Lesson 25: Matrix Multiplication and Addition

Student Outcomes

- Students work with 2×2 matrices as transformations of the plane.
- Students combine matrices using matrix multiplication and addition.
- Students understand the role of the zero matrix in matrix addition.

Lesson Notes

In Lesson 24, students continued to explore matrices and their connection to transformations. In this lesson, students work with the zero matrix and discover that it is the additive identity matrix with a role similar to 0 in the real number system. We will focus on the result of performing one transformation followed by another and discover

If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and M is given by $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then $ML\begin{pmatrix} x \\ y \end{pmatrix}$ is the same as applying the matrix $\begin{pmatrix} pa+rb & pc+rd \\ qa+sb & qc+sd \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$.

This motivates our definition of matrix multiplication. **N-VM.C.8** is introduced in this lesson but treated more fully in Module 2.

Classwork

Opening Exercise (8 minutes)

Allow students time to complete the Opening Exercise independently. Encourage students to think/write independently, chat with a partner, then share as a class.

Opening Exercise

Consider the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ that undergoes a series of two transformations: a dilation of scale factor 4 followed by a reflection about the horizontal axis.

a. **What matrix produces the dilation of scale factor 4? What is the coordinate of the point after the dilation?**

The dilation matrix is $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \end{pmatrix}$

The coordinate is now $\begin{pmatrix} 16 \\ 4 \end{pmatrix}$.

b. What matrix produces the reflection about the horizontal axis? What is the coordinate of the point after the reflection?

The reflection matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \end{pmatrix}$

The coordinate is now $\begin{pmatrix} 16 \\ -4 \end{pmatrix}$.

c. Could we have produced both the dilation and the reflection using a single matrix? If so, what matrix would both dilate by a scale factor of 4 and produce a reflection about the horizontal axis? Show that the matrix you came up with combines these two matrices.

MP.3

Yes, by using the matrix $\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$.

$\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \end{pmatrix}$

The dilation matrix was $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. *The rotation matrix was* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *The product of these matrices gives the matrix that produces a dilation and then a rotation.* $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$

- How did you come up with the dilation matrix?
  - *I know that a dilation matrix is in the form* $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ *where* k *the scale factor is.*
- How did you come up with the reflection matrix?
  - *I know that the matrix* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *reflects coordinates about the horizontal axis.*
- How did you come up with a matrix that was both a dilation and a reflection?
  - *I knew that I wanted to multiply both the* x *and* y *by a factor of* 4, *and that I also wanted to multiply the* y *by* -1; *from that, I combined the two matrices to get* $\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$.
- In what sense did we combine the two matrices?
  - *We multiplied* 4×-1.
- We know that transformations are produced through matrix multiplication. What if we have more than one transformation? Could we multiply the two transformation matrices together first instead of completing the transformations in two separate steps?

MP.7

Write this on the board:

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

- This should be equivalent to applying the dilation and then the reflection. So what should the product equal?
  - $\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$

MP.7

- Ask students to think about how we multiply a 2×2 matrix and a 2×1 matrix. Based on the fact that this product should be $\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$ and what you know about multiplying a matrix by a vector, develop an explanation for how to multiply these two matrices together.
  - $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 \times 1 + 0 \times 0 & 4 \times 0 + 0 \times -1 \\ 0 \times 1 + 0 \times 0 & 4 \times 0 + 4 \times -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$
- Does this technique align with our earlier definition of multiplying a 2×2 times a 2×1?
  - *Yes. We follow the same process to get the numbers in column 2 that we did to get the numbers in column 1—multiplying each row by the numbers in the column and then adding.*
- Can matrices of any size be multiplied together? For example, can you multiply $\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 4 & 5 \end{pmatrix}$? Why or why not?
  - *No, the number of rows and columns do not match up.*
- What must be true about the dimensions of matrices in order for them to be able to be multiplied?
  - *The number of columns of the first matrix must equal the number of rows of the second matrix.*

Example 1 (7 minutes): Is Matrix Multiplication Commutative?

Conduct each part of the example as a think-pair-share. Allow students time to think of the answer independently, pair with a partner to discuss, and then share as a class.

Example 1: Is Matrix Multiplication Commutative?

a. **Take the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ through the following transformations: a rotation of $\frac{\pi}{2}$ and a reflection across the y-axis.**

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b. **Will the resulting point be the same if the order of the transformations is reversed?**

No. If the reflection is applied first followed by the rotation, the resulting point is $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

c. **Are transformations commutative?**

Not necessarily. The order in which the transformations are applied can affect the results in some cases.

d. **Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Find AB and then BA.**

$AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Scaffolding:

- For students who are struggling, start with a review of the commutative property of multiplication.

  If a and b are real numbers, then $a \times b = b \times a$.
- Have them highlight or circle each row and column as they multiply.
- What does it mean for multiplication of real numbers to be commutative? Explain with an example.
- Can you think of an operation that is not commutative? Explain with an example.

e. **Is matrix multiplication commutative?**

No. $AB \neq BA$

f. **If we apply matrix AB to the point $\begin{pmatrix}2\\1\end{pmatrix}$, in what order are the transformations applied.**

$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix}$

The reflection is applied first followed by the dilation.

g. **If we apply matrix BA to the point $\begin{pmatrix}2\\1\end{pmatrix}$, in what order are the transformations applied.**

$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

The dilation is applied first followed by the reflection.

h. **Can we apply $\begin{pmatrix}2\\1\end{pmatrix}$ to matrix BA?**

$\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

No, $\begin{pmatrix}2\\1\end{pmatrix}$ *has two rows and one column. It would have to have two columns to be able to multiply by the* 2×2 *matrix* BA.

Exercises 1–3 (10 minutes)

Allow students time to work on the exercises either independently or in groups. Circulate around the room providing assistance as needed, particularly watching for students who are struggling with matrix multiplication.

Exercises 1– 3

1. **Let $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $M = \begin{pmatrix}4 & -6\\3 & -2\end{pmatrix}$.**

a. **Find IM.**

$\begin{pmatrix}4 & -6\\3 & -2\end{pmatrix}$

b. **Find MI.**

$\begin{pmatrix}4 & -6\\3 & -2\end{pmatrix}$

c. **Do these results make sense based on what you know about the matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$?**

Yes. The matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ *is the identity matrix. It corresponds to a transformation that leaves points alone. Therefore, geometrically we must have* $IM = MI = M$.

2. **Calculate AB, then BA. Is matrix multiplication commutative?**

   a. $A = \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

   $AB = \begin{pmatrix} 2\times 1 + 0\times 0 & 2\times 5 + 0\times 1 \\ -2\times 1 + 3\times 0 & -2\times 5 + 3\times 1 \end{pmatrix} = \begin{pmatrix} 2 & 10 \\ -2 & -7 \end{pmatrix}$

   $BA = \begin{pmatrix} 1\times 2 + 5\times -2 & 1\times 0 + 5\times 3 \\ 0\times 2 + 1\times -2 & 0\times 0 + 1\times 3 \end{pmatrix} = \begin{pmatrix} -8 & 15 \\ -2 & 3 \end{pmatrix}$

   $AB \neq BA$; matrix multiplication is not commutative.

   b. $A = \begin{pmatrix} -10 & 1 \\ 3 & 7 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$

   $AB = \begin{pmatrix} -10\times -3 + 1\times 4 & -10\times 2 + 1\times -1 \\ 3\times -3 + 7\times 4 & 3\times 2 + 7\times -1 \end{pmatrix} = \begin{pmatrix} 34 & -21 \\ 19 & -1 \end{pmatrix}$

   $BA = \begin{pmatrix} -3\times -10 + 2\times 3 & -3\times 1 + 2\times 7 \\ 4\times -10 + -1\times 3 & 4\times 1 + -1\times 7 \end{pmatrix} = \begin{pmatrix} 36 & 11 \\ -43 & -3 \end{pmatrix}$

   $AB \neq BA$; matrix multiplication is not commutative.

3. **Write a matrix that would perform the following transformations in this order: a rotation of $180°$, a dilation by a scale factor of 4, and a reflection across the horizontal axis. Use the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to illustrate that your matrix is correct.**

   $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$

   $\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$

Scaffolding:

For students who like a challenge, pose one of the following:

- Multiply two 3×3 matrices.
- Could we use our definition for multiplying matrices to multiply a 2×1 times a 2×2?

Example 2 (5 minutes): More Operations on Matrices

Discuss this question as a class before completing the example.

- We know that there is matrix multiplication. Does it seem logical that there would be matrix addition?
- If $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix}$, explain how to add matrices.
  - *Add the numbers that are in the same position in the corresponding matrices.*
- Each number within a matrix is called an element. In order to add matrices, the elements in the same row and same column are added. We refer to elements in the same row and same column as corresponding elements. So to recap, in order to add matrices, we add corresponding elements.
- How would you subtract matrices?
  - *Subtract corresponding elements.*
- Find $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$.
  - $\begin{pmatrix} 1 & -5 \\ -2 & 2 \end{pmatrix}$
- Can you add the following matrices: $\begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}$? Explain your answer.
  - *No, the matrix is not the same size, so there are not corresponding elements.*

- When can matrices be added and subtracted?
  - *When they are the same size.*
- In Lesson 24, we studied the multiplicative identity matrix. In what ways is the identity matrix similar to the number 1 within the set of real numbers? Why?
  - *The identity matrix is similar to the number 1 in the real number system because any number times 1 is itself, and any matrix times the identity matrix is itself.*
- What is the additive identity in the real number system? Why?
  - 0*, because any number plus 0 is itself.*
- What matrix would you hypothesize has the same effect in matrix addition? Write the matrix.
  - $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *is the additive identity matrix because any matrix plus this matrix is itself.*
- We call this the zero matrix. What would be the impact of multiplying a matrix by the zero matrix? How does the impact of multiplying a matrix by the zero matrix compare to multiplying by zero in the real number system?
  - *Any number multiplied by 0 is 0, and any matrix multiplied by the zero matrix has all terms of 0.*
- Note that it is difficult to know what matrix addition means geometrically in terms of transformations, but we will see a natural interpretation of matrix addition in Module 3.

Example 2: More Operations on Matrices

Find the sum. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2+1 & 0+5 \\ -2+0 & 3+1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix}$

Find the difference. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-1 & 0-5 \\ -2-0 & 3-1 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ -2 & 2 \end{pmatrix}$

Find the sum. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2+0 & 0+0 \\ -2+0 & 3+0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix}$

Exercises 4–5 (5 minutes)

Allow students time to work on the exercises either independently or in groups. Circulate around the room providing assistance as needed.

Exercises 4–5

4. **Express each of the following as a single matrix.**

   a. $\begin{pmatrix} 6 & -3 \\ 10 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 8 \\ 3 & -12 \end{pmatrix}$

   $\begin{pmatrix} 4 & 5 \\ 13 & -13 \end{pmatrix}$

b. $\begin{pmatrix} -2 & 7 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$\begin{pmatrix} 23 \\ 0 \end{pmatrix}$

c. $\begin{pmatrix} 8 & 5 \\ 0 & 15 \end{pmatrix} - \begin{pmatrix} 4 & -6 \\ -3 & 18 \end{pmatrix}$

$\begin{pmatrix} 4 & 11 \\ 3 & -3 \end{pmatrix}$

5. **In arithmetic, the additive identity says that for some number a, $a + 0 = 0 + a = 0$. What would be an additive identity in matrix arithmetic?**

We would use the zero matrix. $\begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Closing (5 minutes)

Ask students to summarize what they have learned about matrix multiplication and addition either in writing or orally.

- When we multiply two matrices, what is the geometric interpretation?
  - *It is a series of transformations.*
- Can all matrices be multiplied? Why or why not?
  - *Matrices can be multiplied if the number of columns of the first matrix is equal to the number of rows of the second matrix.*

MP.6

- Can two matrices be combined through addition? If so, explain how.
  - *Yes, if the matrices are the same size, they can be added by adding corresponding elements.*

Lesson Summary

- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and M is given by $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then $ML\begin{pmatrix} x \\ y \end{pmatrix}$ is the same as applying the matrix $\begin{pmatrix} pa + rb & pc + rd \\ qa + sb & qc + sd \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$.
- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and I is given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then I acts as a multiplicative identity and $IL = LI = L$.
- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and O is given by $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then O acts as an additive identity and $O + L = L + O = L$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 25: Matrix Multiplication and Addition

Exit Ticket

1. Carmine has never seen matrices before but must quickly understand how to add, subtract, and multiply matrices. Explain the following problems to Carmine.

   a. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$

   b. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -5 & 8 \end{pmatrix}$

   c. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 12 & -7 \\ 20 & -16 \end{pmatrix}$

2. Explain to Carmine the significance of the zero matrix and the multiplicative identity matrix.

Exit Ticket Sample Solutions

1. Carmine has never seen matrices before but must quickly understand how to add, subtract, and multiply matrices. Explain the following problems to Carmine.

   a. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$

   To add matrices, add the corresponding elements. So, add the 2 and the 0 because they are both in the first row, first column.

   b. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -5 & 8 \end{pmatrix}$

   To subtract matrices, subtract the corresponding elements. So, subtract the 0 from the 2 because they are both in the first row, first column.

   c. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 12 & -7 \\ 20 & -16 \end{pmatrix}$

   To multiply matrices, multiply the elements in the first row by the elements in the first column, and then add the products together.

   $$\begin{pmatrix} 2 \times 0 + 3 \times 4 & 2 \times 1 + 3 \times -3 \\ -1 \times 0 + 5 \times 4 & -1 \times 1 + 5 \times -3 \end{pmatrix} = \begin{pmatrix} 12 & -7 \\ 20 & -16 \end{pmatrix}$$

2. Explain to Carmine the significance of the zero matrix and the multiplicative identity matrix.

   The zero matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and is similar to 0 in the real number system. Any matrix added to the zero matrix is itself, and any matrix multiplied by the zero matrix has all terms of 0.

   The multiplicative identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and is similar to 1 in the real number system. Any matrix times the multiplicative identity matrix is itself.

Problem Set Sample Solutions

1. What type of transformation is shown in the following examples? What is the resulting matrix?

   a. $\begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   It is a pure rotation. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is rotated π radians, and the image is $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

   b. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   It is a pure rotation. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is rotated $\frac{\pi}{2}$ radians, and the image is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

   c. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   It is dilation with a factor of 3. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is dilated by a factor of 3, and the image is $\begin{pmatrix} 9 \\ 6 \end{pmatrix}$.

d. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

It is a reflection about the y-axis. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is reflected about the y-axis, and the image is $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

e. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

It is a reflection about x-axis. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is reflected about the x-axis, and the image is $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

f. $\begin{pmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

It is a pure rotation. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is rotated 2π radians, and the image is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

g. $\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

It is a pure rotation. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is rotated $\frac{\pi}{4}$ radians, and the image is $\begin{pmatrix} \sqrt{2} \\ 5\sqrt{2} \end{pmatrix}$.

h. $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

It is a pure rotation. The point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is rotated $\frac{\pi}{6}$ radians, and the image is $\begin{pmatrix} \frac{3\sqrt{2}}{2} - 1 \\ \frac{3}{2} - \sqrt{3} \end{pmatrix}$.

2. Calculate each of the following products.

a. $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 13 \\ 22 \end{pmatrix}$

b. $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 11 & 6 \\ 17 & 8 \end{pmatrix}$

c. $\begin{pmatrix} -1 & -3 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 6 \\ 10 \end{pmatrix}$

d. $\begin{pmatrix} -3 & -1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$

$\begin{pmatrix} 6 & 6 \\ 8 & 10 \end{pmatrix}$

e. $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$

f. $\begin{pmatrix}3 & -1\\4 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\-2 & -3\end{pmatrix}$

$\begin{pmatrix}5 & 3\\8 & 6\end{pmatrix}$

3. **Calculate each sum or difference.**

a. $\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}+\begin{pmatrix}4 & 2\\3 & 1\end{pmatrix}$

$\begin{pmatrix}5 & 5\\5 & 5\end{pmatrix}$

b. $\begin{pmatrix}-4 & -5\\-6 & -7\end{pmatrix}+\begin{pmatrix}-2 & 3\\-1 & 4\end{pmatrix}$

$\begin{pmatrix}-6 & -2\\-7 & -3\end{pmatrix}$

c. $\begin{pmatrix}-5\\4\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}$

$\begin{pmatrix}-2\\6\end{pmatrix}$

d. $\begin{pmatrix}3\\-5\end{pmatrix}-\begin{pmatrix}7\\9\end{pmatrix}$

$\begin{pmatrix}-4\\-14\end{pmatrix}$

e. $\begin{pmatrix}-4 & -5\\-6 & -7\end{pmatrix}-\begin{pmatrix}-2 & 3\\-1 & 4\end{pmatrix}$

$\begin{pmatrix}-2 & -8\\-5 & -11\end{pmatrix}$

4. **In video game programming, Fahad translates a car, whose coordinate is $\begin{pmatrix}1\\1\end{pmatrix}$, 2 units up and 4 units to the right, rotates it $\frac{\pi}{2}$ radians counterclockwise, reflects it about the x-axis, reflects it about the y-axis rotates it $\frac{\pi}{2}$ radians counterclockwise, and finally translates it 4 units down and 2 units to the left. What point represents the final location of the car?**

$\begin{pmatrix}1\\1\end{pmatrix}\rightarrow\begin{pmatrix}1+4\\1+2\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}-3\\5\end{pmatrix}$

$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-3\\5\end{pmatrix}=\begin{pmatrix}-3\\-5\end{pmatrix}$

$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}-3\\5\,-\end{pmatrix}=\begin{pmatrix}3\\-5\end{pmatrix}$

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}3\\5\,-\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$

$\begin{pmatrix}5\\3\end{pmatrix}\rightarrow\begin{pmatrix}5-4\\3-2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}.$

The final location of the car is its initial starting point.