Lesson 2: Wishful Thinking—Does Linearity Hold?

Student Outcomes

- Students learn when ideal linearity properties do and do not hold for classes of functions studied in previous years.
- Students develop familiarity with linearity conditions.

Lesson Notes

This is second day of a two-day lesson in which we look at common mistakes that students make, all based on assuming linearity holds for all functions. In Lesson 1, students were introduced to a new definition of linearity. $f(x)$ is a linear transformation if $f(x+y)=f(x)+f(y)$ and $f(kx)=kf(x)$. Students continue to explore linearity by looking at common student mistakes. In Lesson 1, students explored polynomials and radical equations. Lesson 2 extends this exploration to trigonometric, rational, and logarithmic functions. The last exercise in this lesson, Exercise 4, has no real solutions, leading to the discovery of complex solutions, and this launches the study of complex numbers. This study will include operations on complex numbers as well as using the conjugates to find moduli and quotients.

Classwork

In this Exploratory Challenge, students will work individually while discussing the steps as a class. The exercises will be completed in pairs with the class coming together at the end to present their findings and to watch a video.

Opening Exercise (8 minutes)

In the last problem of the Problem Set from Lesson 1, students were asked to use what they had learned in Lesson 1 and then to think back to some mistakes that they had made in the past simplifying or expanding functions and to show that the mistakes were based on false assumptions. In this Opening Exercise, we want students to give us examples of some of their misconceptions. Have students put the examples on the board. We will study some of these directly in Lesson 2, and others you can assign as part of classwork, homework, or as extensions.

- In the last problem of the Problem Set from Lesson 1, you were asked to think back to some mistakes that you had made in the past simplifying or expanding functions. Show me some examples that you wrote down, and I will ask some of you to put your work on the board.
    - *Answers will vary but could include mistakes such as* $\sin(x+y)=\sin(x)+\sin(y)$, $\log(2a)=2\log(a)$, $10^{a+b}=10^a+10^b$, $\frac{1}{a+b}=\frac{1}{a}+\frac{1}{b}$, *and many others.*

Note: Emphasize that these examples are errors, not true mathematical statements.

Pick a couple of the simpler examples of mistakes that will not be covered in class, and talk about those, going through the steps of Lesson 1. For each example of a mistake, have students verify with numbers that the equation is not true for all real numbers, and then find a solution that is true for all real numbers. Indicate to students the statements that will be reviewed in class. List other statements that may be reviewed later.

Exploratory Challenge (14 minutes)

- In Lesson 1, we discovered that not all functions are linear transformations. Today, we will study some different functions.
- Let's start by looking at a trigonometric function. Is $f(x) = \sin x$ a linear transformation? Explain why or why not.
  - *Answers may vary, and students may be unsure. Proceed to the next question. No,* $\sin(x+y) \neq \sin(x) + \sin(y)$ *and* $\sin(ax) \neq a \sin x$.
- Does $\sin(x+y) = \sin(x) + \sin(y)$ for all real values of x and y?
  - *Answers may vary.*
- Substitute some values of x and y in to this equation.

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Scaffolding:

Students may need a reminder of how to convert between radians and degrees and critical trigonometric function values. Create a chart for students to complete that lists degrees, radian measure, $\sin x$, and $\cos x$. A copy of a table follows this lesson in the Student Materials.

Some students may use degrees and others radians. Allow students to choose. Alternatively, assign half of the students to use degrees and the other half to use radians. Compare answers.

- Did anyone find values of x and y that produced a true statement?
  - *Answers will vary but could include* $x = 0°, y = 0°$ *or* $x = 180°, y = 180°$ *or* $x = 0°, y = 90°$ *or the equivalent in radians* $x = 0, y = 0$ *or* $x = \pi, y = \pi$ *or* $x = 0, y = \frac{\pi}{2}$.
- If you used degrees, compare your answers to a neighbor who used radians. What do you notice?
  - *The answers will be the same but a different measure. For example,* $x = 180°, y = 180°$ *is the same as* $x = \pi, y = \pi$ *because* $180° = \pi \text{ rad}$.
- Did anyone find values of x and y that produced a false statement? Explain.
  - *Answers will vary but could include* $x = 45°, y = 45°$ *or* $x = 30°, y = 30°$ *or* $x = 30°, y = 60°$ *or the equivalent in radians* $x = \frac{\pi}{4}, y = \frac{\pi}{4}$ *or* $x = \frac{\pi}{6}, y = \frac{\pi}{6}$ *or* $x = \frac{\pi}{6}, y = \frac{\pi}{3}$. *For example,* $\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$, *but* $\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$, *so the statement is false.*
- Is this function a linear transformation? Explain this to your neighbor.
  - *This function is not a linear transformation because* $\sin(x+y) \neq \sin(x) + \sin(y)$ *for all real numbers.*

Exercises 1–5 (15 minutes)

In the exercises below, allow students to work through the problems in pairs. Circulate and give students help as needed. Call the class back together, and have groups present their results. You can assign all groups Exercises 1–4. For advanced groups, ask students to find the imaginary solutions to Exercise 4, and/or assign some of the more complicated examples that students brought to class from the Lesson 1 Problem Set and presented in the Opening Exercise.

Scaffolding:

- For advanced learners, have students determine the general solution that will work for all real numbers.
- Monitor group work, and target some groups with more specific questions to help them with the algebra needed. Students may need a reminder of the properties of logarithms such as $a\log(x) = \log(x^a)$.
- Some groups may need to complete the trigonometry value table before starting the exercises.

Exercises 1–5

MP.3

1. **Let $f(x) = \sin x$. Does $f(2x) = 2f(x)$ for all values of x? Is it true for any values of x? Show work to justify your answer.**

   No. If $x = \frac{\pi}{2}$, $\sin\left(2\left(\frac{\pi}{2}\right)\right) = \sin(\pi) = 0$, but $2\sin\left(\frac{\pi}{2}\right) = 2(1) = 2$, so the statement does not hold for every value of x. It is true anytime $\sin(x) = 0$, so for $x = 0, x = \pm\pi, x = \pm 2\pi$.

2. **Let $f(x) = \log(x)$. Find a value for a such that $f(2a) = 2f(a)$. Is there one? Show work to justify your answer.**

$$\log(2a) = 2\log(a)$$
$$\log(2a) = \log(a^2)$$
$$2a = a^2$$
$$a^2 - 2a = 0$$
$$a(a-2) = 0$$

   Thus, $a = 2$ or $a = 0$. Because 0 is not in the domain of the logarithmic function, the only solution is $a = 2$.

3. **Let $f(x) = 10^x$. Show that $f(a+b) = f(a) + f(b)$ is true for $a = b = \log(2)$ and that it is not true for $a = b = 2$.**

   For $a = b = \log(2)$

   $f(a+b) = 10^{(\log(2)+\log(2))} = 10^{2\log(2)} = 10^{\log(2^2)} = 2^2 = 4$

   $f(a) + f(b) = 10^{\log 2} + 10^{\log 2} = 2 + 2 = 4$

   Therefore, $f(a+b) = f(a) + f(b)$.

   For $a = b = 2$

   $f(a+b) = 10^{2+2} = 10^4 = 10{,}000$

   $f(a) + f(b) = 10^2 + 10^2 = 100 + 100 = 200$

   Therefore, $f(a+b) \neq f(a) + f(b)$

4. **Let $f(x) = \frac{1}{x}$. Are there any real numbers a and b so that $f(a+b) = f(a) + f(b)$? Explain.**

   Neither a nor b can equal zero since they are in the denominator of the rational expressions.

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$$
$$\frac{1}{a+b}ab(a+b) = \frac{1}{a}ab(a+b) + \frac{1}{b}ab(a+b)$$
$$ab = a(a+b) + b(a+b)$$
$$ab = a^2 + ab + ab + b^2$$
$$ab = a^2 + 2ab + b^2$$
$$ab = (a+b)^2$$

   This means that ab must be a positive number. Simplifying further, we get $0 = a^2 + ab + b^2$.

   The sum of three positive numbers will never equal zero, so there are no real solutions for a and b.

5. **What do your findings from these Exercises illustrate about the linearity of these functions? Explain.**

   Answers will vary but should address that in each case, the function is not a linear transformation because it does not hold to the conditions $f(a+b) = f(a) + f(b)$ and $f(cx) = c(f(x))$ for all real-numbered inputs.

Closing (3 minutes)

As a class, have a discussion using the following questions.

- What did you notice about the solutions of trigonometric functions? Why?
  - *There are more solutions that work for trigonometric functions because they are cyclical.*
- Which functions were hardest to find solutions that worked? Why?
  - *Answers will vary, but many students may say logarithmic or exponential functions.*
- Are $a = 0$ and/or $b = 0$ always solutions? Explain.
  - *No, it depends on the function.*
  - *For example,* $\cos(0+0) \neq \cos(0) + \cos(0)$. $10^{(0+0)} \neq 10^0 + 10^0$.
- Are trigonometric, exponential, and logarithmic functions linear transformations? Explain.
  - *No, they do not meet the conditions required for linearity:*
    $f(a+b) = f(a) + f(b)$ *and* $f(cx) = c\big(f(x)\big)$ *for all real-numbered inputs.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 2: Wishful Thinking—Does Linearity Hold?

Exit Ticket

1. Koshi says that he knows that $\sin(x+y) = \sin(x) + \sin(y)$ because he has plugged in multiple values for x and y and they all work. He has tried $x = 0°$ and $y = 0°$, but he says that usually works, so he also tried $x = 45°$ and $y = 180°$, $x = 90°$ and $y = 270°$, and several others. Is Koshi correct? Explain your answer.

2. Is $f(x) = \sin x$ a linear transformation? Why or why not?

Exit Ticket Sample Solutions

1. Koshi says that he knows that $\sin(x+y) = \sin(x) + \sin(y)$ because he has plugged in multiple values for x and y and they all work. He has tried $x = 0°$ and $y = 0°$, but he says that usually works, so he also tried $x = 45°$ and $y = 180°$, $x = 90°$ and $y = 270°$, and several others. Is Koshi correct? Explain your answer.

   Koshi is not correct. He happened to pick values that worked, most giving at least one value of $\sin(x) = 0$. If he had chosen other values such as $x = 30°$ and $y = 60°$, $\sin(30° + 60°) = \sin(90°) = 1$, but $\sin(30°) + \sin(60°) = \frac{1}{2} + \frac{\sqrt{3}}{2}$, so the statement that $\sin(30° + 60°) = \sin(30°) + \sin(60°)$ is false.

2. Is $f(x) = \sin x$ a linear transformation? Why or why not?

   No. $\sin(x+y) \neq \sin(x) + \sin(y)$ and $\sin(ax) \neq a \sin x$.

Problem Set Sample Solutions

Assign students some or all of the functions to investigate. Problems 1–4 are all trigonometric functions, Problem 5 is a rational function, and Problems 6 and 7 are logarithmic functions. These can be divided up. Problem 8 sets up Lesson 3 but is quite challenging.

Examine the equations given in Problems 1–4, and show that the functions $f(x) = \cos x$ and $f(x) = \tan x$ are not linear transformations by demonstrating that they do not satisfy the conditions indicated for all real numbers. Then, find values of x and/or y for which the statement holds true.

1. $\cos(x+y) = \cos(x) + \cos(y)$

   Answers that prove the statement false will vary but could include $x = 0$ and $y = 0$.

   This statement is true when $x = 1.9455$, or $111.47°$, and $y = 1.9455$, or $111.47°$. This will be difficult for students to find without technology.

2. $\cos(2x) = 2\cos(x)$

   Answers that prove the statement false will vary, but could include $x = 0$ or $x = \frac{\pi}{2}$.

   This statement is true when $x = 1.9455$, or $111.47°$. This will be difficult for students to find without technology.

3. $\tan(x+y) = \tan(x) + \tan(y)$

   Answers that prove the statement false will vary, but could include $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$.

   This statement is true when $x = 0$ and $y = 0$.

4. $\tan(2x) = 2\tan(x)$

   Answers that prove the statement false will vary, but could include $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$.

   This statement is true when $x = 0$ and $y = 0$.

5. Let $f(x) = \frac{1}{x^2}$, are there any real numbers a and b so that $f(a+b) = f(a) + f(b)$? Explain.

Neither a nor b can equal zero since they are in the denominator of the fractions.

If $f(a+b) = f(a) + f(b)$, then $\frac{1}{(a+b)^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Multiplying each term by $a^2b^2(a+b)^2$, we get

$$a^2b^2(a+b)^2\frac{1}{(a+b)^2} = a^2b^2(a+b)^2\frac{1}{a^2} + a^2b^2(a+b)^2\frac{1}{b^2}$$
$$a^2b^2 = b^2(a+b)^2 + a^2(a+b)^2$$
$$a^2b^2 = (a^2+b^2)(a+b)^2$$
$$a^2b^2 = a^4 + 2a^3b + 2a^2b^2 + 2ab^3 + b^4$$
$$a^2b^2 = a^4 + 2ab(a^2+b^2) + 2a^2b^2 + b^4$$
$$0 = a^4 + 2ab(a^2+b^2) + a^2b^2 + b^4$$

The terms a^4, a^2b^2, and b^4 are positive because they are even-numbered powers of nonzero numbers. We established in the lesson that $ab = (a+b)^2$and, therefore, is also positive.

The product $2ab(a^2+b^2)$ must then also be positive.

The sum of four positive numbers will never equal zero, so there are no real solutions for a and b.

6. Let $f(x) = \log x$, find values of a such that $f(3a) = 3f(a)$.

$$\log(3a) = 3\log(a)$$
$$\log(3a) = \log(a)^3$$
$$e\log(3a) = e\log(a)^3$$
$$3a = a^3$$
$$3 = a^2$$
$$a = \sqrt{3}$$

This is true for the value of a when $3a = a^3$ that is in the domain, which is $a = \sqrt{3}$.

7. Let $f(x) = \log x$, find values of a such that $f(ka) = kf(a)$.

This is true for the values of a when $ka = a^k$ that are in the domain of the function.

8. Based on your results from the previous two problems, form a conjecture about whether $f(x) = \log x$ represents a linear transformation.

The function is not an example of a linear transformation. The condition $f(ka) = kf(a)$ does not hold for all values of a, for example, nonzero values of c and $a = 1$.

9. Let $f(x) = ax^2 + bx + c$.

a. Describe the set of all values for a, b, and c that make $f(x+y) = f(x) + f(y)$ valid for all real numbers x and y.

This will be challenging for students but we want them to realize that $a = 0$, $c = 0$, and any real number b. They may understand that $a = 0$, but $c = 0$ could be more challenging. The point is that it is unusual for functions to satisfy this condition for all real values of x and y. This will be discussed in detail in Lesson 3.

$$f(x+y) = a(x+y)^2 + b(x+y) + c = ax^2 + 2axy + ay^2 + bx + by + c$$
$$f(x) + f(y) = ax^2 + bx + c + ay^2 + by + c$$

$$f(x+y) = f(x) + f(y)$$
$$ax^2 + 2axy + ay^2 + bx + by + c = ax^2 + bx + c + ay^2 + by + c$$
$$2axy + c = c + c$$
$$2axy = c$$

Therefore, the set of values that will satisfy this equation for all real numbers x and y is $a = 0$, any real number b, and $c = 0$.

b. What does your result indicate about the linearity of quadratic functions?

Answers will vary but should address that quadratic functions are not linear transformations, since they only meet the condition $f(x+y) = f(x) + f(y)$ when $a = 0$.

Trigonometry Table

Angles Measure (x Degrees)	Angle Measure (x Radians)	$\sin(x)$	$\cos(x)$
0	0	0	1
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
90	$\frac{\pi}{2}$	1	0