Lesson 20: Exploiting the Connection to Cartesian Coordinates

Classwork

Opening Exercise

a. Find a complex number w so that the transformation $L_1(z) = wz$ produces a clockwise rotation by $1°$ about the origin with no dilation.

b. Find a complex number w so that the transformation $L_2(z) = wz$ produces a dilation with scale factor 0.1 with no rotation.

Exercises 1–4

1.

a. Find values of a and b so that $L_1(x, y) = (ax - by, bx + ay)$ has the effect of dilation with scale factor 2 and no rotation.

b. Evaluate $L_1\big(L_1(x,y)\big)$, and identify the resulting transformation.

2.

a. Find values of a and b so that $L_2(x,y) = (ax - by, bx + ay)$ has the effect of rotation about the origin by $180°$ counterclockwise and no dilation.

b. Evaluate $L_2\big(L_2(x,y)\big)$, and identify the resulting transformation.

3.

a. Find values of a and b so that $L_3(x,y) = (ax - by, bx + ay)$ has the effect of rotation about the origin by $90°$ counterclockwise and no dilation.

b. Evaluate $L_3\big(L_3(x,y)\big)$, and identify the resulting transformation.

4.

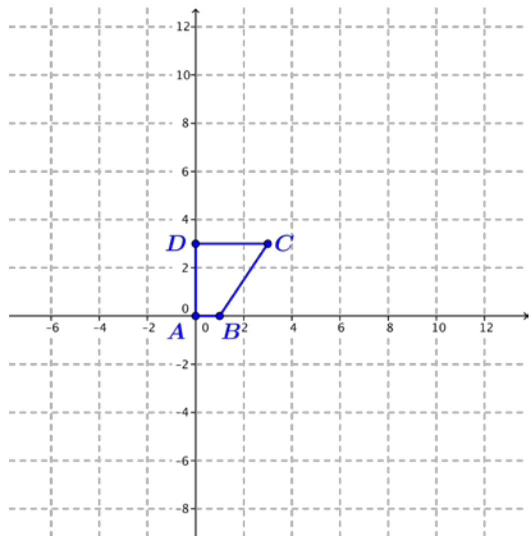
a. Find values of a and b so that $L_3(x,y) = (ax - by, bx + ay)$ has the effect of rotation about the origin by $45°$ counterclockwise and no dilation.

b. Evaluate $L_4\big(L_4(x,y)\big)$, and identify the resulting transformation.

Exercises 5–6

5. The figure below shows a quadrilateral with vertices $A(0,0)$, $B(1,0)$, $C(3,3)$, and $D(0,3)$.

   a. Transform each vertex under $L_5 = (3x + y, 3y - x)$, and plot the transformed vertices on the figure.

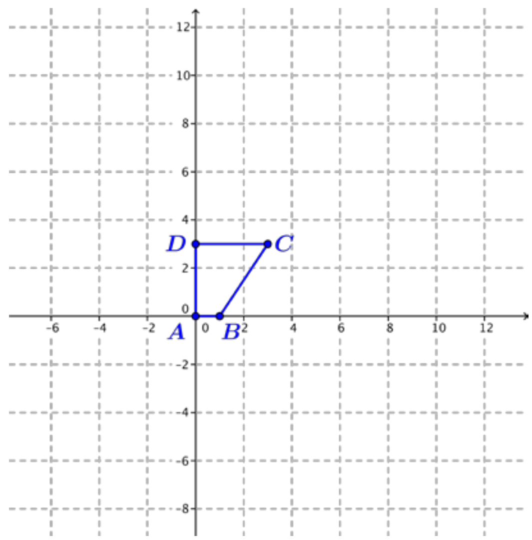


   b. Does L_5 represent a rotation and dilation? If so, estimate the amount of rotation and the scale factor from your figure.

   c. If L_5 represents a rotation and dilation, calculate the amount of rotation and the scale factor from the formula for L_5. Do your numbers agree with your estimate in part (b)? If not, explain why there are no values of a and b so that $L_5(x, y) = (ax - by, bx + ay)$.

6. The figure below shows a figure with vertices $A(0,0)$, $B(1,0)$, $C(3,3)$, and $D(0,3)$.

   a. Transform each vertex under $L_6 = (2x + 2y, 2x - 2y)$, and plot the transformed vertices on the figure.



   b. Does L_6 represent a rotation and dilation? If so, estimate the amount of rotation and the scale factor from your figure.

   c. If L_5 represents a rotation and dilation, calculate the amount of rotation and the scale factor from the formula for L_6. Do your numbers agree with your estimate in part (b)? If not, explain why there are no values of a and b so that $L_6(x, y) = (ax - by, bx + ay)$.

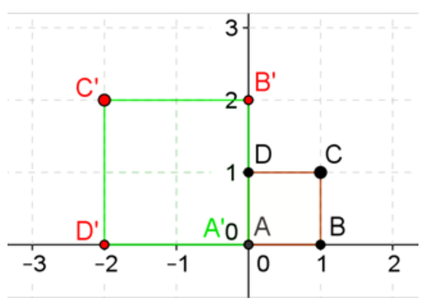
Lesson Summary

For real numbers a and b, the transformation $L(x, y) = (ax - by, bx + ay)$ corresponds to a counterclockwise rotation by $\arg(a + bi)$ about the origin and dilation with scale factor $\sqrt{a^2 + b^2}$.

Problem Set

1. Find real numbers a and b so that the transformation $L(x, y) = (ax - by, bx + ay)$ produces the specified rotation and dilation.
   a. Rotation by $270°$ counterclockwise and dilation by scale factor $\frac{1}{2}$.
   b. Rotation by $135°$ counterclockwise and dilation by scale factor $\sqrt{2}$.
   c. Rotation by $45°$ clockwise and dilation by scale factor 10.
   d. Rotation by $540°$ counterclockwise and dilation by scale factor 4.

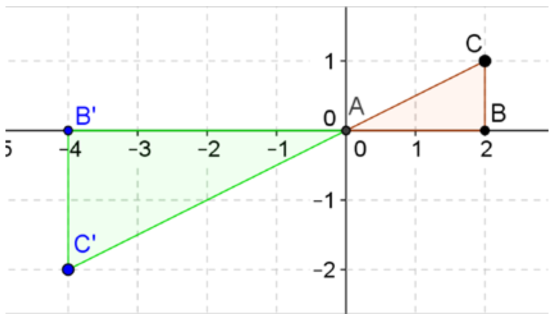
2. Determine if the following transformations represent a rotation and dilation. If so, identify the scale factor and the amount of rotation.



   a. $L(x, y) = (3x + 4y, 4x + 3y)$
   b. $L(x, y) = (-5x + 12y, -12x - 5y)$
   c. $L(x, y) = (3x + 3y, -3y + 3x)$

3. Grace and Lily have a different point of view about the transformation on cube $ABCD$ that is shown above. Grace states that it is a reflection about the imaginary axis and a dilation of factor of 2. However, Lily argues it should be a $90°$ counterclockwise rotation about the origin with a dilation of a factor of 2.
   a. Who is correct? Justify your answer.
   b. Represent the above transformation in the form $L(x, y) = (ax - by, bx + ay)$.

4. Grace and Lily still have a different point of view on this transformation on triangle ABC shown above. Grace states that it is reflected about the real axis first, then reflected about the imaginary axis, and then is dilated with a factor of 2. However, Lily asserts that it is a $180°$ counterclockwise rotation about the origin with a dilation of a factor of 2.



   a. Who is correct? Justify your answer.
   b. Represent the above transformation in the form $L(x,y) = (ax - by, bx + ay)$.

5. Given $z = \sqrt{3} + i$.
   a. Find the complex number w that will cause a rotation with the same number of degrees as z without a dilation.
   b. Can you come up with a general formula $L(x,y) = (ax - by, bx + ay)$ for any complex number $z = x + yi$ to represent this condition?