Lesson 3: Which Real Number Functions Define a Linear Transformation?

Classwork

Opening Exercises

Recall from the previous two lessons that a linear transformation is a function $f$ that satisfies two conditions:
(1) $f\left(x+y\right)=f\left(x\right)+f(y)$ and (2) $f\left(kx\right)=kf(x)$. Here, $k$ refers to any real number, and $x$ and $y$ represent arbitrary elements in the domain of $f$.

1. Let $f\left(x\right)=x^{2}$. Is $f$ a linear transformation? Explain why or why not.
2. Let $g\left(x\right)=\sqrt{x}$. Is $g$ a linear transformation? Explain why or why not.

Problem Set

1. Suppose you have a linear transformation $f:R\rightarrow R$, where $f\left(2\right)=1$ and $f\left(4\right)=2$.
	1. Use the addition property to compute $f\left(6\right)$, $f\left(8\right)$, $f\left(10\right)$, and $f\left(12\right)$.
	2. Find $f\left(20\right)$, $f(24)$, and $f\left(30\right)$. Show your work.
	3. Find $f\left(-2\right)$, $f(-4)$, and $f\left(-8\right)$. Show your work.
	4. Find a formula for $f(x)$.
	5. Draw the graph of the function $f(x)$.
2. The symbol $Z$ represents the set of integers, and so $g:Z\rightarrow Z$ represents a function that takes integers as inputs and produces integers as outputs. Suppose that a function $g:Z\rightarrow Z$ satisfies $g\left(a+b\right)=g\left(a\right)+g(b)$ for all integers $a$ and $b$. Is there necessarily an integer $k$ such that $g\left(n\right)=kn$ for all integer inputs $n$?
	1. Let $k=g(1)$. Compute $g(2)$ and $g(3)$.
	2. Let $n$ be any positive integer. Compute $g(n)$.
	3. Now consider $g(0)$. Since $g\left(0\right)=g(0+0)$, what can you conclude about $g(0)$?
	4. Lastly, use the fact that $g\left(n+-n\right)=g(0)$ to learn something about $g(-n)$, where $n$ is any positive integer.
	5. Use your work above to prove that $g\left(n\right)=kn$ for every integer $n$. Be sure to consider the fact that $n$ could be positive, negative, or $0$.
3. In the following problems, be sure to consider all kinds of functions: polynomial, rational, trigonometric, exponential, logarithmic, etc.
	1. Give an example of a function $f:R\rightarrow R$ that satisfies $f\left(x∙y\right)=f\left(x\right)+f(y)$.
	2. Give an example of a function $g:R\rightarrow R$ that satisfies $g\left(x+y\right)=g\left(x\right)∙g(y)$.
	3. Give an example of a function $h:R\rightarrow R$ that satisfies $h\left(x∙y\right)=h\left(x\right)∙h(y)$.