Lesson 18: Exploiting the Connection to Trigonometry

Classwork

Opening Exercise

a. Identify the modulus and argument of each complex number, and then rewrite it in rectangular form.

i. $2\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$

ii. $5\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$

iii. $3\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)\right)$

iv. $3\left(\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right)$

v. $1(\cos(\pi)+i\sin(\pi))$

b. What is the argument and modulus of each complex number? Explain how you know.

i. $2-2i$

ii. $3\sqrt{3} + 3i$

iii. $-1 - \sqrt{3}i$

iv. $-5i$

v. 1

Exploratory Challenge /Exercises 1–12

1. Rewrite each expression as a complex number in rectangular form.

   a. $(1 + i)^2$

   b. $(1 + i)^3$

   c. $(1 + i)^4$

2. Complete the table below showing the rectangular form of each number and its modulus and argument.

Power of $(\mathbf{1}+\boldsymbol{i})$	Rectangular Form	Modulus	Argument
$(1+i)^0$			
$(1+i)^1$			
$(1+i)^2$			
$(1+i)^3$			
$(1+i)^4$			

3. What patterns do you notice each time you multiply by another factor of $(1+i)$?

4. Graph each power of $1+i$ shown in the table on the same coordinate grid. Describe the location of these numbers in relation to one another using transformations.

5. Predict what the modulus and argument of $(1+i)^5$ would be without actually performing the multiplication. Explain how you made your prediction.

6. Graph $(1+i)^5$ in the complex plane using the transformations you described in Exercise 5.

7. Write each number in polar form using the modulus and argument you calculated in Exercise 4.

   $(1+i)^0$

   $(1+i)^1$

   $(1+i)^2$

   $(1+i)^3$

   $(1+i)^4$

8. Use the patterns you have observed to write $(1+i)^5$ in polar form, and then convert it to rectangular form.

9. What is the polar form of $(1+i)^{20}$? What is the modulus of $(1+i)^{20}$? What is its argument? Explain why $(1+i)^{20}$ is a real number.

10. If z has modulus r and argument θ, what is the modulus and argument of z^2? Write the number z^2 in polar form.

11. If z has modulus r and argument θ, what is the modulus and argument of z^n where n is a nonnegative integer? Write the number z^n in polar form. Explain how you got your answer.

12. Recall that $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$. Explain why it would make sense that formula holds for all integer values of n.

Exercises 13–14

13. Compute $\left(\frac{1-i}{\sqrt{2}}\right)^7$ and write it as a complex number in the form $a + bi$ where a and b are real numbers.

14. Write $\left(1 + \sqrt{3}i\right)^6$, and write it as a complex number in the form $a + bi$ where a and b are real numbers.

Lesson Summary

Given a complex number z with modulus r and argument θ, the nth power of z is given by $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ where n is an integer.

Problem Set

1. Write the complex number in $a + bi$ form where a and b are real numbers.
   a. $2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$
   b. $3(\cos(210°) + i\sin(210°))$
   c. $\left(\sqrt{2}\right)^{10}\left(\cos\left(\frac{15\pi}{4}\right) + i\sin\left(\frac{15\pi}{4}\right)\right)$
   d. $\cos(9\pi) + i\sin(9\pi)$
   e. $4^3\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$
   f. $6(\cos(480°) + i\sin(480°))$

2. Use the formula discovered in this lesson to compute each power of z. Verify that the formula works by expanding and multiplying the rectangular form and rewriting it in the form $a + bi$ where a and b are real numbers.
   a. $\left(1 + \sqrt{3}i\right)^3$
   b. $(-1 + i)^4$
   c. $(2 + 2i)^5$
   d. $(2 - 2i)^{-2}$
   e. $\left(\sqrt{3} - i\right)^4$
   f. $\left(3\sqrt{3} - 3i\right)^6$

3. Given $z = -1 - i$, graph the first five powers of z by applying your knowledge of the geometric effect of multiplication by a complex number. Explain how you determined the location of each in the coordinate plane.

4. Use your work from Problem 3 to determine three values of n for which $(-1 - i)^n$ is a multiple of $-1 - i$.

5. Find the indicated power of the complex number, and write your answer in form $a + bi$ where a and b are real numbers.
   a. $\left[2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\right]^3$
   b. $\left[\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right]^{10}$

c. $\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)^6$

d. $\left[\frac{1}{3}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)\right]^4$

e. $\left[4\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)\right]^{-4}$