Lesson 21: The Hunt for Better Notation

Classwork

Opening Exercise

Suppose that $L\_{1}\left(x,y\right)=(2x-3y, 3x+2y)$ and $L\_{2}\left(x,y\right)=(3x+4y, -4y+3x)$.
Find the result of performing $L\_{1}$ and then $L\_{2}$ on a point $(p,q)$. That is, find $L\_{2}\left(L\_{1}\left(p,q\right)\right)$.

Exercises 1–2

1. Calculate each of the following products.
	1. $\left(\begin{matrix}3&-2\\-1&4\end{matrix}\right)\left(\begin{matrix}1\\5\end{matrix}\right)$
	2. $\left(\begin{matrix}3&3\\3&3\end{matrix}\right)\left(\begin{matrix}4\\-4\end{matrix}\right)$
	3. $\left(\begin{matrix}2&-4\\5&-1\end{matrix}\right)\left(\begin{matrix}3\\-2\end{matrix}\right)$
2. Find a value of $k$ so that $\left(\begin{matrix}1&2\\k&1\end{matrix}\right)\left(\begin{matrix}3\\-1\end{matrix}\right)=\left(\begin{matrix}1\\11\end{matrix}\right)$.

Exercises 3–9

1. Find a matrix $\left(\begin{matrix}a&b\\c&d\end{matrix}\right)$ so that we can represent the transformation $L(x,y)=(2x-3y, 3x+2y)$ by
$L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$.
2. If a transformation $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$ has the geometric effect of rotation and dilation, do you know about the values $a, b, c$, and $d$?
3. Describe the form of a matrix $\left(\begin{matrix}a&b\\c&d\end{matrix}\right)$ so that the transformation $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$ has the geometric effect of only dilation by a scale factor $r$.
4. Describe the form of a matrix $\left(\begin{matrix}a&b\\c&d\end{matrix}\right)$ so that the transformation $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$ has the geometric effect of only rotation by $θ$. Describe the matrix in terms of $θ$.
5. Describe the form of a matrix $\left(\begin{matrix}a&b\\c&d\end{matrix}\right)$ so that the transformation $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$ has the geometric effect of rotation by $θ$ and dilation with scale factor $r$. Describe the matrix in terms of $θ$ and $r$.
6. Suppose that we have a transformation $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}1&2\\0&1\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$.
	1. Does this transformation have the geometric effect of rotation and dilation?
	2. Transform each of the points $A=\left(\begin{matrix}0\\0\end{matrix}\right)$, $B=\left(\begin{matrix}1\\0\end{matrix}\right), C=\left(\begin{matrix}1\\1\end{matrix}\right)$, and $D=\left(\begin{matrix}0\\1\end{matrix}\right)$ and plot the images in the plane shown.



1. Describe the geometric effect of the transformation $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}1&0\\0&1\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$.

Lesson Summary

For real numbers $a, b, c$, and $d$, the transformation $L\left(x,y\right)=(ax+by, cx+dy)$ can be represented using matrix multiplication by $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$, where $\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}ax+by\\cx+dy\end{matrix}\right)$ and the $\left(\begin{matrix}x\\y\end{matrix}\right)$ represents the point $(x,y)$ in the plane.

* The transformation is a counterclockwise rotation by $θ$ if and only if the matrix representation is
$L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}cos(θ)&-sin(θ)\\sin(θ)&cos(θ)\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$.
* The transformation is a dilation with scale factor $k$ if and only if the matrix representation is
$L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}k&0\\0&k\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$.
* The transformation is a counterclockwise rotation by $arg\left(a+bi\right)$ and dilation with scale factor $|a+bi|$ if and only if the matrix representation is $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}a&-b\\b&a\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$. If we let $r=\left|a+bi\right|$ and
$θ=arg(a+bi)$, then the matrix representation is $L\left(\begin{matrix}x\\y\end{matrix}\right)=\left(\begin{matrix}r cos(θ)&-r sin(θ)\\r sin(θ)&r cos(θ)\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$.

Problem Set

1. Perform the indicated multiplication.
	1. $\left(\begin{matrix}1&2\\4&8\end{matrix}\right)\left(\begin{array}{c}3\\-2\end{array}\right)$
	2. $\left(\begin{matrix}3&5\\-2&-6\end{matrix}\right)\left(\begin{array}{c}2\\4\end{array}\right)$
	3. $\left(\begin{matrix}1&1\\1&-1\end{matrix}\right)\left(\begin{array}{c}6\\8\end{array}\right)$
	4. $\left(\begin{matrix}5&7\\4&9\end{matrix}\right)\left(\begin{array}{c}10\\100\end{array}\right)$
	5. $\left(\begin{matrix}4&2\\3&7\end{matrix}\right)\left(\begin{array}{c}-3\\1\end{array}\right)$
	6. $\left(\begin{matrix}6&4\\9&6\end{matrix}\right)\left(\begin{array}{c}2\\-3\end{array}\right)$
	7. $\left(\begin{matrix}cos(θ)&-sin(θ)\\sin(θ)&cos\left(θ\right)\end{matrix}\right)\left(\begin{array}{c}x\\y\end{array}\right)$
	8. $\left(\begin{matrix}π&1\\1&-π\end{matrix}\right)\left(\begin{array}{c}10\\7\end{array}\right)$
2. Find a value of $k$ so that $\left(\begin{matrix}k&3\\4&k\end{matrix}\right)\left(\begin{array}{c}4\\5\end{array}\right)=\left(\begin{array}{c}7\\6\end{array}\right)$.
3. Find values of $k$ and $m$ so that $\left(\begin{matrix}k&3\\-2&m\end{matrix}\right)\left(\begin{array}{c}5\\4\end{array}\right)=\left(\begin{array}{c}7\\-10\end{array}\right)$.
4. Find values of $k$ and $m$ so that $\left(\begin{matrix}1&2\\-2&5\end{matrix}\right)\left(\begin{array}{c}k\\m\end{array}\right)=\left(\begin{array}{c}0\\-9\end{array}\right)$.
5. Write the following transformations using matrix multiplication.
	1. $L\left(x,y\right)=(3x-2y, 4x-5y)$
	2. $L\left(x,y\right)=\left(6x+10y,-2x+y\right)$
	3. $L\left(x,y\right)=\left(25x+10y,8x-64y\right)$
	4. $L\left(x,y\right)=\left(πx-y,-2x+3y\right)$
	5. $L\left(x,y\right)=(10x,100x)$
	6. $L\left(x,y\right)=(2y,7x)$
6. Identify whether or not the following transformations have the geometric effect of rotation only, dilation only, rotation and dilation only, or none of these.
	1. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}3&-2\\4&-5\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	2. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}42&0\\0&42\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	3. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}-4&-2\\2&-4\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	4. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}5&-1\\-1&5\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	5. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}-7&1\\1&7\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	6. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}0&-2\\2&0\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
7. Create a matrix representation of a linear transformation that has the specified geometric effect.
	1. Dilation by a factor of $4$ and no rotation.
	2. Rotation by $180°$ and no dilation.
	3. Rotation by $-\frac{π}{2}rad$ and dilation by a scale factor of $3$.
	4. Rotation by $30°$ and dilation by a scale factor of $4$.
8. Identify the geometric effect of the following transformations. Justify your answer.
	1. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}-\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	2. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}0&-5\\5&0\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	3. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}-10&0\\0&-10\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$
	4. $L\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{matrix}6&6\sqrt{3}\\-6\sqrt{3}&6\end{matrix}\right)\left(\begin{matrix}x\\y\end{matrix}\right)$