Lesson 6: Complex Numbers as Vectors

Classwork

Opening Exercises

Perform the indicated arithmetic operations for complex numbers $z = -4 + 5i$ and $w = -1 - 2i$.

a. $z + w$

b. $z - w$

c. $z + 2w$

d. $z - z$

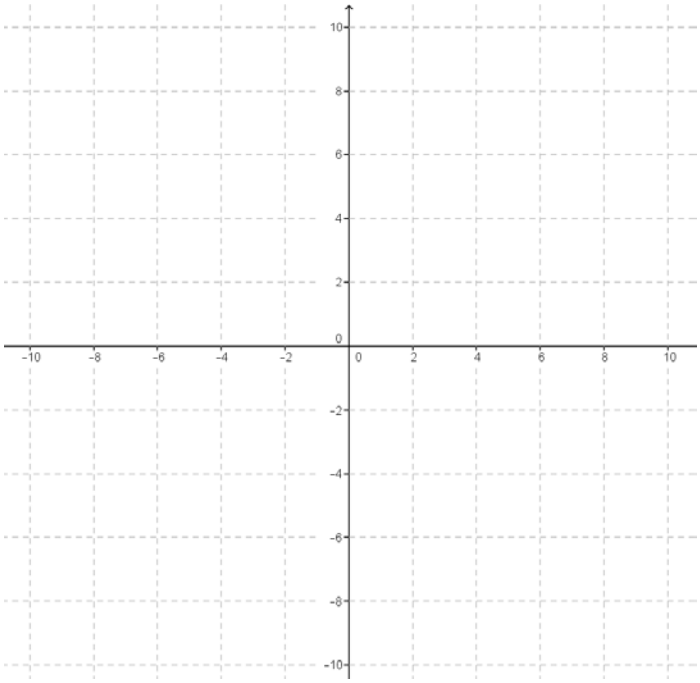
e. Explain how you add and subtract complex numbers.

Exercise 1

1. The length of the vector that represents $z_1 = 6 - 8i$ is 10 because $\sqrt{6^2 + (-8)^2} = \sqrt{100} = 10$.

   a. Find at least seven other complex numbers that can be represented as vectors that have length 10.

   b. Draw the vectors on the coordinate axes provided below.

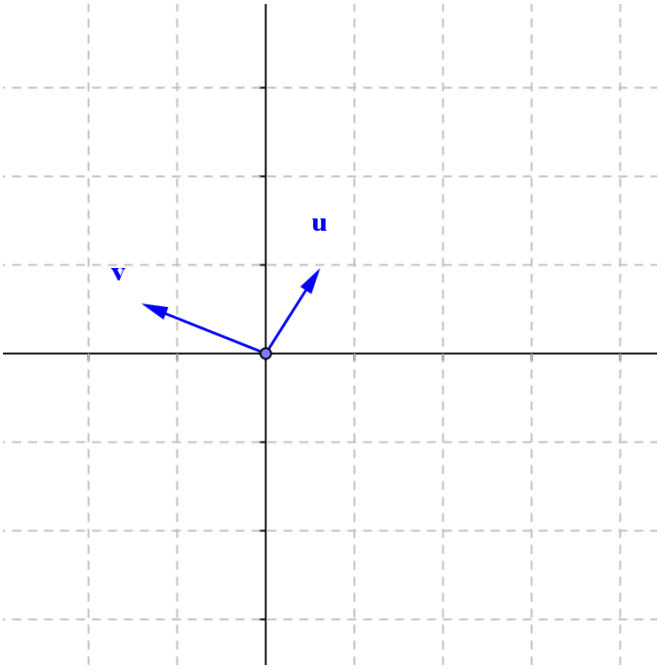


   c. What do you observe about all of these vectors?

2. In the Opening Exercises, we computed $z + 2w$. Calculate this sum using vectors.

3. In the Opening Exercises, we also computed $z - z$. Calculate this sum using vectors.

4. For the vectors u and v pictured below, draw the specified sum or difference on the coordinate axes provided.
   a. $u + v$
   b. $v - u$
   c. $2u - v$
   d. $-u - 3v$



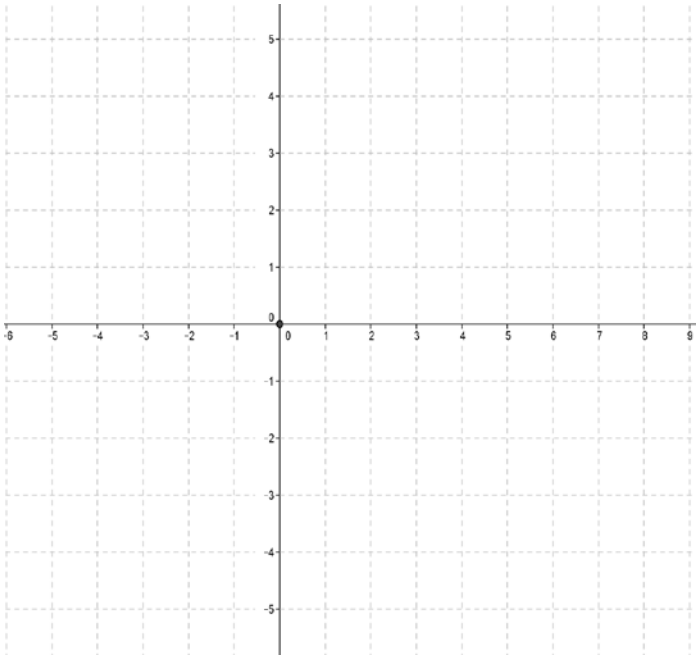
5. Find the sum of $4 + i$ and $-3 + 2i$ geometrically.

6. Show that $(7 + 2i) - (4 - i) = 3 + 3i$ by representing the complex numbers as vectors.

Problem Set

1. Let $z = 1 + i$ and $w = 1 - 3i$. Find the following. Express your answers in $a + bi$ form.
   a. $z + w$
   b. $z - w$
   c. $4w$
   d. $3z + w$
   e. $-w - 2z$
   f. What is the length of the vector representing z?
   g. What is the length of the vector representing w?

2. Let $u = 3 + 2i$, $v = 1 + i$, and $w = -2 - i$. Find the following. Express your answer in $a + bi$ form, and represent the result in the plane.
   a. $u - 2v$
   b. $u - 2w$
   c. $u + v + w$
   d. $u - v + w$
   e. What is the length of the vector representing u?
   f. What is the length of the vector representing $u - v + w$?



3. Find the sum of $-2 - 4i$ and $5 + 3i$ geometrically.

4. Show that $(-5 - 6i) - (-8 - 4i) = 3 - 2i$ by representing the complex numbers as vectors.

5. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, and $z_3 = a_3 + b_3 i$. Prove the following using algebra or by showing with vectors.
   a. $z_1 + z_2 = z_2 + z_1$
   b. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

6. Let $z = -3 - 4i$ and $w = -3 + 4i$.
   a. Draw vectors representing z and w on the same set of axes.
   b. What are the lengths of the vectors representing z and w?
   c. Find a new vector, u_z, such that u_z is equal to z divided by the length of the vector representing z.
   d. Find u_w, such that u_w is equal to w divided by the length of the vector representing w.
   e. Draw vectors representing u_z and u_w on the same set of axes as part (a).
   f. What are the lengths of the vectors representing u_z and u_w?
   g. Compare the vectors representing u_z to z and u_w to w. What do you notice?
   h. What is the value of u_z times u_w?
   i. What does your answer to part (h) tell you about the relationship between u_z and u_w?

7. Let $z = a + bi$.
   a. Let u_z be represented by the vector in the direction of z with length 1. How can you find u_z? What is the value of u_z?
   b. Let u_w be the complex number that when multiplied by u_z, the product is 1. What is the value of u_w?
   c. What number could we multiply z by to get a product of 1?

8. Let $z = -3 + 5i$.
   a. Draw a picture representing $z + w = 8 + 2i$.
   b. What is the value of w?