrix A the inverse of matrix B? Show your work and explain your answer.

2. What is the determinant of matrix B? Of matrix A?

Exit Ticket Sample Solutions

$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

1. Is matrix A the inverse of matrix B? Show your work and explain your answer.

   No, the product of the two matrices is not the identity matrix.

   $\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

2. What is the determinant of matrix B? Of matrix A?

   The determinant of matrix $A = |(4)(3) - (-2)(-1)| = 10$.

   The determinant of matrix $B = |(3)(4) - (2)(1)| = 10$.

Problem Set Sample Solutions

1. In this lesson, we learned $R_\theta R_{-\theta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Chad was saying that he found an easy way to find the inverse matrix, which is: $R_{-\theta} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{R_\theta}$. His argument is that if we have $2x = 1$, then $x = \frac{1}{2}$.

   a. Is Chad correct? Explain your reason.

      Chad is not correct. Matrices cannot be divided.

   b. If Chad is not correct, what is the correct way to find the inverse matrix?

      To find the inverse of $R_{-\theta}$, calculate the determinant, switch the terms on the forward diagonal and change the signs on the back diagonal, then divide all terms by the absolute value of the determinant.

2. Find the inverse matrix and verify it.

   a. $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

      $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

      $\begin{bmatrix} 3a+2b & 3c+2d \\ 7a+5b & 7c+5d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $3a+2b=1$, $3c+2d=0$, $7a+5b=0$, $7c+5d=1$,

      solve a, b, c, d: $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

      Verify: $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 15-14 & -6+6 \\ 35-35 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix}-2 & -1\\ 3 & 1\end{bmatrix}$

$\begin{bmatrix}-3 & -1\\ 3 & 1\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}-3a-b & -3c-d\\ 3a+b & 3c+d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\quad -3a-b=1,\quad -3c-d=0,\quad 3a+b=0,\quad 3c+d=1,$

solve a,b,c,d: $\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}1 & 1\\ -3 & -2\end{bmatrix}$

Verify: $\begin{bmatrix}-2 & -1\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -3 & -2\end{bmatrix}=\begin{bmatrix}-2+3 & -2+2\\ 3-3 & 3-2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

c. $\begin{bmatrix}3 & -3\\ -2 & 2\end{bmatrix}$

The determinant is 0*; therefore, there is no inverse matrix.*

d. $\begin{bmatrix}0 & 1\\ -1 & 3\end{bmatrix}$

$\begin{bmatrix}0 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}b & d\\ -a+3b & -c+3d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\quad b=1,\quad d=0,\quad -a+3b=0,\quad -c+3d=1,$

solve a,b,c,d: $\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}3 & -1\\ 1 & 0\end{bmatrix}$

Verify: $\begin{bmatrix}0 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}3 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0+1 & 0\\ -3+3 & 1+0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

e. $\begin{bmatrix}4 & 1\\ 2 & 1\end{bmatrix}$

$\begin{bmatrix}4 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}4a+b & 4c+d\\ 2a+b & 2c+d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\quad 4a+b=1,\quad \begin{matrix}4a+b & 4c+d\\ 2a+b & 2c+d\end{matrix}=0,\quad 2a+b=0,\quad 2c+d=1,$

solve a,b,c,d: $\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -1 & 2\end{bmatrix}$

Verify: $\begin{bmatrix}4 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2-1 & -2+2\\ 1-1 & -1+2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

3. Find the starting point $\begin{bmatrix}x\\ y\end{bmatrix}$ if

a. the point $\begin{bmatrix}4\\ 2\end{bmatrix}$ is the image of a pure dilation with a factor of 2.

$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\frac{4}{2}\\ \frac{2}{2}\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}$

b. the point $\begin{bmatrix}4\\2\end{bmatrix}$ is the image of a pure dilation with a factor of $\frac{1}{2}$.

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\dfrac{4}{\frac{1}{2}}\\\dfrac{2}{\frac{1}{2}}\end{bmatrix}=\begin{bmatrix}8\\4\end{bmatrix}$$

c. the point $\begin{bmatrix}-10\\35\end{bmatrix}$ is the image of a pure dilation with a factor of 5.

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\dfrac{-10}{5}\\\dfrac{35}{5}\end{bmatrix}=\begin{bmatrix}-2\\7\end{bmatrix}$$

d. the point $\begin{bmatrix}\frac{4}{9}\\\frac{16}{21}\end{bmatrix}$ is the image of a pure dilation with a factor of $\frac{2}{3}$.

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\dfrac{\frac{4}{9}}{\frac{2}{3}}\\\dfrac{\frac{16}{21}}{\frac{2}{3}}\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\\\frac{8}{7}\end{bmatrix}$$

4. Find the starting point if

a. $3+2i$ is the image of a reflection about the real axis.

$\bar{z}=3-2i$

b. $3+2i$ is the image of a reflection about the imaginary axis.

$-\bar{z}=-(\overline{3+2i})=-(3-2i)=-3+2i$

c. $3+2i$ is the image of a reflection about the real axis and then the imaginary axis.

$-\bar{\bar{z}}=-(\overline{\overline{3+2i}})=-(\overline{3-2i})=-(3+2i)=-3-2i$

d. $-3-2i$ is the image of a π radians counterclockwise rotation.

$\begin{bmatrix}x\\y\end{bmatrix}=\frac{3+2i}{i\cdot i}=3+2i.$

5. Let's call the pure counterclockwise rotation of the matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ as R_θ, and the "undo" of the pure rotation is $\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}$ as $R_{-\theta}$.

a. Simplify $\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}$.

$$\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

b. What would you get if you multiply R_{θ} to $R_{-\theta}$?

$$R_{\theta} \times R_{-\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta \cdot \sin\theta - \sin\theta \cdot \cos\theta \\ \sin\theta \cdot \cos\theta - \cos\theta \cdot \sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. Write the matrix if you want to rotate $\frac{\pi}{2}$ radians counterclockwise.

$$\begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

d. Write the matrix if you want to rotate $\frac{\pi}{2}$ radians clockwise.

$$\begin{bmatrix} \cos\left(-\frac{\pi}{2}\right) & -\sin\left(-\frac{\pi}{2}\right) \\ \sin\left(-\frac{\pi}{2}\right) & \cos\left(-\frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

e. Write the matrix if you want to rotate $\frac{\pi}{6}$ radians counterclockwise.

$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

f. Write the matrix if you want to rotate $\frac{\pi}{4}$ radians counterclockwise.

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

g. If the point $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is the image of $\frac{\pi}{4}$ radians counterclockwise rotation, find the starting point $\begin{bmatrix} x \\ y \end{bmatrix}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

h. If the point $\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$ is the image of $\frac{\pi}{6}$ radians counterclockwise rotation, find the starting point $\begin{bmatrix} x \\ y \end{bmatrix}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$