Lesson 17: The Geometric Effect of Multiplying by a Reciprocal

Classwork

Opening Exercise

Given $w = 1 + i$. What is $\arg(w)$ and $|w|$? Explain how you got your answer.

Exploratory Challenge 1/Exercises 1–9

1. Describe the geometric effect of the transformation $L(z) = (1 + i)z$.

2. Describe a way to undo the effect of the transformation $L(z) = (1 + i)z$.

3. Given that $0 \leq \arg(z) < 2\pi$ for any complex number, how could you describe any clockwise rotation of θ as an argument of a complex number?

4. Write a complex number in polar form that describes a rotation and dilation that will undo multiplication by $(1 + i)$, and then convert it to rectangular form.

5. In a previous lesson you learned that to undo multiplication by $1+i$, you would multiply by the reciprocal $\frac{1}{1+i}$. Write the complex number $\frac{1}{1+i}$ in rectangular form $z = a + bi$ where a and b are real numbers.

6. How do your answers to Exercises 4 and 5 compare? What does that tell you about the geometric effect of multiplication by the reciprocal of a complex number?

7. Jimmy states the following:

   Multiplication by $\frac{1}{a+bi}$ *has the reverse geometric effect of multiplication by* $+ bi$.

   Do you agree or disagree? Use your work on the previous exercises to support your reasoning.

8. Show that the following statement is true when $z = 2 - 2\sqrt{3}i$:

   The reciprocal of a complex number z *with modulus* r *and argument* θ *is* $\frac{1}{z}$ *with modulus* $\frac{1}{r}$ *and argument* $2\pi - \theta$.

9. Explain using transformations why $z \cdot \frac{1}{z} = 1$.

Exploratory Challenge 2/Exercise 10

10. Complete the graphic organizer below to summarize your work with complex numbers so far.

Operation	Geometric Transformation	Example. Illustrate algebraically and geometrically Let $z = 3 - 3i$ and $w = -2i$
Addition $z + w$		
Subtraction $z - w$		
Conjugate of z		
Multiplication $w \cdot z$		
Reciprocal of z		
Division $\frac{w}{z}$		

Exercises 11–13

Let $z = -1 + i$ and let $w = 2i$. Describe each complex number as a transformation of z and then write the number in rectangular form.

11. $w\bar{z}$

12. $\frac{1}{\bar{z}}$

13. $\overline{w + z}$

Problem Set

1. Describe the geometric effect of multiplying z by the reciprocal of each complex number listed below.
   a. $w_1 = 3\mathrm{i}$
   b. $w_2 = -2$
   c. $w_3 = \sqrt{3} + i$
   d. $w_4 = 1 - \sqrt{3}i$

2. Let $z = -2 - 2\sqrt{3}i$. Show that the geometric transformations you described in Problem 1 really produce the correct complex number by performing the indicated operation and determining the argument and modulus of each number.
   a. $\frac{-2-2\sqrt{3}i}{w_1}$
   b. $\frac{-2-2\sqrt{3}i}{w_2}$
   c. $\frac{-2-2\sqrt{3}i}{w_3}$
   d. $\frac{-2-2\sqrt{3}i}{w_4}$

3. In Exercise 12 of this lesson you described the complex number $\frac{1}{\bar{z}}$ as a transformation of z for a specific complex number z. Show that this transformation always produces a dilation of $z = a + bi$.

4. Does $L(z) = \frac{1}{z}$ satisfy the conditions that $L(z + w) = L(z) + L(w)$ and $L(mz) = mL(z)$ which makes it a linear transformation? Justify your answer.

5. Show that $L(z) = w\left(\overline{\frac{1}{w}z}\right)$ describes a reflection of z about the line containing the origin and w for $z = 3i$ and $w = 1 + i$.

6. Describe the geometric effect of each transformation function on z where z, w, and a are complex numbers.
   a. $L_1(z) = \frac{z-w}{a}$
   b. $L_2(z) = \overline{\left(\frac{z-w}{a}\right)}$
   c. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)}$
   d. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)} + w$

7. Verify your answers to Problem 6 if $z = 1 - i$, $w = 2i$, and $a = -1 - i$.
   a. $L_1(z) = \frac{z-w}{a}$

b. $L_2(z) = \overline{\left(\frac{z-w}{a}\right)}$

c. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)}$

d. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)} + w$