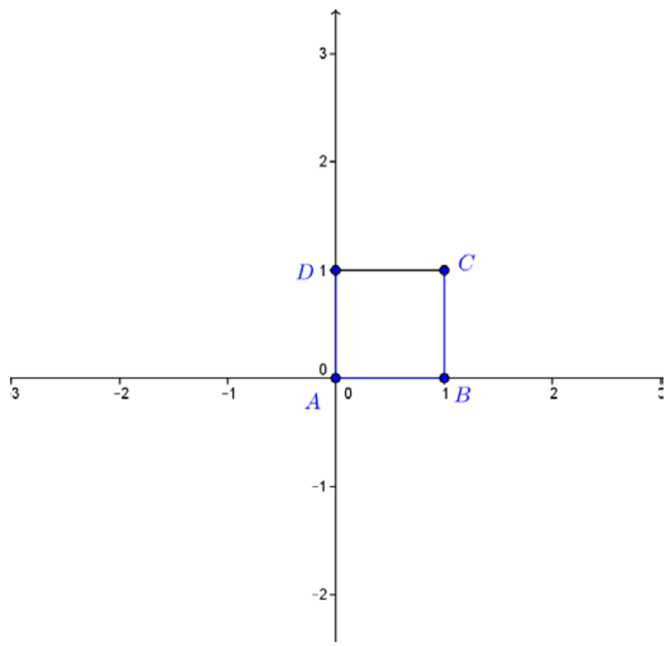
Lesson 14: Discovering the Geometric Effect of Complex Multiplication

Classwork

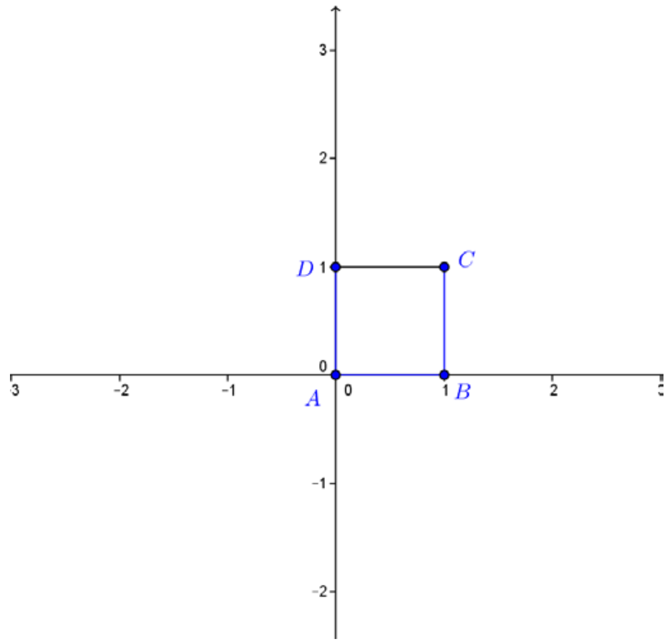
Exercises

The vertices $A(0,0)$, $B(1,0)$, $C(1,1)$, and $D(0,1)$ of a unit square can be represented by the complex numbers $A = 0$, $B = 1$, $C = 1 + i$, and $D = i$.

1. Let $L_1(z) = -z$.
   a. Calculate $A' = L_1(A)$, $B' = L_1(B)$, $C' = L_1(C)$, and $D' = L_1(D)$. Plot these four points on the axes.
   b. Describe the geometric effect of the linear transformation $L_1(z) = -z$ on the square $ABCD$.

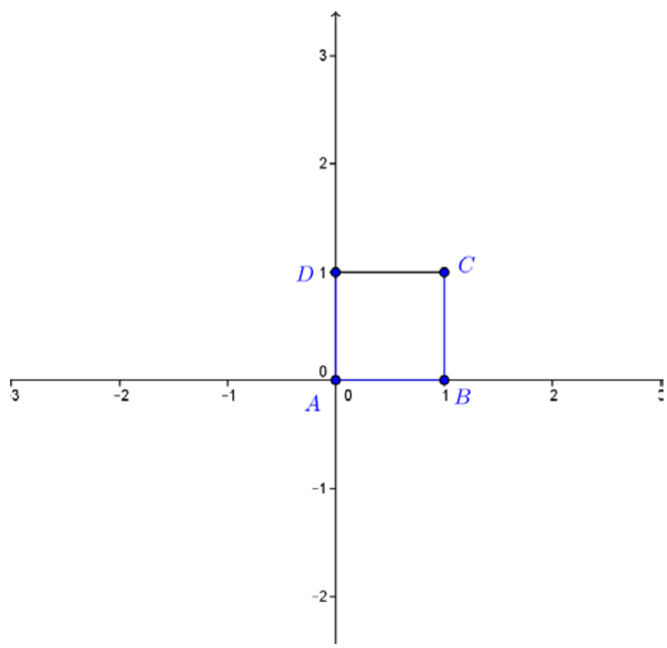


2. Let $L_2(z) = 2z$.
   a. Calculate $A' = L_2(A)$, $B' = L_2(B)$, $C' = L_2(C)$, and $D' = L_2(D)$. Plot these four points on the axes.



   b. Describe the geometric effect of the linear transformation $L_2(z) = 2z$ on the square $ABCD$.

3. Let $L_3(z) = iz$.
   a. Calculate $A' = L_3(A)$, $B' = L_3(B)$, $C' = L_3(C)$, and $D' = L_3(D)$. Plot these four points on the axes.

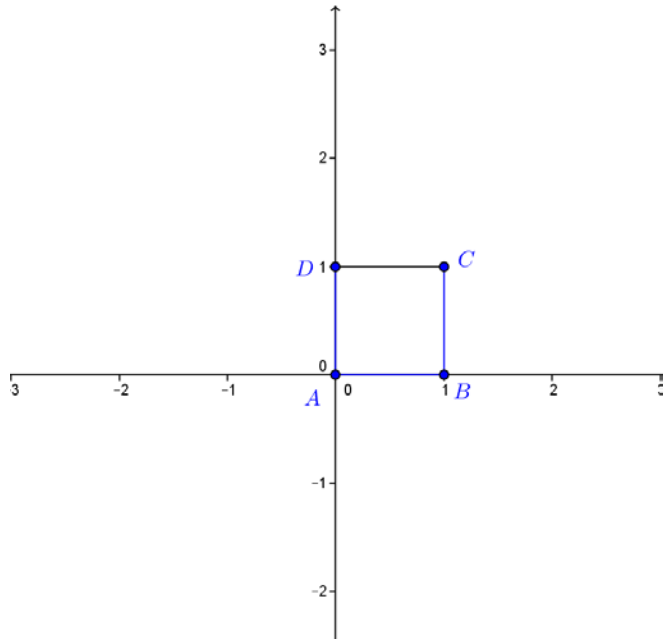


   b. Describe the geometric effect of the linear transformation $L_3(z) = iz$ on the square $ABCD$.

4. Let $L_4(z) = (2i)z$.

   a. Calculate $A' = L_4(A)$, $B' = L_4(B)$, $C' = L_4(C)$, and $D' = L_4(D)$. Plot these four points on the axes.

   b. Describe the geometric effect of the linear transformation $L_4(z) = (2i)z$ on the square $ABCD$.



5. Explain how transformations L_2, L_3, and L_4 are related.

6. We will continue to use the unit square $ABCD$ with $A = 0$, $B = 1$, $C = 1 + I$, $D = i$ for this exercise.

   a. What is the geometric effect of the transformation $L(z) = 5z$ on the unit square?

   b. What is the geometric effect of the transformation $L(z) = (5i)z$ on the unit square?

c. What is the geometric effect of the transformation $L(z) = (5i^2)z$ on the unit square?

d. What is the geometric effect of the transformation $L(z) = (5i^3)z$ on the unit square?

e. What is the geometric effect of the transformation $L(z) = (5i^4)z$ on the unit square?

f. What is the geometric effect of the transformation $L(z) = (5i^5)z$ on the unit square?

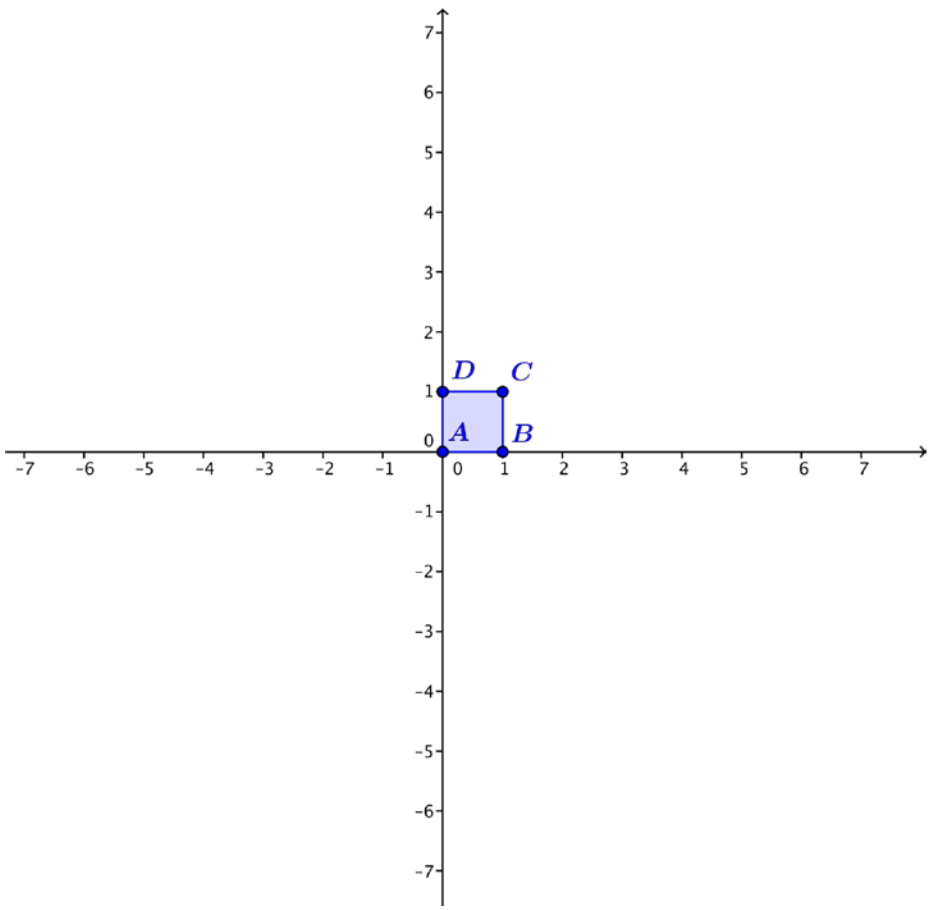
g. What is the geometric effect of the transformation $L(z) = (5i^n)z$ on the unit square, for some integer $n \geq 0$?

Exploratory Challenge

Your group has been assigned either to the 1-team, 2-team, 3-team, or 4-team. Each team will answer the questions below for the transformation that corresponds to their team number:

$$L_1(z) = (3 + 4i)z$$
$$L_2(z) = (-3 + 4i)z$$
$$L_3(z) = (-3 - 4i)z$$
$$L_4(z) = (3 - 4i)z.$$

The unit square unit square $ABCD$ with $A = 0$, $B = 1$, $C = 1 + i$, $D = i$ is shown below. Apply your transformation to the vertices of the square $ABCD$ and plot the transformed points A', B', C', and D' on the same coordinate axes.



For the 1-team: a. Why is $B' = 3 + 4i$? b. What is the argument of $3 + 4i$? c. What is the modulus of $3 + 4i$?	For the 2-team: a. Why is $B' = -3 + 4i$? b. What is the argument of $-3 + 4i$? c. What is the modulus of $-3 + 4i$?
For the 3-team: a. Why is $B' = -3 - 4i$? b. What is the argument of $-3 - 4i$? c. What is the modulus of $-3 - 4i$?	For the 4-team: a. Why is $B' = 3 - 4i$? b. What is the argument of $3 - 4i$? c. What is the modulus of $3 - 4i$?

All groups should also answer the following:

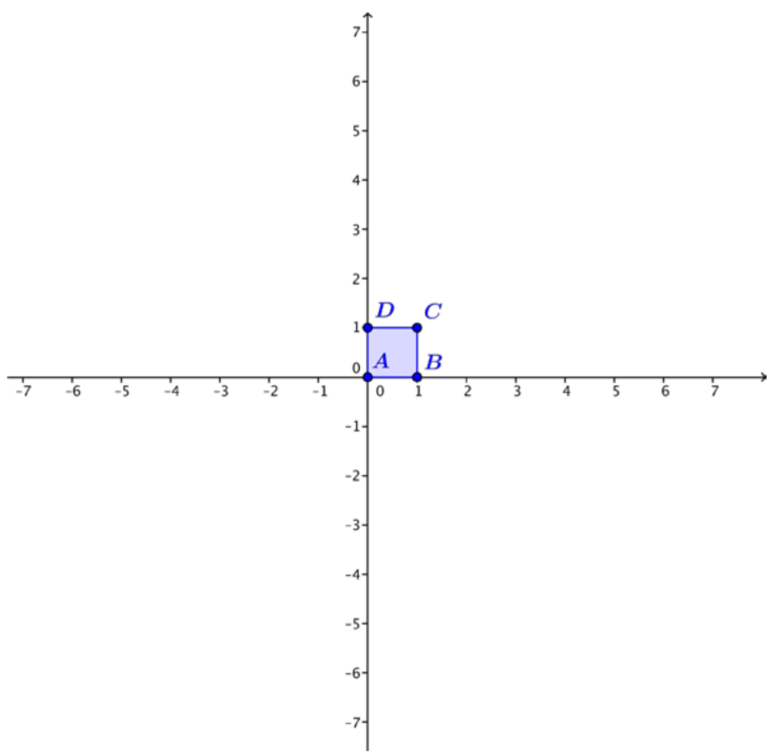
a. Describe the amount the square has been rotated counterclockwise.

b. What is the dilation factor of the square? Explain how you know.

c. What is the geometric effect of your transformation L_1, L_2, L_3, or L_4 on the unit square $ABCD$?

d. Make a conjecture: What do you expect to be the geometric effect of the transformation $L(z) = (2+i)z$ on the unit square $ABCD$?

e. Test your conjecture with the unit square on the axes below.



Problem Set

1. Find the modulus and argument for each of the following complex numbers.
   a. $z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$
   b. $z_2 = 2 + 2\sqrt{3}i$
   c. $z_3 = -3 + 5i$
   d. $z_4 = -2 - 2i$
   e. $z_5 = 4 - 4i$
   f. $z_6 = 3 - 6i$

2. For parts (a)–(c), determine the geometric effect of the specified transformation.
   a. $L(z) = -3z$
   b. $L(z) = -100z$
   c. $L(z) = -\frac{1}{3}z$
   d. Describe the geometric effect of the transformation $L(z) = az$ for any negative real number a.

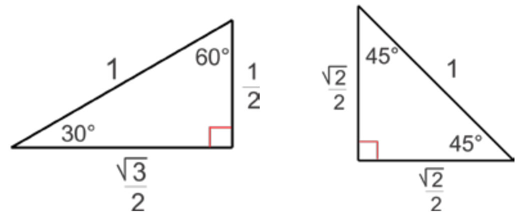
3. For parts (a)–(c), determine the geometric effect of the specified transformation.
   a. $L(z) = (-3i)z$
   b. $L(z) = (-100i)z$
   c. $L(z) = \left(-\frac{1}{3}i\right)z$
   d. Describe the geometric effect of the transformation $L(z) = (bi)z$ for any negative real number b.

4. Suppose that we have two linear transformations $L_1(z) = 3z$ and $L_2(z) = (5i)z$.
   a. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?
   b. What is the geometric effect of first performing transformation L_2, and then performing transformation L_1?
   c. Are your answers to parts (a) and (b) the same or different? Explain how you know.

5. Suppose that we have two linear transformations $L_1(z) = (4 + 3i)z$ and $L_2(z) = -z$. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?

6. Suppose that we have two linear transformations $L_1(z) = (3 - 4i)z$ and $L_2(z) = -z$. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?

7. Explain the geometric effect of the linear transformation $L(z) = (a - bi)z$, where a and b are positive real numbers.



8. In Geometry, we learned the special angles of a right triangle whose hypotenuse is 1 unit. The figures are shown above. Describe the geometric effect of the following transformations.

   a. $L_1(z) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) z$
   b. $L_2(z) = \left(2 + 2\sqrt{3}i\right)z$
   c. $L_3(z) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) z$
   d. $L_4(z) = (4 + 4i)z$

9. Recall that a function L is a linear transformation if all z and w in the domain of L and all constants a meet the following two conditions:

   i. $L(z + w) = L(z) + L(w)$
   ii. $L(az) = aL(z)$

   Show that the following functions meet the definition of a linear transformation.

   a. $L_1(z) = 4z$
   b. $L_2(z) = iz$
   c. $L_3(z) = (4 + i)z$

10. The vertices $A(0, 0)$, $B(1, 0)$, $C(1, 1)$, $D(0, 1)$ of a unit square can be represented by the complex numbers $A = 0$, $B = 1$, $C = 1 + i$, $D = i$. We learned that multiplication of those complex numbers by i rotates the unit square by $90°$ counterclockwise. What do you need to multiply by so that the unit square will be rotated by $90°$ clockwise?