



Lesson 6: Complex Numbers as Vectors

Student Outcomes

- Students represent complex numbers as vectors.
- Students represent complex number addition and subtraction geometrically using vectors.

Lesson Notes

Students studied vectors as directed line segments in Grade 8, and in this lesson, we use vectors to represent complex numbers in the coordinate plane. This representation presents a geometric interpretation of addition and subtraction of complex numbers and is needed to make the case in Lesson 15 that when multiplying two complex numbers z and w , the argument of the product is the sum of the arguments: $\arg(zw) = \arg(z) + \arg(w)$.

This lesson aligns with **N-CN.B.5**: Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

The following vocabulary terms from Grade 8 are needed in this lesson:

VECTOR: A *vector* associated to the directed line segment $\overrightarrow{AB}$ is any directed segment that is congruent to the directed segment $\overrightarrow{AB}$ using only translations of the plane.

DIRECTED SEGMENT: A *directed segment* $\overrightarrow{AB}$ is the line segment $\overline{AB}$ together with a direction given by connecting an initial point A to a terminal point B .

The study of vectors will form a vital part of this course; notation for vectors varies across different contexts and curricula. These materials will refer to a vector as $\mathbf{v}$ (lowercase, bold, non-italicized) or $\langle 4, 5 \rangle$ or in column format, $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ or $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

We will use “let $\mathbf{v} = \langle 4, 5 \rangle$ ” to establish a name for the vector $\langle 4, 5 \rangle$.

This curriculum will avoid stating $\mathbf{v} = \langle 4, 5 \rangle$ without the word *let* preceding the equation when naming a vector unless it is absolutely clear from the context that we are naming a vector. However, we will continue to use the “=” to describe vector equations, like $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$, as we have done with equations throughout all other grades.

We will refer to the vector from A to B as “vector $\overrightarrow{AB}$ ”—notice, this is a ray with a full arrow. This notation is consistent with the way vectors are introduced in Grade 8 and is also widely used in post-secondary textbooks to describe both a ray and a vector depending on the context. To avoid confusion in this curriculum, the context will be provided or strongly implied, so it will be clear whether the full arrow indicates a vector or a ray. For example, when referring to a ray from A passing through B , we will say “ray $\overrightarrow{AB}$ ” and when referring to a vector from A to B , we will say “vector $\overrightarrow{AB}$.” Students should be encouraged to think about the context of the problem and not just rely on a hasty inference based on the symbol.

The magnitude of a vector will be signified as $\|\mathbf{v}\|$ (lowercase, bold, non-italicized).

Classwork

Opening Exercise (4 minutes)

The Opening Exercise reviews complex number arithmetic. We will revisit this example later in the lesson when we study the geometric interpretation of complex addition and subtraction using a vector representation of complex numbers. Students should work on these exercises either individually or in pairs.

Opening Exercise

Perform the indicated arithmetic operations for complex numbers $z = -4 + 5i$ and $w = -1 - 2i$.

a. $z + w$

$$z + w = -5 + 3i$$

b. $z - w$

$$z - w = -3 + 7i$$

c. $z + 2w$

$$z + 2w = -6 + i$$

d. $z - z$

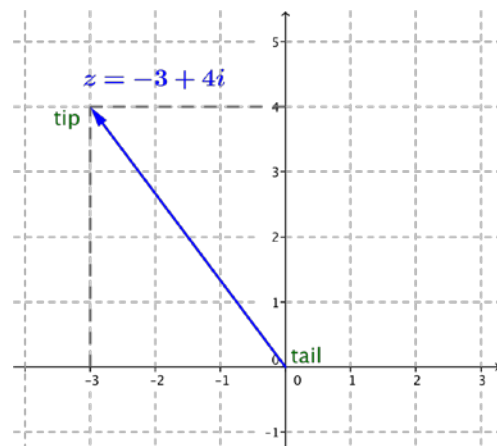
$$z - z = 0 + 0i$$

e. Explain how you add and subtract complex numbers.

Add or subtract the real components and the imaginary components separately.

Discussion (6 minutes)

- In Lesson 5, we represented a complex number $a + bi$ as the point (a, b) in the coordinate plane. Another way we can represent a complex number in the coordinate plane is as a vector. Recall the definition of a vector from Grade 8, which is that a vector $\overrightarrow{AB}$ is a directed segment from point A in the plane to point B, which we draw as an arrow from point A to point B. Since we can represent a complex number $z = a + bi$ as the point (a, b) in the plane, and we can use a vector to represent the directed segment from the origin to the point (a, b) , we can represent a complex number as a vector in the plane. The vector representing the complex number $z = -3 + 4i$ is shown.
- The length of a vector $\overrightarrow{AB}$ is the distance from the tail A of the vector to the tip B. For our purposes, the tail is the origin, and the tip is the point $z = (a + bi)$ in the coordinate plane.



- A vector consists of a length and a direction. To get from point A to point B , you move a distance AB in the direction of the vector $\overrightarrow{AB}$. So, to move from the origin to point $z = a + bi$, we move the length of $a + bi$ in the direction of the line from the origin to (a, b) . (This idea will be important later in the lesson when we use vectors to add and subtract complex numbers.)
- What is the length of the vector that represents the complex number $z_1 = -3 + 4i$?
 - Since $\sqrt{(-3 - 0)^2 + (4 - 0)^2} = \sqrt{9 + 16} = 5$, the length of the vector that represents z_1 is 5.
- What is the length of the vector that represents the complex number $z_2 = 2 - 7i$?
 - Since $\sqrt{(2 - 0)^2 + (-7 - 0)^2} = \sqrt{4 + 49} = \sqrt{53}$, the length of the vector that represents z_2 is $\sqrt{53}$.
- What is the length of the vector that represents the complex number $z_3 = a + bi$?
 - Since $\sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$, the length of the vector that represents z_3 is $\sqrt{a^2 + b^2}$.

Exercise 1 (8 minutes)

Have students work Exercise 1 in pairs or small groups. Circulate to be sure that students are correctly plotting the complex numbers in the plane and correctly computing the lengths of the resulting vectors.

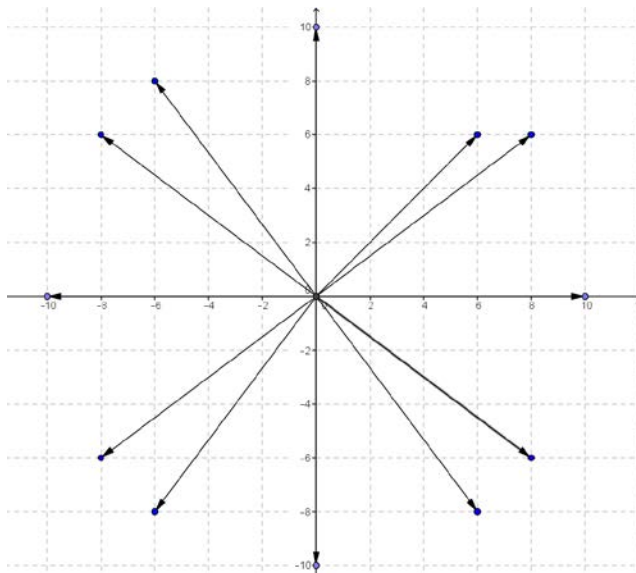
Exercises

1. The length of the vector that represents $z_1 = 6 - 8i$ is 10 because $\sqrt{6^2 + (-8)^2} = \sqrt{100} = 10$.

- a. Find at least seven other complex numbers that can be represented as vectors that have length 10.

There are an infinite number of complex numbers that meet this criteria; the most obvious are $10, 6 + 8i, 8 + 6i, 10i, -6 + 8i, -8 + 6i, -10, -8 - 6i, -6 - 8i, -10i$, and $8 - 6i$. The associated vectors for these numbers are shown in the sample response for part (b).

- b. Draw the vectors on the coordinate axes provided below.



Scaffolding:

Students struggling to find these values may want to work with the more general formula $a^2 + b^2 = 100$. Choose either a or b to create an equation they can solve. This also helps students see the relation to a circle in part (c).

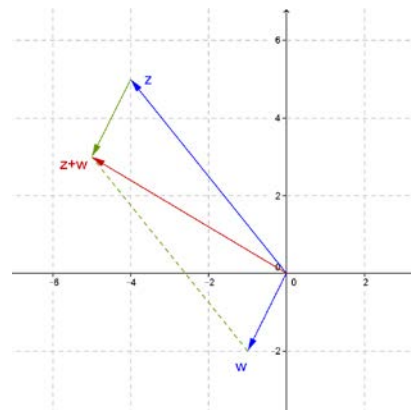
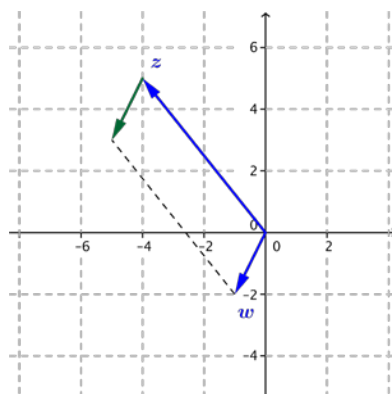
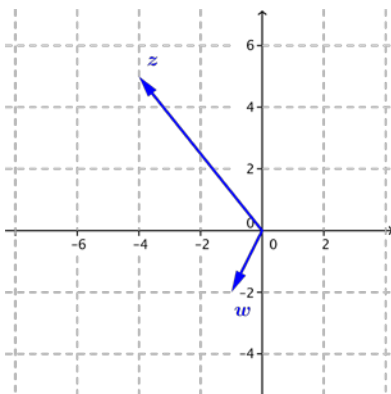
- c. What do you observe about all of these vectors?

Students should observe that the tips of the vectors lie on the circle of radius 10 centered at the origin.

Discussion (10 minutes)

Proceed slowly through this discussion, drawing plenty of figures to clarify the process of adding two vectors.

- Using vectors, how can we add two complex numbers? We know from the Opening Exercise that if $z = -4 + 5i$ and $w = -1 - 2i$, $z + w = -5 + 3i$. How could we show this is true using vectors?
 - *We know that if we think of vectors as a length and a direction, the sum $z + w$ is the distance we need to move from the origin in the direction of the vector $z + w$ to get to the point $z + w$. We can get from the origin to point $z + w$ by first moving from the origin to point z and then moving from point z to point w .*
- Using coordinates, this means that to find $z + w$, we do the following:
 1. Starting at the origin, move 4 units left and 5 units up to the tip of the vector representing z .
 2. From point z , move 1 unit left and two units down.
 3. The resulting point is $z + w$.
- Using vectors, we can locate point $z + w$ by the tip-to-tail method: Translate the vector that represents w so that the tip of z is at the same point as the tail of the new vector that represents w . The tip of this new translated vector is the sum $z + w$. See the sequence of graphs below.



- How would we find $z - w$ using vectors?
 - *We could think of $z - w$ as $z + (-w)$ since we already know how to add vectors.*
- Thinking of a vector as a length and a direction, how does $-w$ relate to w ?
 - *The vector $-w$ would have the same length as w but the opposite direction.*

Scaffolding:

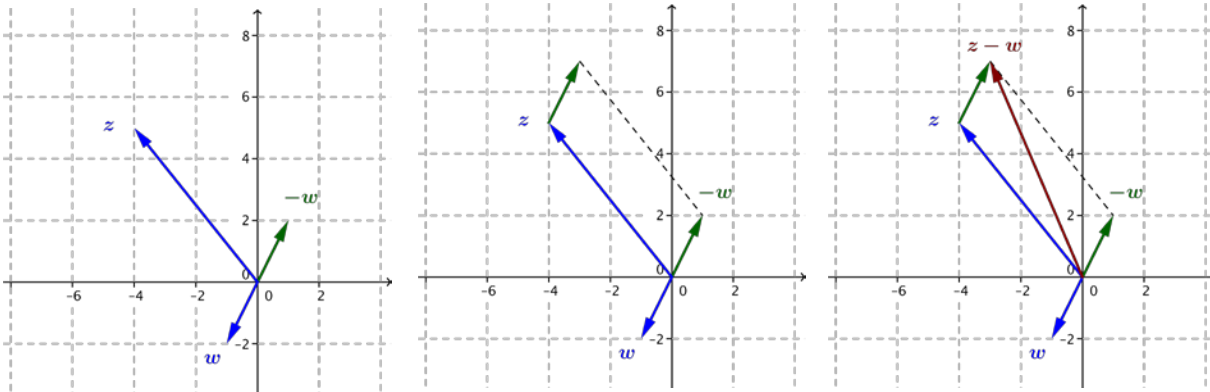
For advanced students, discuss how to find $z - w$ using the original parallelogram; that is, $z - w$ is the vector from the tip of w to the tip of z and then translated to the origin. Discuss how this is the same as subtraction in one dimension.

MP.7

MP.2
&
MP.3

MP.7

- Yes. So, we need to add $-w$ to z . To find $-w$, we reverse its direction. See the sequence of graphs below.



- In the Opening Exercise, we found that $z - w = -3 + 7i$. Does that agree with our calculation using vectors?
 - Yes.

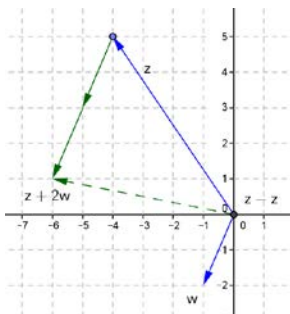
Exercises 2–6 (8 minutes)

Have students work on these exercises in pairs or small groups.

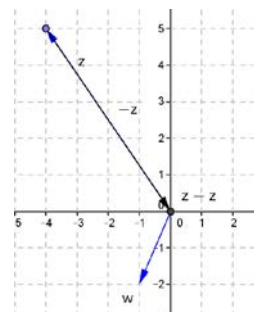
Scaffolding:

Model an example such as $2z + w$ for struggling students before asking them to work on these exercises.

2. In the Opening Exercise, we computed $z + 2w$. Calculate this sum using vectors.

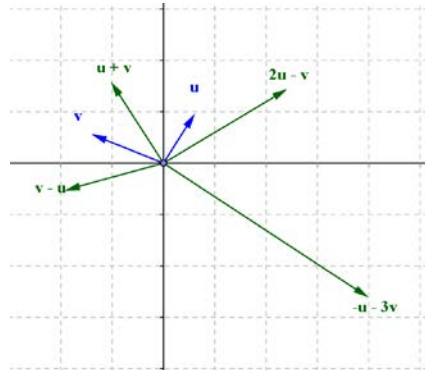


3. In the Opening Exercise, we also computed $z - z$. Calculate this sum using vectors.



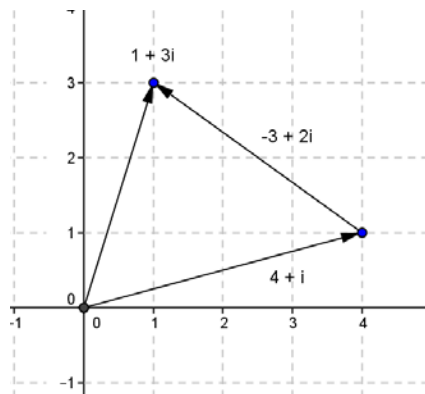
4. For the vectors u and v pictured below, draw the specified sum or difference on the coordinate axes provided.

- $u + v$
- $v - u$
- $2u - v$
- $-u - 3v$

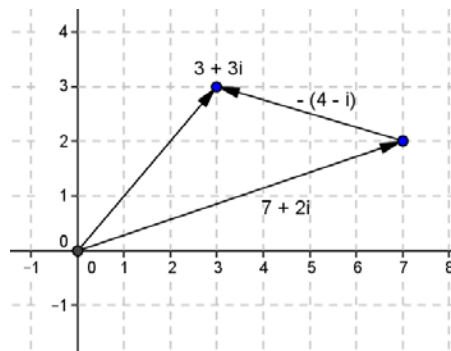


5. Find the sum of $4 + i$ and $-3 + 2i$ geometrically.

$1 + 3i$



6. Show that $(7 + 2i) - (4 - i) = 3 + 3i$ by representing the complex numbers as vectors.



Closing (4 minutes)

Ask students to write in their journal or notebook to explain the process of representing a complex number by a vector and the processes for adding and subtracting two vectors. Key points are summarized in the box below.

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 6: Complex Numbers as Vectors

Exit Ticket

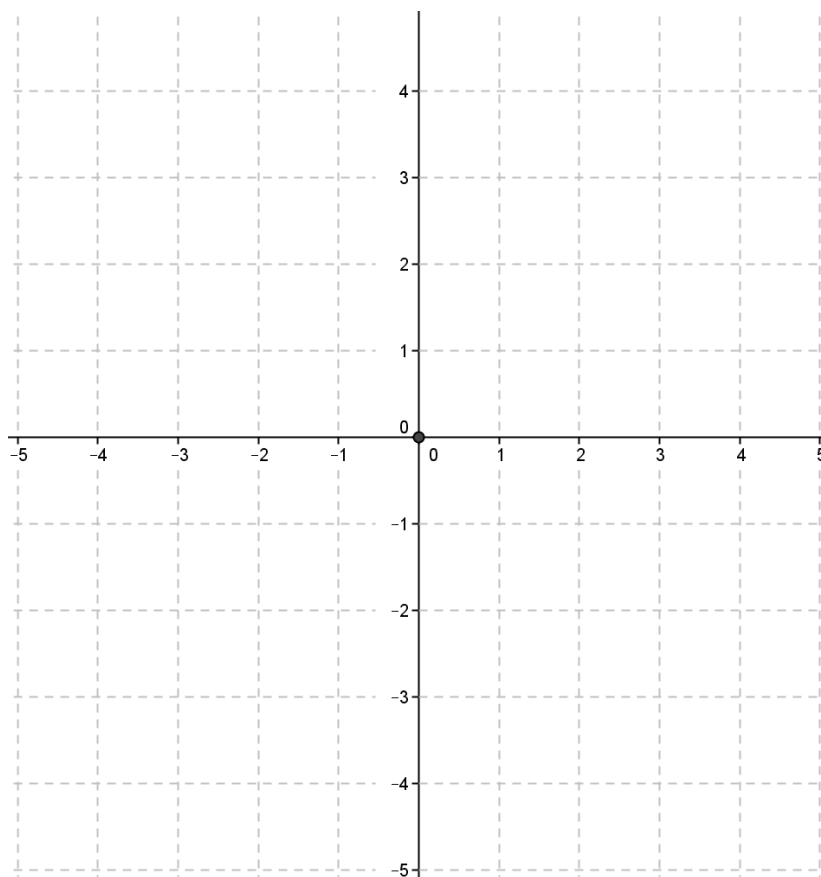
Let $z = -1 + 2i$ and $w = 2 + i$. Find the following, and verify each geometrically by graphing z , w , and each result.

a. $z + w$

b. $z - w$

c. $2z - w$

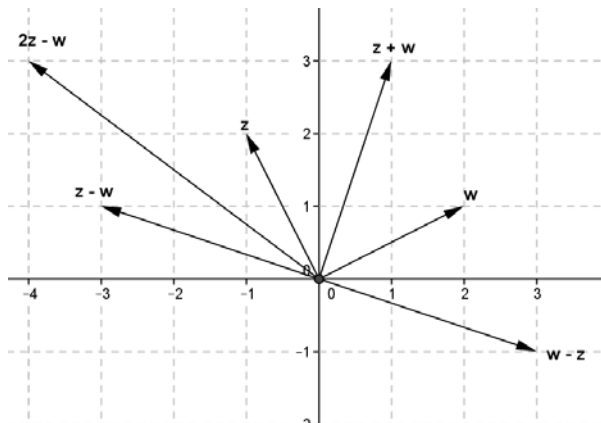
d. $w - z$



Exit Ticket Sample Solutions

Let $z = -1 + 2i$ and $w = 2 + i$. Find the following, and verify geometrically by graphing z , w , and each result.

- a. $z + w$
 $1 + 3i$
- b. $z - w$
 $-3 + i$
- c. $2z - w$
 $-4 + 3i$
- d. $w - z$
 $3 - i$



Problem Set Sample Solutions

1. Let $z = 1 + i$ and $w = 1 - 3i$. Find the following. Express your answers in $a + bi$ form.

- a. $z + w$
 $1 + i + 1 - 3i = 2 - 2i$
- b. $z - w$
 $1 + i - (1 - 3i) = 1 + i - 1 + 3i$
 $= 0 + 4i$
- c. $4w$
 $4(1 - 3i) = 4 - 12i$
- d. $3z + w$
 $3(1 + i) + 1 - 3i = 3 + 3i + 1 - 3i$
 $= 4 + 0i$
- e. $-w - 2z$
 $-(1 - 3i) - 2(1 + i) = -1 + 3i - 2 - 2i$
 $= -3 + i$
- f. What is the length of the vector representing z ?

The length of the vector representing z is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

- g. What is the length of the vector representing w ?

The length of the vector representing w is $\sqrt{1^2 + (-3)^2} = \sqrt{10}$.

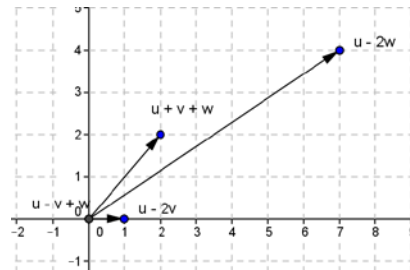
2. Let $u = 3 + 2i$, $v = 1 + i$, and $w = -2 - i$. Find the following. Express your answer in $a + bi$ form, and represent the result in the plane.

- a. $u - 2v$

$$\begin{aligned} 3 + 2i - 2(1 + i) &= 3 + 2i - 2 - 2i \\ &= 1 + 0i \end{aligned}$$

- b. $u - 2w$

$$\begin{aligned} 3 + 2i - 2(-2 - i) &= 3 + 2i + 4 + 2i \\ &= 7 + 4i \end{aligned}$$



- c. $u + v + w$

$$3 + 2i + 1 + i - 2 - i = 2 + 2i$$

- d. $u - v + w$

$$\begin{aligned} 3 + 2i - (1 + i) - 2 - i &= 3 + 2i - 1 - i - 2 - i \\ &= 0 + 0i \end{aligned}$$

- e. What is the length of the vector representing u ?

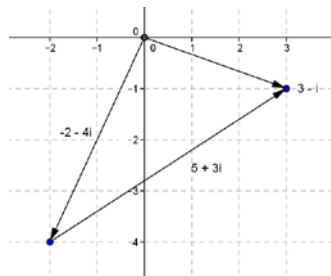
The length of the vector representing u is $\sqrt{3^2 + 2^2} = \sqrt{13}$.

- f. What is the length of the vector representing $u - v + w$?

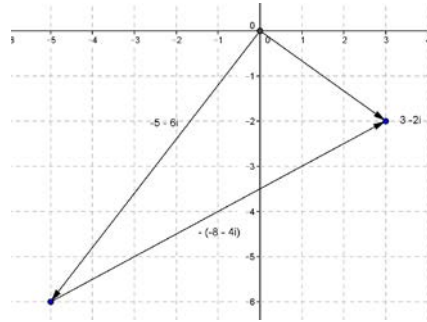
The length of the vector representing $u - v + w = \sqrt{0^2 + 0^2} = \sqrt{0} = 0$.

3. Find the sum of $-2 - 4i$ and $5 + 3i$ geometrically.

$$3 - i$$



4. Show that $(-5 - 6i) - (-8 - 4i) = 3 - 2i$ by representing the complex numbers as vectors.



5. Let $z_1 = a_1 + b_1i$, $z_2 = a_2 + b_2i$, and $z_3 = a_3 + b_3i$. Prove the following using algebra or by showing with vectors.

a. $z_1 + z_2 = z_2 + z_1$

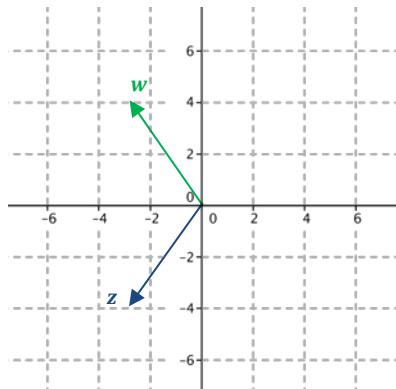
$$\begin{aligned} z_1 + z_2 &= (a_1 + b_1i) + (a_2 + b_2i) \\ &= (a_2 + b_2i) + (a_1 + b_1i) \\ &= z_2 + z_1 \end{aligned}$$

b. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$$\begin{aligned} z_1 + (z_2 + z_3) &= (a_1 + b_1i) + ((a_2 + b_2i) + (a_3 + b_3i)) \\ &= ((a_1 + b_1i) + (a_2 + b_2i)) + (a_3 + b_3i) \\ &= (z_1 + z_2) + z_3 \end{aligned}$$

6. Let $z = -3 - 4i$ and $w = -3 + 4i$.

- a. Draw vectors representing z and w on the same set of axes.



- b. What are the lengths of the vectors representing z and w ?

The length of the vector representing z is $\sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$.

The length of the vector representing w is $\sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$.

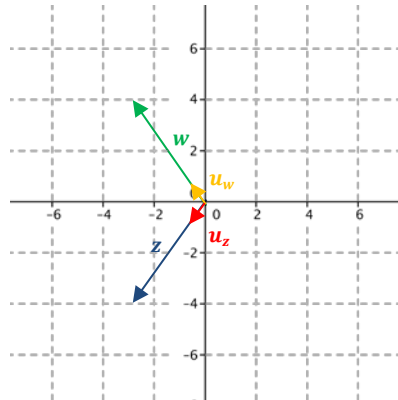
- c. Find a new vector, u_z , such that u_z is equal to z divided by the length of the vector representing z .

$$u_z = \frac{-3 - 4i}{5} = \frac{-3}{5} - \frac{4}{5}i$$

- d. Find u_w , such that u_w is equal to w divided by the length of the vector representing w .

$$u_w = \frac{-3 + 4i}{5} = \frac{-3}{5} + \frac{4}{5}i$$

- e. Draw vectors representing u_z and u_w on the same set of axes as part (a).



- f. What are the lengths of the vectors representing u_z and u_w ?

The length of the vector representing u_z is $\sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$.

The length of the vector representing u_w is $\sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$.

- g. Compare the vectors representing u_z to z and u_w to w . What do you notice?

The vectors representing u_z and u_w are in the same direction as z and w , respectively, but their lengths are only 1.

- h. What is the value of u_z times u_w ?

$$\left(\frac{3}{5} - \frac{4}{5}i\right)\left(\frac{3}{5} + \frac{4}{5}i\right) = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}i\right)^2 = \frac{9}{25} + \frac{16}{25} = 1$$

- i. What does your answer to part (h) tell you about the relationship between u_z and u_w ?

Since their product is 1, we know that u_z and u_w are reciprocals of each other.

7. Let $z = a + bi$.

- a. Let u_z be represented by the vector in the direction of z with length 1. How can you find u_z ? What is the value of u_z ?

Find the length of z , and then divide z by its length.

$$u_z = \frac{a + bi}{\sqrt{a^2 + b^2}}$$

- b. Let u_w be the complex number that when multiplied by u_z , the product is 1. What is the value of u_w ?

From Problem 4, we expect $u_w = \frac{a-bi}{\sqrt{a^2+b^2}}$. Multiplying, we get

$$\begin{aligned} \frac{a+bi}{\sqrt{a^2+b^2}} \cdot \frac{a-bi}{\sqrt{a^2+b^2}} &= \frac{a^2 - (bi)^2}{a^2 + b^2} \\ &= \frac{a^2 + b^2}{a^2 + b^2} \\ &= 1 \end{aligned}$$

- c. What number could we multiply z by to get a product of 1?

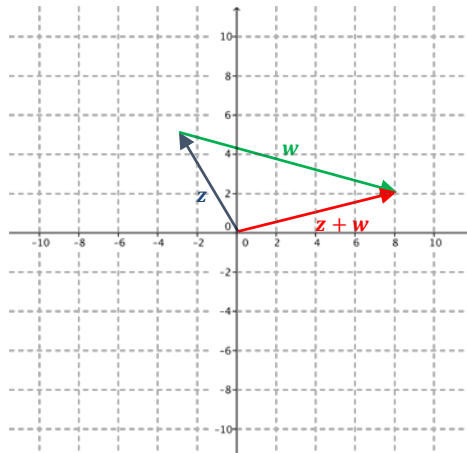
Since we know that u_z is equal to z divided by the length of z and that $u_z \cdot u_w = 1$. We get

$$z \cdot \frac{1}{\sqrt{a^2+b^2}} \cdot \frac{a-bi}{\sqrt{a^2+b^2}} = z \cdot \frac{a-bi}{a^2+b^2} = 1$$

So, multiplying z by $\frac{a-bi}{a^2+b^2}$ will result in a product of 1.

8. Let $z = -3 + 5i$.

- a. Draw a picture representing $z + w = 8 + 2i$.



- b. What is the value of w ?

$$w = 11 - 3i$$