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Complex Numbers and Transformations

[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Precalculus and Advanced Topics • Module 1

Complex Numbers and Transformations

OVERVIEW

Module 1 sets the stage for expanding students' understanding of transformations by first exploring the notion of linearity in an algebraic context ("Which familiar algebraic functions are linear?"). This quickly leads to a return to the study of complex numbers and a study of linear transformations in the complex plane. Thus, Module 1 builds on standards **N-CN.A.1** and **N-CN.A.2** introduced in the Algebra II course and standards **G-CO.A.2**, **G-CO.A.4**, and **G-CO.A.5** introduced in the Geometry course.

Topic A opens with a study of common misconceptions by asking questions such as "For which numbers a and b does $(a+b)^2 = a^2 + b^2$ happen to hold?"; "Are there numbers a and b for which $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$?"; and so on. This second equation has only complex solutions, which launches a study of quotients of complex numbers and the use of conjugates to find moduli and quotients (**N-CN.A.3**). The topic ends by classifying real and complex functions that satisfy linearity conditions. (A function L is linear if, and only if, there is a real or complex value w such that $L(z) = wz$ for all real or complex z.) Complex number multiplication is emphasized in the last lesson.

In Topic B, students develop an understanding that when complex numbers are considered points in the Cartesian plane, complex number multiplication has the geometric effect of a rotation followed by a dilation in the complex plan. This is a concept that has been developed since Algebra II and builds upon standards **N-CN.A.1** and **N-CN.A.2**, which, when introduced, were accompanied with the observation that multiplication by i has the geometric effect of rotating a given complex number $90°$ about the origin in a counterclockwise direction. The algebraic inverse of a complex number (its reciprocal) provides the inverse geometric operation. Analysis of the angle of rotation and the scale of the dilation brings a return to topics in trigonometry first introduced in Geometry (**G-SRT.C.6**, **G-SRT.C.7**, **G-SRT.C.8**) and expanded on in Algebra II (**F-TF.A.1**, **F-TF.A.2**, **F-TF.C.8**). It also reinforces the geometric interpretation of the modulus of a complex number and introduces the notion of the argument of a complex number.

The theme of Topic C is to highlight the effectiveness of changing notations and the power provided by certain notations such as matrices. By exploiting the connection to trigonometry, students see how much complex arithmetic is simplified. By regarding complex numbers as points in the Cartesian plane, students can begin to write analytic formulas for translations, rotations, and dilations in the plane and revisit the ideas of high school Geometry (**G-CO.A.2**, **G-CO.A.4**, **G-CO.A.5**) in this light. Taking this work one step further, students develop the 2×2 matrix notation for planar transformations represented by complex number arithmetic. This work sheds light on how geometry software and video games efficiently perform rigid motion calculations. Finally, the flexibility implied by 2×2 matrix notation allows students to study additional matrix transformations (shears, for example) that do not necessarily arise from our original complex number thinking context.

In Topic C, the study of vectors and matrices is introduced through a coherent connection to transformations and complex numbers. Students learn to see matrices as representing transformations in the plane and develop understanding of multiplication of a matrix by a vector as a transformation acting on a point in the plane (**N-VM.C.11**, **N-VM.C.12**). While more formal study of multiplication of matrices will occur in Module 2, in Topic C, students are exposed to initial ideas of multiplying 2×2 matrices including a geometric interpretation of matrix invertibility and the meaning of the zero and identity matrices (**N-VM.C.8**, **N-VM.C.10**). **N-VM.C.8** is introduced in a strictly geometric context and is expanded upon more formally in Module 2. **N-VM.C.8** will be assessed secondarily, in the context of other standards but not directly, in the Mid- and End-of-Module Assessments until Module 2.

The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic C.

Focus Standards

Perform arithmetic operations with complex numbers.

N-CN.A.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

N-CN.B.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.B.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $\left(-1+\sqrt{3}i\right)^3 = 8$ *because* $\left(-1+\sqrt{3}i\right)$ *has modulus* 2 *and argument* $120°$.

N-CN.B.6 (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Perform operations on matrices and use matrices in applications.

N-VM.C.8 (+) Add, subtract, and multiply matrices of appropriate dimensions.

N-VM.C.10[2] (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N-VM.C.12 (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

[2] N.VM and G.CO standards are included in the context of defining transformations of the plane rigorously using complex numbers and 2×2 matrices and linking rotations and reflections to multiplication by complex number and/or by 2×2 matrices to show how geometry software and video games work.

Foundational Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.★

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of* P *and a factor not depending on* P.

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is* 15%.

Create equations that describe numbers or relationships.★

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R.

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Experiment with transformations in the plane.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Focus Standards for Mathematical Practice

MP.2 **Reason abstractly and quantitatively.** Students come to recognize that multiplication by a complex number corresponds to the geometric action of a rotation and dilation from the origin in the complex plane. Students apply this knowledge to understand that multiplication by the reciprocal provides the inverse geometric operation to a rotation and dilation. Much of the module is dedicated to helping students quantify the rotations and dilations in increasingly abstract ways so they do not depend on the ability to visualize the transformation. That is, they reach a point where they do not need a specific geometric model in mind to think about a rotation or dilation. Instead, they can make generalizations about the rotation or dilation based on the problems they have previously solved.

MP.3 **Construct viable arguments and critique the reasoning of others.** Throughout the module, students study examples of work by algebra students. This work includes a number of common mistakes that algebra students make, but it is up to the student to decide about the validity of the argument. Deciding on the validity of the argument focuses the students on justification and argumentation as they work to decide when purported algebraic identities do or do not hold. In cases where they decide that the given student work is incorrect, the students work to develop the correct general algebraic results and justify them by reflecting on what they perceived as incorrect about the original student solution.

MP.4 **Model with mathematics.** As students work through the module, they become attuned to the geometric effect that occurs in the context of complex multiplication. However, initially it is unclear to them why multiplication by complex numbers entails specific geometric effects. In the module, the students create a model of computer animation in the plane. The focus of the mathematics in the computer animation is such that the students come to see rotating and translating as dependent on matrix operations and the addition of 2×1 vectors. Thus, their understanding becomes more formal with the notion of complex numbers.

Terminology

New or Recently Introduced Terms

- **Argument** (The *argument* of the complex number z is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray (i.e., the ray corresponding to the positive real axis) to the ray from the origin through the complex number z in the complex plane. The argument of z is denoted $\arg(z)$.)
- **Bound Vector** (A *bound vector* is a directed line segment (an *arrow*). For example, the directed line segment $\overrightarrow{AB}$ is a bound vector whose initial point (or *tail*) is A and terminal point (or *tip*) is B.

  Bound vectors are *bound* to a particular location in space. A bound vector $\overrightarrow{AB}$ has a magnitude given by the length of $\overline{AB}$ and direction given by the ray $\overrightarrow{AB}$. Many times only the magnitude and direction of a bound vector matters, not its position in space. In that case, we consider any translation of that bound vector to represent the same free vector.)
- **Complex Number** (A *complex number* is a number that can be represented by a point in the complex plane. A complex number can be expressed in two forms:
  1. The *rectangular form* of a complex number z is $a + bi$ where z corresponds to the point (a, b) in the complex plane, and i is the imaginary unit. The number a is called the *real part* of $a + bi$ and the number b is called the *imaginary part* of $a + bi$. Note that both the real and imaginary parts of a complex number are themselves real numbers.
  2. For $z \neq 0$, the *polar form* of a complex number z is $r(\cos(\theta) + i\sin(\theta))$ where $r = |z|$ and $\theta = \arg(z)$, and i is the imaginary unit.)
- **Complex Plane** (The *complex plane* is a Cartesian plane equipped with addition and multiplication operators defined on ordered pairs by the following:
  - Addition: $(a, b) + (c, d) = (a + c, b + d)$.

    When expressed in rectangular form, if $z = a + bi$ and $w = c + di$, then $z + w = (a + c) + (b + d)i$.
  - Multiplication: $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

    When expressed in rectangular form, if $z = a + bi$ and $w = c + di$, then $z \cdot w = (ac - bd) + (ad + bc)i$. The horizontal axis corresponding to points of the form $(x, 0)$ is called the real axis, and a vertical axis corresponding to points of the form $(0, y)$ is called the imaginary axis.)
- **Conjugate** (The *conjugate* of a complex number of the form $a + bi$ is $a - bi$. The conjugate of z is denoted $\overline{z}$.)
- **Determinant of 2×2 Matrix** (The *determinant of the* 2×2 *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number computed by evaluating $ad - bc$, and is denoted by $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.)

- **Determinant of 3×3 Matrix** (The *determinant of the* 3×3 *matrix* $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the number computed by evaluating the expression,
$$a_{11} \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12} \det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + a_{13} \det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right),$$
and is denoted by $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$.)
- **Directed Graph** (A *directed graph* is an ordered pair $D = (V, E)$ with
  - V a set whose elements are called *vertices* or *nodes*, and
  - E a set of ordered pairs of vertices, called *arcs* or *directed edges*.)
- **Directed Segment** (A *directed segment* $\overrightarrow{AB}$ is the line segment AB together with a direction given by connecting an initial point A to a terminal point B.)
- **Free Vector** (A *free vector* is the equivalence class of all directed line segments (*arrows*) that are equivalent to each other by translation. For example, scientists often use free vectors to describe physical quantities that have magnitude and direction only, *freely* placing an arrow with the given magnitude and direction anywhere in a diagram where it is needed. For any directed line segment in the equivalence class defining a free vector, the directed line segment is said to be a *representation* of the free vector or is said to *represent* the free vector.)
- **Identity Matrix** (The $n \times n$ *identity matrix* is the matrix whose entry in row i and column i for $1 \leq i \leq n$ is 1, and whose entries in row i and column j for $1 \leq i, j \leq n$ and $i \neq j$ are all zero. The identity matrix is denoted by I.)
- **Imaginary Axis** (See *complex plane*.)
- **Imaginary Number** (An *imaginary number* is a complex number that can be expressed in the form bi where b is a real number.)
- **Imaginary Part** (See *complex number*.)
- **Imaginary Unit** (The *imaginary unit*, denoted by i, is the number corresponding to the point $(0,1)$ in the complex plane.)
- **Incidence Matrix** (The *incidence matrix of a network diagram* is the $n \times n$ matrix such that the entry in row i and column j is the number of edges that start at node i and end at node j.)
- **Inverse Matrix** (An $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix B so that $AB = BA = I$, where I is the $n \times n$ identity matrix. The matrix B, when it exists, is unique and is called the *inverse of* A and is denoted by A^{-1}.)
- **Linear Function** (A function $f: \mathbb{R} \to \mathbb{R}$ is called a *linear function* if it is a polynomial function of degree one; that is, a function with real number domain and range that can be put into the form $f(x) = mx + b$ for real numbers m and b. A linear function of the form $f(x) = mx + b$ is a linear transformation only if $b = 0$.)

- **Linear Transformation** (A function $L: \mathbb{R}^n \to \mathbb{R}^n$ for a positive integer n is a *linear transformation* if the following two properties hold:
  - $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and
  - $L(k\mathbf{x}) = k \cdot L(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{R}$,

  where $\mathbf{x} \in \mathbb{R}^n$ means that $\mathbf{x}$ is a point in $\mathbb{R}^n$.)
- **Linear Transformation Induced by Matrix A** (Given a 2×2 matrix A, the *linear transformation induced by matrix A* is the linear transformation L given by the formula $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$. Given a 3×3 matrix A, the *linear transformation induced by matrix A* is the linear transformation L given by the formula $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.)
- **Matrix** (An $m \times n$ *matrix* is an ordered list of nm real numbers, $a_{11}, a_{12}, \dots, a_{1n}, a_{21}, a_{22}, \dots a_{2n}, \dots, a_{m1}, a_{m2}, \dots a_{mn}$, organized in a rectangular array of m rows and n columns: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$. The number a_{ij} is called the *entry in row i and column j.)*
- **Matrix Difference** (Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij} and let B be an $m \times n$ matrix whose entry in row i and column j is b_{ij}. Then the *matrix difference* $A - B$ is the $m \times n$ matrix whose entry in row i and column j is $a_{ij} - b_{ij}$.)
- **Matrix Product** (Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij} and let B be an $n \times p$ matrix whose entry in row i and column j is b_{ij}. Then the *matrix product* AB is the $m \times p$ matrix whose entry in row i and column j is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.)
- **Matrix Scalar Multiplication** (Let k be a real number and let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}. Then the *scalar product* $k \cdot A$ is the $m \times n$ matrix whose entry in row i and column j is $k \cdot a_{ij}$.)
- **Matrix Sum** (Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij} and let B be an $m \times n$ matrix whose entry in row i and column j is b_{ij}. Then the *matrix sum* $A + B$ is the $m \times n$ matrix whose entry in row i and column j is $a_{ij} + b_{ij}$.)
- **Modulus** (The *modulus* of a complex number z, denoted $|z|$, is the distance from the origin to the point corresponding to z in the complex plane. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.)
- **Network Diagram** (A *network diagram* is a graphical representation of a directed graph where the n vertices are drawn as circles with each circle labeled by a number 1 through n, and the directed edges are drawn as segments or arcs with arrow pointing from the tail vertex to the head vertex.)

- **Opposite Vector** (For a vector $\vec{v}$ represented by the directed line segment $\overrightarrow{AB}$, the *opposite vector*, denoted $-\vec{v}$, is the vector represented by the directed line segment $\overrightarrow{BA}$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $-\vec{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}$.)
- **Polar Form of a Complex Number** (The *polar form of a complex number* z is $r(\cos(\theta) + i\sin(\theta))$ where $r = |z|$ and $\theta = \arg(z)$.)
- **Position Vector** (For a point $P(v_1, v_2, \dots, v_n)$ in $\mathbb{R}^n$, the *position vector* $\vec{v}$, denoted by $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ or $\langle v_1, v_2, \dots, v_n \rangle$, is a free vector $\vec{v}$ that is represented by the directed line segment $\overrightarrow{OP}$ from the origin $O(0,0,0,\dots,0)$ to the point P. The real number v_i is called the i^{th} *component* of the vector $\vec{v}$.)
- **Real Coordinate Space** (For a positive integer n, the n-dimensional *real coordinate space*, denoted $\mathbb{R}^n$, is the set of all n-tuple of real numbers equipped with a distance function d that satisfies
$$d[(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)] = \sqrt{(y_1 - x_1)^2 + (y_2 - x_1)^2 + \dots + (y_n - x_n)^2}$$
for any two points in the space. One-dimensional real coordinate space is called a *number line* and the two-dimensional real coordinate space is called the *Cartesian plane.)*
- **Rectangular Form of a Complex Number** (The *rectangular form of a complex number* z is $a + bi$ where z corresponds to the point (a, b) in the complex plane, and i is the imaginary unit. The number a is called the *real part* of $a + bi$ and the number b is called the *imaginary part* of $a + bi$.)
- **Translation by a Vector in Real Coordinate Space** (A *translation by a vector* $\vec{v}$ *in* $\mathbb{R}^n$ is the translation transformation $T_{\vec{v}}: \mathbb{R}^n \to \mathbb{R}^n$ given by the map that takes $\vec{x} \mapsto \vec{x} + \vec{v}$ for all $\vec{x}$ in $\mathbb{R}^n$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $T_{\vec{v}}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \\ \vdots \\ x_n + v_n \end{bmatrix}$ for all $\vec{x}$ in $\mathbb{R}^n$.)

- **Vector Addition** (For vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, the sum $\vec{v}+\vec{w}$ is the vector whose i^{th} component is the sum of the i^{th} components of $\vec{v}$ and $\vec{w}$ for $1 \le i \le n$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, then $\vec{v}+\vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$.)
- **Vector Subtraction** (For vectors $\vec{v}$ and $\vec{w}$, the difference $\vec{v}-\vec{w}$ is the sum of $\vec{v}$ and the opposite of $\vec{w}$; that is, $\vec{v}-\vec{w} = \vec{v}+(-\vec{w})$. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, then $\vec{v}-\vec{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \\ \vdots \\ v_n - w_n \end{bmatrix}$.)
- **Vector Magnitude** (The *magnitude* or *length* of a vector $\vec{v}$, denoted $|\vec{v}|$ or $\|\vec{v}\|$, is the length of any directed line segment that represents the vector. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$, which is the distance from the origin to the associated point $P(v_1, v_2, \ldots, v_n)$.)
- **Vector Scalar Multiplication** (For a vector $\vec{v}$ in $\mathbb{R}^n$ and a real number k, the scalar product $k \cdot \vec{v}$ is the vector whose i^{th} component is the product of k and the i^{th} component of $\vec{v}$ for $1 \le i \le n$. If k is a real number and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $k \cdot \vec{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}$.)
- **Vector Representation of a Complex Number** (The *vector representation of a complex number* z is the position vector $\vec{z}$ associated to the point z in the complex plane. If $z = a + bi$ for two real numbers a and b, then $\vec{z} = \begin{bmatrix} a \\ b \end{bmatrix}$.)
- **Zero Matrix** (The $m \times n$ *zero matrix* is the $m \times n$ matrix in which all entries are equal to zero. For example, the 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.)
- **Zero Vector** (The *zero vector* in $\mathbb{R}^n$ is the vector in which each component is equal to zero. For example, the zero vector in $\mathbb{R}^2$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the zero vector in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.)

Familiar Terms and Symbols[3]

- Dilation
- Rectangular Form
- Rotation
- Translation

Suggested Tools and Representations

- Geometer's Sketchpad software
- Graphing calculator
- Wolfram Alpha software

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-CN.A.3, N-CN.B.4, N-CN.B.5, N-CN.B.6
End-of-Module Assessment Task	After Topic C	Constructed response with rubric	N-CN.B.4, N-CN.B.5, N-VM.C.8, N-VM.C.10, N-VM.C.11, N-VM.C.12

[3] These are terms and symbols students have seen previously.