Lesson 7: Complex Number Division

Student Outcomes

- Students determine the multiplicative inverse of a complex number.
- Students determine the conjugate of a complex number.

Lesson Notes

This is the first day of a two-day lesson on complex number division and applying this knowledge to further questions about linearity. In this lesson, students find the multiplicative inverse of a complex number. Students see the connection between the conjugate of a complex number and its multiplicative inverse. This sets the stage for our study of complex number division in Lesson 8.

Classwork

Opening Exercise (5 minutes)

To get ready for our work in this lesson, we will review complex number operations that students have previously studied in Algebra II, as well as $a + bi$ form. For our work in Lessons 7 and 8, students need to understand the real and imaginary components of complex numbers.

Opening Exercise

Perform the indicated operations. Write your answer in $a + bi$ form. Identify the real part of your answer and the imaginary part of your answer.

a. $(2 + 3i) + (-7 - 4i)$

$-5 - i$, -5 is real, and $-i$ is imaginary.

b. $i^2(-4i)$

$4i$, there is no real component, and $4i$ is imaginary.

c. $3i - (-2 + 5i)$

$2 - 2i$, 2 is real, and $-2i$ is imaginary.

d. $(3 - 2i)(-7 + 4i)$

$-13 + 26i$, -13 is real, and $26i$ is imaginary.

e. $(-4 - 5i)(-4 + 5i)$

41, 41 is real, and there is no imaginary component.

Discussion (5 minutes)

- In real number arithmetic, what is the multiplicative inverse of 5?
  - $\frac{1}{5}$
- How do you know? In other words, what is a multiplicative inverse?
  - $5\left(\frac{1}{5}\right) = 1$, *a number times its multiplicative inverse is always equal to* 1.
- The role of the multiplicative inverse is to get back to the identity.
- Is there a multiplicative inverse of i?

Allow students to really think about this and discuss this among themselves. Then follow with the question below.

- Is there a complex number z such that $z \cdot i = 1$?
  - $\frac{1}{i}$
- Can you find another way to say $\frac{1}{i}$? Explain your answer.
  - $-i$ *because* $i \cdot -i = -(i^2) = -(-1) = 1$. *Students could also mention* i^3 *as a possibility*.
- In today's lesson, we will look further at the multiplicative inverse of complex numbers.

Scaffolding:

- If students do not see the pattern, have them do a few additional examples. Find the multiplicative inverses of $-1+2i$, $-2-7i$, $3+10i$, and $4-i$.
- To help students see the pattern of the multiplicative inverse, have them compare the inverses of $2+3i$, $2-3i$, $-2+3i$, and $-2-3i$.
- For advanced students, have them work independently in pairs through the examples and exercises without leading questions. Be sure to check to make sure their general formula is correct before they begin the exercises.

Exercise 1 (2 minutes)

Exercises

1. What is the multiplicative inverse of $2i$?

$$\frac{1}{2i} = \frac{1}{2} \cdot \frac{1}{i} = \frac{1}{2} \cdot (-i) = -\frac{1}{2}i$$

Example 1 (8 minutes)

Students were able to reason what the multiplicative inverse of i was in the Discussion, but the multiplicative inverse of a complex number in the form of $p + qi$ is more difficult to find. In this example, students find the multiplicative inverse of a complex number by multiplying by a complex number in general form and solving the resulting system of equations.

- Does $3 + 4i$ have a multiplicative inverse?
  - *Yes,* $\frac{1}{3+4i}$.
- Is there a complex number $p + qi$ such that $(3 + 4i)(p + qi) = 1$?
  - *Students will have to think about this answer. Give them a couple of minutes, and then proceed with the example.*
- Let's begin by expanding this binomial. What equation do you get?
  - $3p + 3iq + 4ip + 4i^2q = 1$ *so* $3p + 3iq + 4ip - 4q = 1$

- Group the real terms and the complex terms, and rewrite the equation.
  - $(3p - 4q) + (3q + 4p)i = 1$
- Look at the right side of the equation. What do you notice?
  - *The number* 1 *is real, and there is no imaginary component.*
- What would the real terms have to be equal to? The imaginary terms?
  - *The real terms must equal* 1*, and the imaginary terms must equal* 0.
- Set up that system of equations.
  - $3p - 4q = 1$ *and* $4p + 3q = 0$
- Solve this system of equations for p and q.
  - $p = \frac{3}{25}, q = -\frac{4}{25}$
- This suggests that $\frac{1}{3+4i} = ?$
  - $\frac{3}{25} + \frac{-4}{25}i = \frac{3-4i}{25}$
- Does the product of $3 + 4i$ and $\frac{3-4i}{25}$ equal 1? Confirm that they are multiplicative inverses by performing this calculation. Check your work with a neighbor.
  - *Students should confirm that their result was correct.*

Note: Students can use their prior knowledge of conjugates and radicals from Algebra II for a simpler method of find the inverse for Examples 1 and 2. If students see this connection, allow this, but be sure that students see the connection and understand the math behind this concept.

Exercise 2 (3 minutes)

2. Find the multiplicative inverse of $5 + 3i$.

$\frac{5-3i}{34}$

Example 2 (8 minutes)

In Example 2, students look at patterns between the complex numbers and their multiplicative inverses from Example 1 and Exercise 2 and then find the general formula for the multiplicative inverse of any number.

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- Without doing any work, can you tell me what the multiplicative inverse of $3 - 4i$ and $5 - 3i$ would be?
  - $\frac{3+4i}{25}$ *and* $\frac{5+3i}{34}$
- Explain to your neighbor in words how to find the multiplicative inverse of a complex number.
  - *Change the sign between the real and imaginary terms, and then divide by the sum of the square of the coefficients of each term.*
- If $z = a + bi$, do you remember the name of $\bar{z} = a - bi$ from Algebra II?
  - *The conjugate.*

- Let's develop a general formula for the multiplicative inverse of any number of the form $z = a + bi$. Using what we did earlier in this example, what might we do?
  - *Multiply by another complex number $(p + qi)$, and set the product equal to 1.*
- Solve $(a + bi)(p + qi) = 1$. Show each step, and explain your work to your neighbor.
  - $ap + aqi + bpi + bqi^2 = 1$ *Expand the binomial.*
  - $ap + aqi + bpi - bq = 1$ *Simplify the equation.*
  - $ap - bq = 1$ and $aq + bp = 0$ *Set the real terms equal to 1 and the imaginary terms equal to 0.*
  - $p = \frac{a}{a^2 + b^2}$ *and* $q = -\frac{b}{a^2 + b^2}$ *Solve the system of equations for p and q.*
- What is the general formula of the multiplicative inverse of $z = a + bi$?
  - $\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$ *or* $\frac{a - bi}{a^2 + b^2}$
- Does this agree with what you discovered earlier in the example?
  - *Yes.*
- Explain how to find the multiplicative inverse of a complex number using the term *conjugate*.
  - *To find the multiplicative inverse of a complex number, $a + bi$, take the conjugate of the number and divide by $a^2 + b^2$.*

Exercises 3–7 (6 minutes)

In these exercises, students practice using the general formula for finding the multiplicative inverse of a number. The goal is to show students that this formula works for all numbers, real or complex.

State the conjugate of each number, and then using the general formula for the multiplicative inverse of $z = a + bi$, find the multiplicative inverse.

3. $3 + 4i$

$3 - 4i;\ \frac{3 - 4i}{3^2 + 4^2} = \frac{3 - 4i}{25}$

4. $7 - 2i$

$7 + 2i;\ \frac{7 - (-2)i}{7^2 + (-2)^2} = \frac{7 + 2i}{53}$

5. i

$-i;\ \frac{0 - 1i}{0^2 + (1)^2} = \frac{-i}{1} = -i$

6. 2

$2;\ \frac{2 - 0i}{2^2 + 0^2} = \frac{2}{4} = \frac{1}{2}$

7. Show that $a = -1 + \sqrt{3}i$ and $b = 2$ satisfy $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$.

Finding a common denominator of the right side, and then simplifying:

$$\begin{aligned}\frac{1}{a} + \frac{1}{b} &= \frac{1}{-1+\sqrt{3}i} + \frac{1}{2}\\ &= \frac{(2)1}{(2)} + \frac{(-1+\sqrt{3}i)\cdot 1}{(-1+\sqrt{3}i)\cdot 2}\\ &= \frac{1+\sqrt{3}i}{2(-1+\sqrt{3}i)}\\ &= \frac{1+\sqrt{3}i}{2(-1+\sqrt{3}i)}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i}\\ &= \frac{1-3i^2}{2(-1+\sqrt{3}i+\sqrt{3}i-3i^2)}\\ &= \frac{1+3}{2(-1+2\sqrt{3}i+3)}\\ &= \frac{4}{2(2+2\sqrt{3}i)}\\ &= \frac{4}{4(1+\sqrt{3}i)}\\ &= \frac{1}{1+\sqrt{3}i}\end{aligned}$$

$$\begin{aligned}\frac{1}{a+b} &= \frac{1}{-1+\sqrt{3}i+2}\\ &= \frac{1}{1+\sqrt{3}i}\end{aligned}$$

The two expressions are equal for the given values of a and b.

Closing (3 minutes)

Allow students to think about the questions below in pairs, and then pull the class together to wrap up the discussion.

- Was it necessary to use the formula for Exercise 6? Explain.
  - *No, the number 2 is a real number, so the multiplicative inverse was its reciprocal.*
- Look at Exercises 3–6. What patterns did you discover in the formats of the real and complex numbers and their multiplicative inverses?
  - *For any complex number $z = a + bi$, the multiplicative inverse has the format $\frac{a-bi}{a^2+b^2}$ when simplified.*
  - *This formula works for real and complex numbers, but for real numbers it is easier just to find the reciprocal.*

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 7: Complex Number Division

Exit Ticket

1. Find the multiplicative inverse of $3 - 2i$. Verify that your solution is correct by confirming that the product of $3 - 2i$ and its multiplicative inverse is 1.

2. What is the conjugate of $3 - 2i$?

Exit Ticket Sample Solutions

1. **Find the multiplicative inverse of $3-2i$. Verify that your solution is correct by confirming that the product of $3-2i$ and its multiplicative inverse is 1.**

   If $a+bi$ is the multiplicative inverse of $3-2i$, then

   $$(3-2i)(a+bi) = 1+0i(3-2i)(a+bi)$$
   $$3a+3bi-2ai-2bi^2 = 1+0i$$
   $$3a+3bi-2ai+2b = 1+0i.$$

   $3a+2b=1$ *and* $(3b-2a)i=0i$, *so* $3b-2a=0$.

   $a=\frac{3}{13}$, $b=\frac{2}{13}$, *so the multiplicative inverse* $a+bi=\frac{3+2i}{13}$.

   Verification: $(3-2i)\left(\frac{3}{13}+\frac{2i}{13}\right)=\frac{9}{13}+\frac{6i}{13}-\frac{6i}{13}-\frac{4i^2}{13}=\frac{9}{13}+\frac{4}{13}=1$

2. **What is the conjugate of $3-2i$?**

   $3+2i$

Problem Set Sample Solutions

Problems 1 and 2 are easy entry problems that allow students to practice operations on complex numbers and the algebra involved in such operations, including solving systems of equations. These problems also reinforce that complex numbers have a real component and an imaginary component. Problem 3 is more difficult. Most students should attempt part (a), but part (b) is optional and sets the stage for the next lesson. All skills practiced in this Problem Set are essential for success in Lesson 8.

1. **State the conjugate of each complex number. Then find the multiplicative inverse of each number, and verify by multiplying by $a+bi$ and solving a system of equations.**

   a. $-5i$

   $5i$ *is the conjugate.*

   $$-5i(a+bi) = 1$$
   $$-5ai-5bi^2 = 1$$
   $$-5ai+5b = 1$$

   $$-5a=0, 5b=1$$
   $$a=0, b=\frac{1}{5}$$

   $0+\frac{1}{5}i=\frac{1}{5}i$ *is the multiplicative inverse.*

b. $5-\sqrt{3}i$

$\left(5+\sqrt{3}i\right)$ *is the conjugate.*

$$\left(5-\sqrt{3}i\right)(a+bi)=1$$
$$5a+5bi-\sqrt{3}ai-\sqrt{3}bi^2=1$$
$$5a+5bi-\sqrt{3}ai+\sqrt{3}b=1$$

$$5a+\sqrt{3}b=1, 5b-\sqrt{3}a=0$$
$$a=\frac{5}{28}, b=\frac{\sqrt{3}}{28}$$

$\frac{5+\sqrt{3}i}{28}$ *is the multiplicative inverse.*

2. Find the multiplicative inverse of each number, and verify using the general formula to find multiplicative inverses of numbers of the form $z=a+bi$.

a. i^3

$i^3=-i=0-i$

$\frac{0-(-1)i}{0^2+(-1)^2}=\frac{i}{1}=i$ *is the multiplicative inverse.*

b. $\frac{1}{3}$

$\frac{1}{3}=\frac{1}{3}+0i$

$\frac{\frac{1}{3}-0i}{\left(\frac{1}{3}\right)^2+0^2}=\frac{\frac{1}{3}}{\frac{1}{9}}=\frac{1}{3}\cdot\frac{9}{1}=3$ *is the multiplicative inverse.*

c. $\frac{\sqrt{3}-i}{4}$

$\frac{\sqrt{3}-i}{4}=\frac{\sqrt{3}}{4}+\frac{-1}{4}i$

$\frac{\frac{\sqrt{3}}{4}-\left(\frac{-1}{4}\right)i}{\left(\frac{\sqrt{3}}{4}\right)^2+\left(\frac{-1}{4}\right)^2}=\frac{\frac{\sqrt{3}}{4}+\frac{1}{4}i}{\frac{3}{16}+\frac{1}{16}}=\frac{\frac{\sqrt{3}+i}{4}}{\frac{4}{16}}=\frac{\frac{\sqrt{3}+i}{4}}{\frac{1}{4}}=\frac{\sqrt{3}+i}{4}\cdot\frac{4}{1}=\sqrt{3}+i$ *is the multiplicative inverse.*

d. $1+2i$

$\frac{1-2i}{(1)^2+(-2)^2}=\frac{1-2i}{5}=\frac{1}{5}-\frac{2i}{5}$ *is the multiplicative inverse.*

e. $4-3i$

$\frac{4+3i}{(4)^2+(-3)^2}=\frac{4+3i}{25}=\frac{4}{25}+\frac{3i}{25}$ *is the multiplicative inverse.*

f. $2+3i$

$\frac{2-3i}{(2)^2+(-3)^2}=\frac{2-3i}{13}=\frac{2}{13}-\frac{3i}{13}$ *is the multiplicative inverse.*

g. $-5-4i$

$\frac{-5+4i}{(-5)^2+(-4)^2}=\frac{-5+4i}{41}=-\frac{5}{41}+\frac{4i}{41}$ *is the multiplicative inverse.*

h. $-3+2i$

$\frac{-3-2i}{(-3)^2+(2)^2}=\frac{-3-2i}{13}=-\frac{3}{13}-\frac{2i}{13}$ *is the multiplicative inverse.*

i. $\sqrt{2}+i$

$\frac{\sqrt{2}-i}{\left(\sqrt{2}\right)^2+(1)^2}=\frac{\sqrt{2}-i}{3}=\frac{\sqrt{2}}{3}-\frac{i}{3}$ *is the multiplicative inverse.*

j. $3-\sqrt{2}\cdot i$

$\frac{3+\sqrt{2}i}{(3)^2+\left(\sqrt{2}\right)^2}=\frac{3+\sqrt{2}i}{11}=\frac{3}{11}+\frac{\sqrt{2}i}{11}$ *is the multiplicative inverse.*

k. $\sqrt{5}+\sqrt{3}\cdot i$

$\frac{\sqrt{5}-\sqrt{3}i}{\left(\sqrt{5}\right)^2+\left(-\sqrt{3}\right)^2}=\frac{\sqrt{5}-\sqrt{3}i}{8}=\frac{\sqrt{5}}{8}-\frac{\sqrt{3}i}{8}$ *is the multiplicative inverse.*

3. Given $z_1=1+i$ and $z_2=2+3i$.

a. Let $w=z_1\cdot z_2$. Find w and the multiplicative inverse of w.

$w=(1+i)(2+3i)=-1+5i.$

$\frac{-1-5i}{1+25}=-\frac{1}{26}-\frac{5i}{26}$ *is the multiplicative inverse.*

b. Show that the multiplicative inverse of w is the same as the product of the multiplicative inverses of z_1 and z_2.

$z_1=1+i;\frac{1-i}{1+1}=\frac{1-i}{2}$ *is the multiplicative inverse.*

$z_2=2+3i;\frac{2-3i}{4+9}=\frac{2-3i}{13}$ *is the multiplicative inverse.*

$$\begin{aligned} z_1\cdot z_2&=\left(\frac{1-i}{2}\right)\left(\frac{2-3i}{13}\right)\\ &=\frac{2-3i-2i-3}{26}\\ &=\frac{-1-5i}{26}\\ &=-\frac{1}{26}-\frac{5i}{26}\end{aligned}$$