Lesson 1: Introduction to Networks

Classwork

Exploratory Challenge 1

A network diagram depicts interrelated objects by circles that represent the objects and directed edges drawn as segments or arcs between related objects with arrows to denote direction. The network diagram below shows the bus routes that run between four cities, forming a network. The arrows indicate the direction the buses travel.
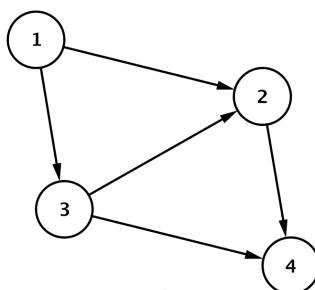

Figure 1

How many ways can you travel from City 1 to City 4? Explain how you know.

What about these bus routes doesn't make sense?

It turns out there was an error in printing the first route map. An updated network diagram showing the bus routes that connect the four cities is shown below in Figure 2. Arrows on both ends of an edge indicate that buses travel in both directions.
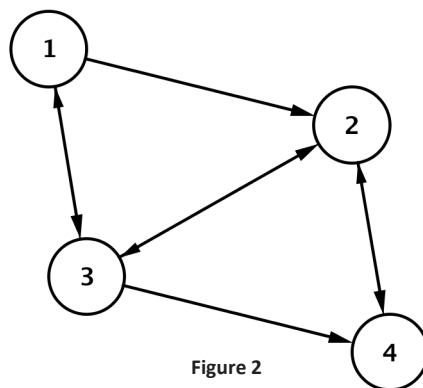

Figure 2

How many ways can you reasonably travel from City 4 to City 1 using the route map in Figure 2? Explain how you know.

Exploratory Challenge 2

A rival bus company offers more routes connecting these four cities as shown in the network diagram in Figure 3.
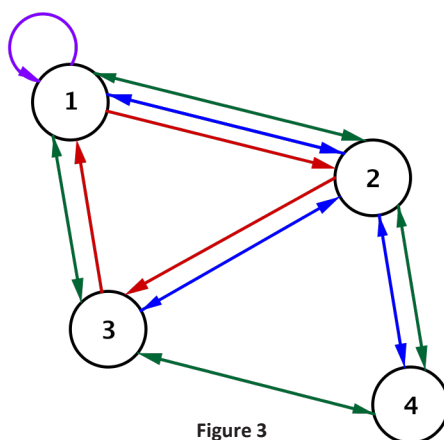

Figure 3

What might the loop at City 1 represent?

How many ways can you travel from City 1 to City 4 if you want to stop in City 2 and make no other stops?

How many possible ways are there to travel from City 1 to City 4 without repeating a city?

Exploratory Challenge 3

We will consider a "direct route" to be a route from one city to another without going through any other city. Organize the number of direct routes from each city into the table shown below. The first row showing the direct routes between City 1 and the other cities is complete for you.

Destination Cities

| Cities of Origin	1	2	3	4
1	1	3	1	0
2				
3				
4				

Exercises 1–7

1. Use the network diagram in Figure 3 to represent the number of direct routes between the four cities in a matrix R.

2. What is the value of $r_{2,3}$? What does it represent in this situation?

3. What is the value of $r_{2,3} \cdot r_{3,1}$, and what does it represent in this situation?

4. Write an expression for the total number of one-stop routes from City 4 and City 1, and determine the number of routes stopping in one city.

5. Do you notice any patterns in the expression for the total number of one-stop routes from City 4 and City 1?

6. Create a network diagram for the matrices shown below. Each matrix represents the number of transportation routes that connect four cities. The rows are the cities you travel from, and the columns are the cities you travel to.

a. $R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

b. $R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

Here is a type of network diagram called an arc diagram.
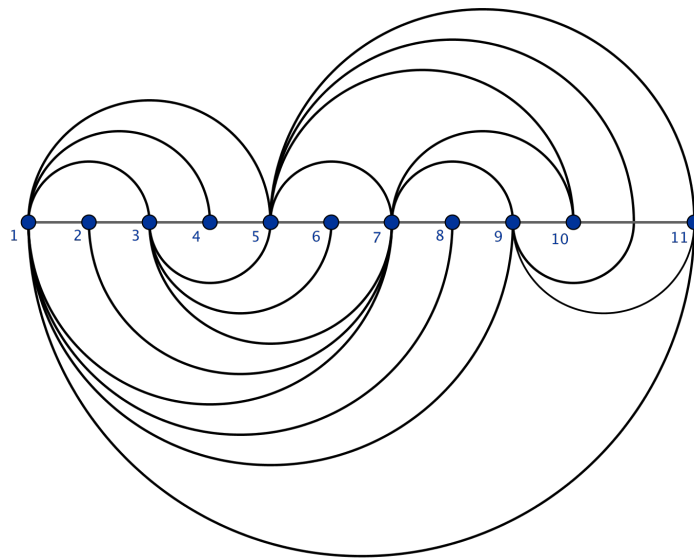
Suppose the points represent eleven students in your mathematics class, numbered 1 through 11. Suppose the arcs above and below the line of vertices 1–11 are the people who are friends on a social network.

7. Complete the matrix that shows which students are friends with each other on this social network. The first row has been completed for you.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix}$$

Number 1 is not friends with number 10. How many ways could number 1 get a message to 10 by only going through one other friend?

a.   Who has the most friends in this network? Explain how you know.

b.   Is everyone in this network connected at least as a friend of a friend? Explain how you know.

c.   What is entry $A_{2,3}$? Explain its meaning in this context.

> **Lesson Summary**
>
> Students organize data and use matrices to represent data in an organized way.
>
> A *network diagram* is a graphical representation of a directed graph where the n vertices are drawn as circles with each circle labeled by a number 1 through n, and the directed edges are drawn as segments or arcs with an arrow pointing from the tail vertex to the head vertex.
>
> A *matrix* is an array of numbers organized into m rows and n columns. A matrix containing m rows and n columns has dimensions $m \times n$. The entry in the first row and first column is referred to as $a_{1,1}$. In general, the entry in the ith row and j^{th} column would be denoted $a_{i,j}$.

Problem Set

1. Consider the railroad map between Cities 1, 2, and 3, as shown.

    a. Create a matrix R to represent the railroad map between Cities 1, 2, and 3.

    b. How many different ways can you travel from City 1 to City 3 without passing through the same city twice?

    c. How many different ways can you travel from City 2 to City 3 without passing through the same city twice?

    d. How many different ways can you travel from City 1 to City 2 with exactly one connecting stop?

    e. Why is this not a reasonable network diagram for a railroad?
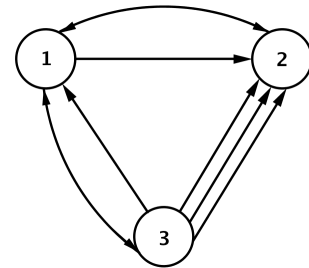
2. Consider the subway map between stations 1, 2, and 3, as shown.

    a. Create a matrix S to represent the subway map between stations 1, 2, and 3.

    b. How many different ways can you travel from station 1 to station 3 without passing through the same station twice?

    c. How many different ways can you travel directly from station 1 to station 3 with no stops?

    d. How many different ways can you travel from station 1 to station 3 with exactly one stop?

    e. How many different ways can you travel from station 1 to station 3 with exactly two stops? Allow for stops at repeated stations.
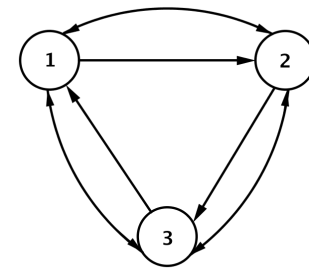
3. Suppose the matrix R represents a railroad map between cities 1, 2, 3, 4, and 5.

$$R = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 3 & 0 & 0 \end{bmatrix}$$

   a. How many different ways can you travel from City 1 to City 3 with exactly one connection?

   b. How many different ways can you travel from City 1 to City 5 with exactly one connection?

   c. How many different ways can you travel from City 2 to City 5 with exactly one connection?

4. Let $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ represent the bus routes between 3 cities.

   a. Draw an example of a network diagram represented by this matrix.

   b. Calculate the matrix $B^2 = BB$.

   c. How many routes are there between City 1 and City 2 with one stop in between?

   d. How many routes are there between City 2 and City 2 with one stop in between?

   e. How many routes are there between City 3 and City 2 with one stop in between?

   f. What is the relationship between your answers to parts (b)-(e)? Formulate a conjecture.

5. Consider the airline flight routes between Cities 1, 2, 3, and 4, as shown.
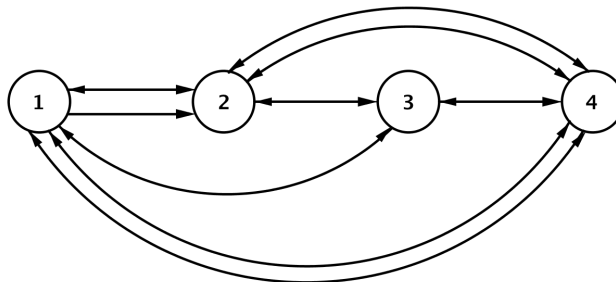
   a. Create a matrix F to represent the flight map between Cities 1, 2, 3, and 4.

   b. How many different routes can you take from City 1 to City 4 with no stops?

   c. How many different routes can you take from City 1 to City 4 with exactly one stop?

   d. How many different routes can you take from City 3 to City 4 with exactly one stop?

   e. How many different routes can you take from City 1 to City 4 with exactly two stops? Allow for routes that include repeated cities.

   f. How many different routes can you take from City 2 to City 4 with exactly two stops? Allow for routes that include repeated cities.

6. Consider the following directed graph representing the number of ways Trenton can get dressed in the morning (only visible options are shown):
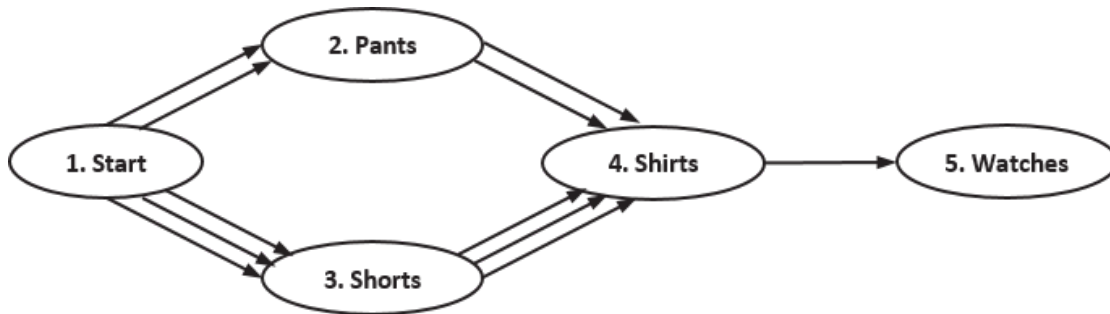
   a. What reasons could there be for there to be three choices for shirts after "traveling" to shorts but only two after traveling to pants?

   b. What could the order of the vertices mean in this situation?

   c. Write a matrix A representing this directed graph.

   d. Delete any rows of zeros in matrix A, and write the new matrix as matrix B. Does deleting this row change the meaning of any of the entries of B? If you had deleted the first column, would the meaning of the entries change? Explain.

   e. Calculate $b_{1,2} \cdot b_{2,4} \cdot b_{4,5}$. What does this product represent?

   f. How many different outfits can Trenton wear assuming he always wears a watch?

7. Recall the network representing bus routes used at the start of the lesson:
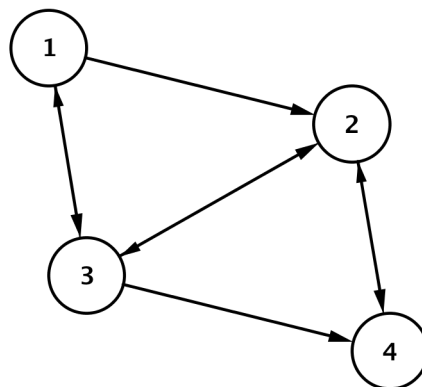
   Faced with competition from rival companies, you have been tasked with considering the option of building a toll road going directly from City 1 to City 4. Once built, the road will provide income in the form of tolls and also enable the implementation of a non-stop bus route to and from City 1 and City 4.

Analysts have provided you with the following information (values are in millions of dollars):

	Start-up costs (expressed as profit)	Projected minimum profit per year	Projected maximum profit per year
Road	−$63	$65	$100
New bus route	−$5	$0.75	$1.25

a. Express this information in a matrix P.

b. What are the dimensions of the matrix?

c. Evaluate $p_{1,1} + p_{1,2}$. What does this sum represent?

d. Solve $p_{1,1} + t \cdot p_{1,2} = 0$ for t. What does the solution represent?

e. Solve $p_{1,1} + t \cdot p_{1,3} = 0$ for t. What does the solution represent?

f. Summarize your results to part (d) and (e).

g. Evaluate $p_{1,1} + p_{2,1}$. What does this sum represent?

h. Solve $p_{1,1} + p_{2,1} + t(p_{1,2} + p_{2,2}) = 0$ for t. What does the solution represent?

i. Make your recommendation. Should the company invest in building the toll road or not? If they build the road, should they also put in a new bus route? Explain your answer.

Lesson 2: Networks and Matrix Arithmetic

Classwork

Opening Exercise

Suppose a subway line also connects the four cities. Here is the subway and bus line network. The bus routes connecting the cities are represented by solid lines, and the subway routes are represented by dashed arcs.
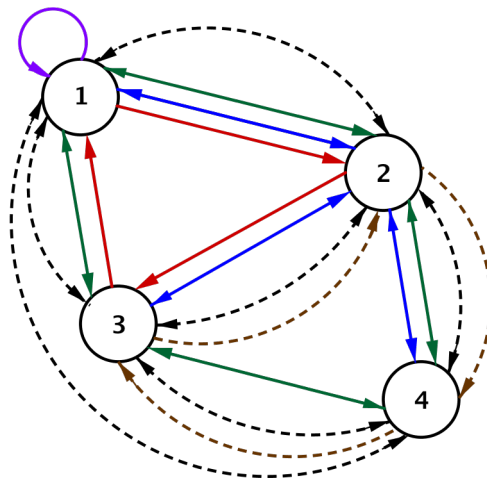
Write a matrix B to represent the bus routes and a matrix S to represent the subway lines connecting the four cities.

Exploratory Challenge/Exercises 1–6: Matrix Arithmetic

Use the network diagram from the Opening Exercise and your answers to help you complete this challenge with your group.

1. Suppose the number of bus routes between each city were doubled.

    a.   What would the new bus route matrix be?

    b.   Mathematicians call this matrix $2B$. Why do you think they call it that?

2.   What would be the meaning of $10B$ in this situation?

3.   Write the matrix $10B$.

4.   Ignore whether or not a line connecting cities represents a bus or subway route.

    a.   Create one matrix that represents all the routes between the cities in this transportation network.

    b.   Why would it be appropriate to call this matrix $B + S$? Explain your reasoning.

5.   What would be the meaning of $4B + 2S$ in this situation?

6.   Write the matrix $4B + 2S$. Show work and explain how you found your answer.

Exercise 7

7. Complete this graphic organizer.

Matrix Operations Graphic Organizer

Operation	Symbols	Describe How to Calculate	Example Using 3×3 Matrices
Scalar Multiplication	kA		
The Sum of Two Matrices	$A + B$		
The Difference of Two Matrices	$A - B$ $= A + (-1)B$		

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Lesson Summary

MATRIX SCALAR MULTIPLICATION: Let k be a real number, and let A be an $m \times n$ matrix whose entry in row i and column j is $a_{i,j}$. Then the *scalar product* $k \cdot A$ is the $m \times n$ matrix whose entry in row i and column j is $k \cdot a_{i,j}$.

MATRIX SUM: Let A be an $m \times n$ matrix whose entry in row i and column j is $a_{i,j}$, and let B be an $m \times n$ matrix whose entry in row i and column j is $b_{i,j}$. Then the *matrix sum $A + B$* is the $m \times n$ matrix whose entry in row i and column j is $a_{i,j} + b_{i,j}$.

MATRIX DIFFERENCE: Let A be an $m \times n$ matrix whose entry in row i and column j is $a_{i,j}$, and let B be an $m \times n$ matrix whose entry in row i and column j is $b_{i,j}$. Then the *matrix difference $A - B$* is the $m \times n$ matrix whose entry in row i and column j is $a_{i,j} - b_{i,j}$.

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Problem Set

1. For the matrices given below, perform each of the following calculations or explain why the calculation is not possible.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 2 & 9 \\ 6 & 1 & 3 \\ -1 & 1 & 0 \end{bmatrix} \qquad\qquad D = \begin{bmatrix} 1 & 6 & 0 \\ 3 & 0 & 2 \\ 1 & 3 & -2 \end{bmatrix}$$

a. $A + B$

b. $2A - B$

c. $A + C$

d. $-2C$

e. $4D - 2C$

f. $3B - 3B$

g. $5B - C$

h. $B - 3A$

i. $C + 10D$

j. $\dfrac{1}{2}C + D$

k. $\dfrac{1}{4}B$

l. $3D - 4A$

m. $\dfrac{1}{3}B - \dfrac{2}{3}A$

2. For the matrices given below, perform each of the following calculations or explain why the calculation is not possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & -1 \\ -1 & 0 \\ 4 & 1 \end{bmatrix}$$

a. $A + 2B$

b. $2A - C$

c. $A + C$

d. $-2C$

e. $4D - 2C$

f. $3D - 3D$

g. $5B - D$

h. $C - 3A$

i. $B + 10D$

j. $\dfrac{1}{2}C + A$

k. $\dfrac{1}{4}B$

l. $3A + 3B$

m. $\dfrac{1}{3}B - \dfrac{2}{3}D$

3. Let

$$A = \begin{bmatrix} 3 & \dfrac{2}{3} \\ -1 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 3 \\ 2 & \dfrac{3}{2} \\ 4 & 1 \end{bmatrix}$$

a. Let $C = 6A + 6B$. Find matrix C.

b. Let $D = 6(A + B)$. Find matrix D.

c. What is the relationship between matrices C and D? Why do you think that is?

4. Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ 3 & -4 \end{bmatrix}$ and X be a 3×2 matrix. If $A + X = \begin{bmatrix} -2 & 3 \\ 4 & 1 \\ 1 & -5 \end{bmatrix}$, then find X.

5. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ represent the bus routes of two companies between three cities.

   a. Let $C = A + B$. Find matrix C. Explain what the resulting matrix and entry $c_{1,3}$ mean in this context.

   b. Let $D = B + A$. Find matrix D. Explain what the resulting matrix and entry $d_{1,3}$ mean in this context.

   c. What is the relationship between matrices C and D? Why do you think that is?

6. Suppose that April's Pet Supply has three stores in Cities 1, 2, and 3. Ben's Pet Mart has two stores in Cities 1 and 2. Each shop sells the same type of dog crates in size 1 (small), 2 (medium), 3 (large), and 4 (extra large).

   April's and Ben's inventory in each city are stored in the tables below.

April's Pet Supply			
	City 1	City 2	City 3
Size 1	3	5	1
Size 2	4	2	9
Size 3	1	4	2
Size 4	0	0	1

Ben's Pet Mart		
	City 1	City 2
Size 1	2	3
Size 2	0	2
Size 3	4	1
Size 4	0	0

   a. Create a matrix A so that $a_{i,j}$ represents the number of crates of size i available in April's store j.

   b. Explain how the matrix $B = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ can represent the dog crate inventory at Ben's Pet Mart.

   c. Suppose that April and Ben merge their inventories. Find a matrix that represents their combined inventory in each of the three cities.

7. Jackie has two businesses she is considering buying and a business plan that could work for both. Consider the tables below, and answer the questions following.

| Horus's One-Stop Warehouse Supply				Re's 24-Hour Distributions		
	If business stays the same	If business improves as projected			If business stays the same	If business improves as projected
Expand to Multiple States	−$75,000,000	$45,000,000		Expand to Multiple States	−$99,000,000	$62,500,000
Invest in Drone Delivery	−$33,000,000	$30,000,000		Invest in Drone Delivery	−$49,000,000	$29,000,000
Close and Sell Out	$20,000,000	$20,000,000		Close and Sell Out	$35,000,000	$35,000,000

a. Create matrices H and R representing the values in the tables above such that the rows represent the different options and the columns represent the different outcomes of each option.

b. Calculate $R - H$. What does $R - H$ represent?

c. Calculate $H + R$. What does $H + R$ represent?

d. Jackie estimates that the economy could cause fluctuations in her numbers by as much as 5% both ways. Find matrices to represent the best and worst case scenarios for Jackie.

e. Which business should Jackie buy? Which of the three options should she choose? Explain your choices.

Lesson 3: Matrix Arithmetic in its Own Right

Classwork

Opening Exercise

The subway and bus line network connecting four cities that we used in Lesson 2 is shown below. The bus routes connecting the cities are represented by solid lines, and the subway routes are represented by dashed lines.
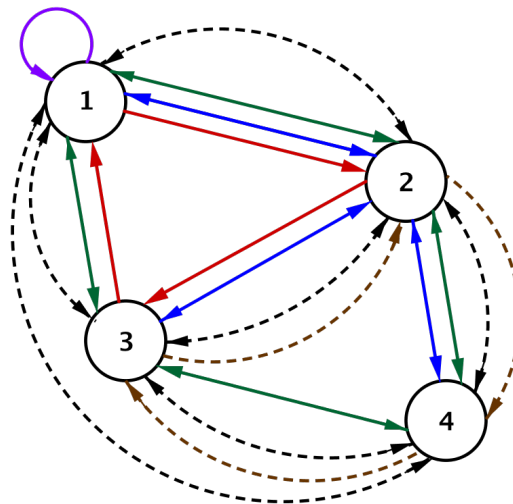
Suppose we want to travel from City 2 to City 1, first by bus and then by subway, with no more than one connecting stop.

a. Complete the chart below showing the number of ways to travel from City 2 to City 1 using first a bus and then the subway. The first row has been completed for you.

First Leg (BUS)		Second Leg (SUBWAY)		Total Ways to Travel
City 2 to City 1:	2	City 1 to City 1:	0	$2 \cdot 0 = 0$
City 2 to City 2:		City 2 to City 1:		
City 2 to City 3:		City 3 to City 1:		
City 2 to City 4:		City 4 to City 1:		

b. How many ways are there to travel from City 2 to City 1, first on a bus and then on a subway? How do you know?

Exploratory Challenge/Exercises 1–12: The Meaning of Matrix Multiplication

Suppose we want to travel between all cities, traveling first by bus and then by subway, with no more than one connecting stop.

1. Use a chart like the one in the Opening Exercise to help you determine the total number of ways to travel from City 1 to City 4 using first a bus and then the subway.

2. Suppose we create a new matrix P to show the number of ways to travel between the cities, first by bus and then by subway, with no more than one connecting stop. Record your answers to Opening Exercises, part (b) and Exercise 1 in this matrix in the appropriate row and column location. We do not yet have enough information to complete the entire matrix. Explain how you decided where to record these numbers in the matrix shown below.

$$P = \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{bmatrix}$$

3. What is the total number of ways to travel from City 3 to City 2 first by bus and then by subway with no more than one connecting stop? Explain how you got your answer and where you would record it in matrix P above.

Recall matrix B, which shows the number of bus lines connecting the cities in this transportation network, and matrix S, which represents the number of subway lines connecting the cities in this transportation network.

$$B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

4. What does the product $b_{1,2}s_{2,4}$ represent in this situation? What is the value of this product?

5. What does $b_{1,4}s_{4,4}$ represent in this situation? What is the value of this product? Does this make sense?

6. Calculate the value of the expression $b_{1,1}s_{1,4} + b_{1,2}s_{2,4} + b_{1,3}s_{3,4} + b_{1,4}s_{4,4}$. What is the meaning of this expression in this situation?

7. Circle the first row of B and the second column of S. How are these entries related to the expression above and your work in Exercise 1?

8. Write an expression that represents the total number of ways you can travel between City 2 and City 1, first by bus and then by subway, with no more than one connecting stop. What is the value of this expression? What is the meaning of the result?

9. Write an expression that represents the total number of ways you can travel between City 4 and City 1, first by bus and then by subway, with no more than one connecting stop. What is the value of this expression? What is the meaning of the result?

10. Complete matrix P that represents the routes connecting the four cities if you travel first by bus and then by subway.

11. Construct a matrix M that represents the routes connecting the four cities if you travel first by subway and then by bus.

12. Should these two matrices be the same? Explain your reasoning.

Exercises 13–16

13. Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$

    a. Construct a matrix Z such that $A + Z = A$. Explain how you got your answer.

    b. Explain why $k \cdot Z = Z$ for any real number k.

    c. The real number 0 has the properties that $a + 0 = 0$ and $a \cdot 0 = 0$ for all real numbers a. Why would mathematicians call Z a zero matrix?

14. Suppose each city had a trolley car that ran a route between tourist destinations. The blue loops represent the trolley car routes. Remember that straight lines indicate bus routes, and dotted lines indicate subway routes.
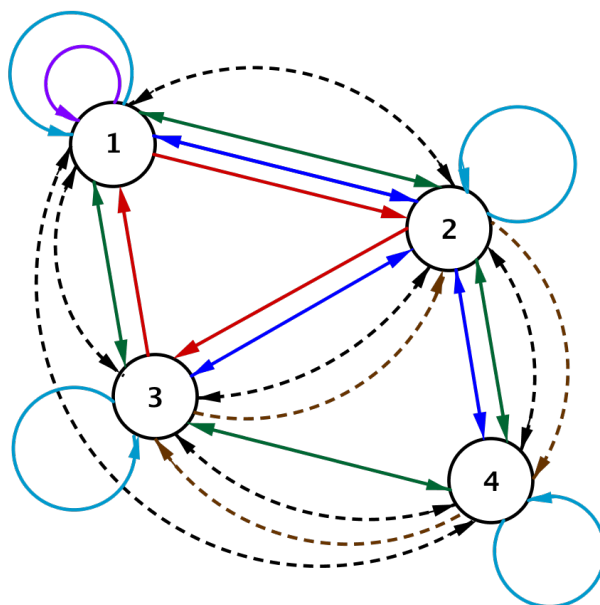
a. Explain why the matrix I shown below would represent the number of routes connecting cities by trolley car in this transportation network.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b. Recall that B is the bus route matrix. Show that $I \cdot B = B$. Explain why this makes sense in terms of the transportation network.

c. The real number 1 has the property that $1 \cdot a = a$ for all real numbers a, and we call 1 the *multiplicative identity.* Why would mathematicians call I an identity matrix?

d. What would be the form of a 2×2 identity matrix? What about a 3×3 identity matrix?

15. In this lesson you learned that the commutative property does not hold for matrix multiplication. This exercise asks you to consider other properties of real numbers applied to matrix arithmetic.

a. Is matrix addition associative? That is, does $(A + B) + C = A + (B + C)$ for matrices A, B, and C that have the same dimensions? Explain your reasoning.

b.   Is matrix multiplication associative? That is, does $(A \cdot B) \cdot C := A \cdot (B \cdot C)$ for matrices A, B, and C for which the multiplication is defined? Explain your reasoning.

c.   Is matrix addition commutative? That is, does $A + B = B + A$ for matrices A and B with the same dimensions?

16. Complete the graphic organizer to summarize your understanding of the product of two matrices.

Operation	Symbols	Describe How to Calculate	Example Using 3×3 Matrices
Matrix Multiplication	$A \cdot B$		

Lesson Summary

MATRIX PRODUCT: Let A be an $m \times n$ matrix whose entry in row i and column j is $a_{i,j}$, and let B be an $n \times p$ matrix whose entry in row i and column j is $b_{i,j}$. Then the matrix product AB is the $m \times p$ matrix whose entry in row i and column j is $a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j}$.

IDENTITY MATRIX: The $n \times n$ *identity matrix* is the matrix whose entry in row i and column i for $1 \le i \le n$ is 1, and whose entries in row i and column j for $1 \le i, j \le n$ and $i \ne j$ are all zero. The identity matrix is denoted by I. The 2×2 identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the 3×3 identity matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If the size of the identity matrix is not explicitly stated, then the size is implied by context.

ZERO MATRIX: The $m \times n$ *zero matrix* is the $m \times n$ matrix in which all entries are equal to zero. For example, the 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If the size of the zero matrix is not specified explicitly, then the size is implied by context.

Problem Set

1. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ represent the bus routes of two companies between 2 cities. Find the product $A \cdot B$, and explain the meaning of the entry in row 1, column 2 of $A \cdot B$ in the context of this scenario.

2. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ represent the bus routes of two companies between three cities.

   a. Let $C = A \cdot B$. Find matrix C, and explain the meaning of entry $c_{1,3}$.

   b. Nina wants to travel from City 3 to City 1 and back home to City 3 by taking a direct bus from Company A on the way to City 1 and a bus from Company B on the way back home to City 3. How many different ways are there for her to make this trip?

   c. Oliver wants to travel from City 2 to City 3 by taking first a bus from Company A and then taking a bus from Company B. How many different ways can he do this?

   d. How many routes can Oliver choose from if travels from City 2 to City 3 by first taking a bus from Company B and then taking a bus from Company A?

3. Recall the bus and trolley matrices from the lesson:

$$B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a. Explain why it makes sense that $BI = IB$ in the context of the problem.

b. Multiply out BI and IB to show $BI = IB$.

c. Consider the multiplication that you did in part (b). What about the arrangement of the entries in the identity matrix causes $BI = B$?

4. Consider the matrices

$$A = \begin{bmatrix} 3 & 1 & -\dfrac{1}{2} \\ 2 & \dfrac{2}{3} & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

a. Multiply AB and BA or explain why you cannot.

b. Would you consider B to be an identity matrix for A? Why or why not?

c. Would you consider $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ an identity matrix for A? Why or why not?

5. We've shown that matrix multiplication is generally not commutative, meaning that as a general rule for two matrices A and B, $A \cdot B \neq B \cdot A$. Explain why $F \cdot G = G \cdot F$ in each of the following examples.

a. $F = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 2 & 6 \\ 4 & 0 \end{bmatrix}$.

b. $F = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c. $F = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

d. $F = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

6. Let I_n be the $n \times n$ identity matrix. For the matrices given below, perform each of the following calculations or explain why the calculation is not possible:

$$A = \begin{bmatrix} \frac{1}{2} & 3 \\ 2 & \frac{2}{3} \end{bmatrix} \qquad B = \begin{bmatrix} 9 & -1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & \sqrt{2} & -2 & \frac{1}{2} \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

a. AB

b. BA

c. AC

d. ABC

e. $ABCD$

f. AD

g. A^2

h. C^2

i. BC^2

j. $ABC + AD$

k. ABI_2

l. AI_2B

m. CI_3B

n. I_2BC

o. $2A + B$

p. $B(I_3 + C)$

q. $B + BC$

r. $4DI_4$

7. Let F be an $m \times n$ matrix. Then what do you know about the dimensions of matrix G in the problems below if each expression has a value?

a. $F + G$

b. FG

c. GF

d. FHG for some matrix H.

8. Consider an $m \times n$ matrix A such that $m \neq n$. Explain why you cannot evaluate A^2.

9. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ represent the routes of three airlines A, B, and C between three cities.

   a. Zane wants to fly from City 1 to City 3 by taking Airline A first and then Airline B second. How many different ways are there for him to travel?

   b. Zane did not like Airline A after the trip to City 3, so on the way home, Zane decides to fly Airline C first and then Airline B second. How many different ways are there for him to travel?

10. Let $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$ represent airline flights of one airline between 4 cities.

   a. We use the notation A^2 to represent the product $A \cdot A$. Calculate A^2. What do the entries in matrix A^2 represent?

   b. Jade wants to fly from City 1 to City 4 with exactly one stop. How many different ways are there for her to travel?

   c. Now Jade wants to fly from City 1 to City 4 with exactly two stops. How many different ways are there for her to choose?

Lesson 4: Linear Transformations Review

Classwork

Exercises 1–2

1. Describe the geometric effect of each mapping.

   a. $L(x) = 9 \cdot x$

   b. $L(x) = -\frac{1}{2} \cdot x$

2. Write the formula for the mappings described.

   a. A dilation that expands each interval to 5 times its original size.

   b. A collapse of the interval to the number 0.

Problem Set

1. Suppose you have a linear transformation $L: \mathbb{R} \to \mathbb{R}$, where $L(3) = 6, L(5) = 10$.
   a. Use the addition property to find $L(6), L(8), L(10)$, and $L(13)$.
   b. Use the multiplication property to find $L(15), L(18)$, and $L(30)$.
   c. Find $L(-3), L(-8)$, and $L(-15)$
   d. Find the formula for $L(x)$.
   e. Draw the graph of the function $L(x)$.

2. A linear transformation $L: \mathbb{R} \to \mathbb{R}$ must have the form of $L(x) = ax$ for some real number a. Consider the interval $[-5,2]$. Describe the geometric effect of the following, and find the new interval.
   a. $L(x) = 5x$
   b. $L(x) = -2x$

3. A linear transformation $L: \mathbb{R} \to \mathbb{R}$ must have the form of $L(x) = ax$ for some real number a. Consider the interval $[-2,6]$. Write the formula for the mapping described, and find the new interval.
   a. A reflection over the origin.
   b. A dilation with a scale of $\sqrt{2}$.
   c. A reflection over the origin and a dilation with a scale of $\frac{1}{2}$.
   d. A collapse of the interval to the number 0.

4. In Module 1, we used 2×2 matrices to do transformations on a square, such as a pure rotation, a pure reflection, a pure dilation, and a rotation with a dilation. Now use those matrices to do transformations on this complex number: $z = 2 + i$. For each transformation below, graph your answers.
   a. A pure dilation with a factor of 2.
   b. A pure $\frac{\pi}{2}$ radians counterclockwise rotation about the origin.
   c. A pure π radians counterclockwise rotation about the origin.
   d. A pure reflection about the real axis.
   e. A pure reflection about the imaginary axis.
   f. A pure reflection about the line $y = x$.
   g. A pure reflection about the line $y = -x$.

5. Wesley noticed that by multiplying the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by a complex number z produces a pure $\frac{\pi}{2}$ radians counterclockwise rotation, and multiplying by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ produces a pure dilation with a factor of 2. So, he thinks he can add these two matrices, which will produce $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and will rotate z by $\frac{\pi}{2}$ radians counterclockwise and dilate z with a factor of 2. Is he correct? Explain your reason.

6. In Module 1, we learned that there is not any real number that will satisfy $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$, which is the addtition property of linear transformation. However, we discussed that some fixed complex numbers might work. Can you find two pairs of complex numbers that will work? Show you work.

7. Suppose L is a complex-number function that satisfies the dream conditions: $L(z + w) = L(z) + L(w)$ and $L(kz) = k(z)$ for all complex numbers z, w, and k. Show $L(z) = mz$ for a fixed complex-number m, the only type of complex-number function that satisfies these conditions?

8. For complex numbers, the linear transformation requires $L(x + y) = L(x) + L(y)$, $L(a \cdot x) = a \cdot x$. Prove that in general $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is a linear transformation, where $\begin{pmatrix} x \\ y \end{pmatrix}$ represents $z = x + yi$.

Lesson 5: Coordinates of Points in Space

Classwork

Opening Exercise

Compute:

a. $(-10 + 9i) + (7 - 5i)$

b. $5 \cdot (2 + 3i)$

c. $\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

d. $-2 \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

Exercises

1.   Let $\mathbf{x} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Compute $\mathbf{z} = \mathbf{x} + \mathbf{y}$, and draw the associated parallelogram.

2. Let $\mathbf{x} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Compute $\mathbf{z} = \mathbf{x} + \mathbf{y}$, and draw the associated parallelogram.

3. Let $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$. Compute $\mathbf{z} = \mathbf{x} + \mathbf{y}$, and draw the associated parallelogram.

4. Let $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Compute $\mathbf{z} = 2\mathbf{x}$, and plot $\mathbf{x}$ and $\mathbf{z}$ in the plane.

5. Let $\mathbf{x} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$. Compute $\mathbf{z} = \frac{1}{3}\mathbf{x}$, and plot $\mathbf{x}$ and $\mathbf{z}$ in the plane.

6. Let $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Compute $\mathbf{z} = -3\mathbf{x}$, and plot $\mathbf{x}$ and $\mathbf{z}$ in the plane.

7. Let $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$. Compute $\mathbf{z} = \mathbf{x} + \mathbf{y}$, and then plot each of these three points.

8. Let $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$. Compute $\mathbf{z} = \mathbf{x} + \mathbf{y}$, and then plot each of these three points.

9. Let $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Compute $\mathbf{z} = 4\mathbf{x}$, and then plot each of the three points.

10. Let $\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$. Compute $\mathbf{z} = -\frac{1}{2}\mathbf{x}$, and then plot each of the three points. Describe what you see.

Problem Set

1. Find the sum of the following complex numbers, and graph them on the complex plane. Trace the parallelogram that is formed by those two complex numbers, the resultant, and the origin. Describe the geometric interpretation.

    a. $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

    b. $\mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

    c. $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

    d. $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

2. Simplify and graph the complex number and the resultant. Describe the geometric effect on the complex number.

    a. $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $k = 2$, $k\mathbf{x} = ?$

    b. $\mathbf{x} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$, $k = -\frac{1}{3}$, $k\mathbf{x} = ?$

    c. $\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $k = 0$, $k\mathbf{x} = ?$

3. Find the sum of the following points, graph the points and the resultant on a 3-dimensional coordinate plane, and describe the geometric interpretation.

    a. $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

    b. $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

    c. $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

4. Simplify the following, graph the point and the resultant on a 3-dimensional coordinate plane, and describe the geometric effect.

    a. $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $k = 2$, $k\mathbf{x} = ?$

    b. $\mathbf{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $k = -1$, $k\mathbf{x} = ?$

5. Find

   a. Any two different points whose sum is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

   b. Any two different points in 3 dimensions whose sum is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

   c. Any two different complex numbers and their sum will create the degenerate parallelogram.

   d. Any two different points in 3 dimensions that their sum lie on the same line.

   e. A point that is mapped to $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ after multiplying -2.

   f. A point that is mapped to $\begin{pmatrix} \frac{1}{2} \\ -2 \\ 4 \end{pmatrix}$ after multiplying $-\frac{2}{3}$.

6. Given $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

   a. Find $\mathbf{x} + \mathbf{y}$ and graph parallelogram that is formed by $\mathbf{x}, \mathbf{y}, \ \mathbf{x} + \mathbf{y}$, and the origin.

   b. Transform the unit square by multiplying it by the matrix $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$, and graph the result.

   c. What did you find from parts (a) and (b)?

   d. What is the area of the parallelogram that is formed by part (a)?

   e. What is the determinant of the matrix $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$?

   f. Based on observation, what can you say about the degenerate parallelograms in part (a) and part (b)?

7. We learned that when multiplying -1 to a complex number z, for example $z = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, the resulting complex number $z_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ will be on the same line but on the opposite side of the origin. What matrix will produce the same effect? Verify your answer.

8. A point $z = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$ is transformed to $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$. The final step of the transformation is adding the complex number $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. Describe a possible transformation that can get this result.

Lesson 6: Linear Transformations as Matrices

Classwork

Opening Exercise

Let $A = \begin{pmatrix} 7 & -2 \\ 5 & -3 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Does this represent a linear transformation? Explain how you know.

Exploratory Challenge 1: The Geometry of 3D Matrix Transformations

a. What matrix in $\mathbb{R}^2$ serves the role of 1 in the real number system? What is that role?

b. What matrix in $\mathbb{R}^2$ serves the role of 0 in the real number system? What is that role?

c.   What is the result of scalar multiplication in $\mathbb{R}^2$?

d.   Given a complex number $a + bi$, what represents the transformation of that point across the real axis?

Exploratory Challenge 2: Properties of Vector Arithmetic

    a.   Is vector addition commutative? That is, does $x + y = y + x$ for each pair of points in $\mathbb{R}^2$? What about points in $\mathbb{R}^3$?

b. Is vector addition associative? That is, does $(x + y) + r = x + (y + r)$ for any three points in $\mathbb{R}^2$? What about points in $\mathbb{R}^3$?

c.   Does the distributive property apply to vector arithmetic? That is, does $k \cdot (x + y) = kx + ky$ for each pair of points in $\mathbb{R}^2$? What about points in $\mathbb{R}^3$?

d.   Is there an identity element for vector addition? That is, can you find a point a in $\mathbb{R}^2$ such that $x + a = x$ for every point x in $\mathbb{R}^2$? What about for$\mathbb{R}^3$?

e. Does each element in $\mathbb{R}^2$ have an additive inverse? That is, if you take a point a in $\mathbb{R}^2$, can you find a second point b such that $a + b = 0$?

Problem Set

1. Show that the associative property, $x + (y + z) = (x + y) + z$, holds for the following.

    a. $x = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, y = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, z = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

    b. $x = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}, z = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

2. Show that the distributive property, $k\,(x + y) = kx + ky$, holds for the following.

    a. $x = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, y = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, k = -2$

    b. $x = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, y = \begin{pmatrix} -4 \\ 6 \\ -7 \end{pmatrix}, k = -3$

3. Compute the following.

    a. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

    b. $\begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

    c. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

4. Let $x = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Compute $L(x) = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \cdot x$, plot the points, and describe the geometric effect to x.

    a. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

    b. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

    c. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

    d. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

5. Let $x = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Compute $L(x) = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \cdot x$. Describe the geometric effect to x.

a. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

c. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

d. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

h. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

6. Find the matrix that will transform the point $x = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ to the following point:

a. $\begin{pmatrix} -4 \\ -12 \\ -8 \end{pmatrix}$

b. $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

7. Find the matrix/matrices that will transform the point $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ to the following point:

a. $\mathbf{x}' = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$

b. $\mathbf{x}' = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

Lesson 7: Linear Transformations Applied to Cubes

Classwork

Opening Exercise

Consider the following matrices: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$
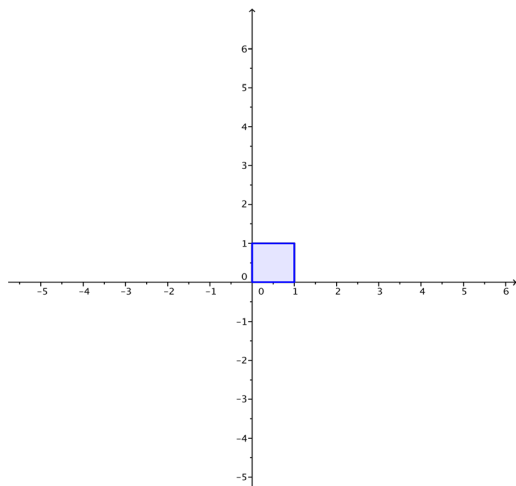
    a.   Compute the following determinants.

        i. $\det(A)$
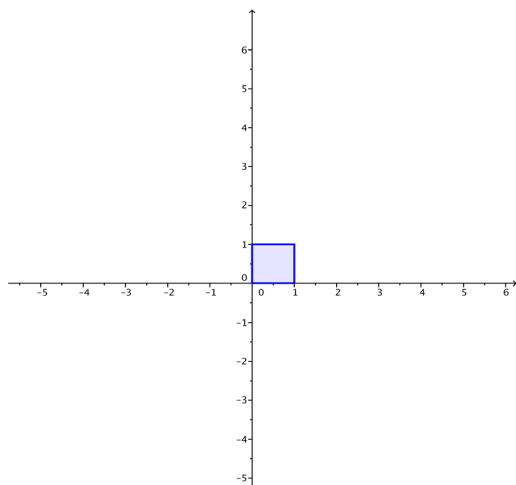
        ii. $\det(B)$

        iii. $\det(C)$

    b.   Sketch the image of the unit square after being transformed by each transformation.
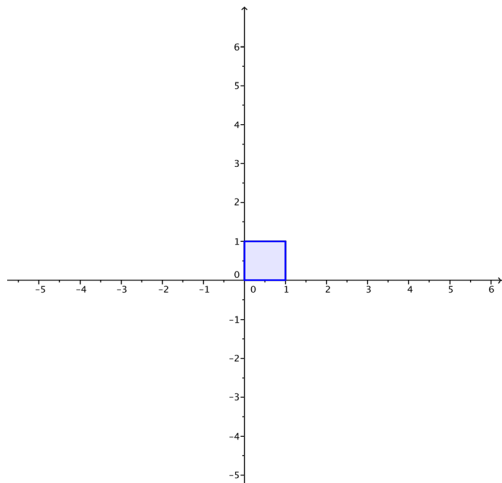
        i. $L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

ii. $L_B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

iii. $L_C\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

c. Find the area of each image of the unit square in Part 2.

d. Explain the connection between the responses to Parts 1 and 3.

Exploratory Challenge 1

For each matrix A given below:

  i. Plot the image of the unit cube under the transformation.

 ii. Find the volume of the image of the unit cube from part (i).

iii. Does the transformation have an inverse? If so, what is the matrix that induces the inverse transformation?

a. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

c. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

d. Describe the geometric effect of a transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for numbers a, b, and c.

Describe when such a transformation is invertible.

Exploratory Challenge 2

a. Make a prediction: What would be the geometric effect of the transformation

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(45°) & -\sin(45°) \\ 0 & \sin(45°) & \cos(45°) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ on the unit cube? Use the GeoGebra demo to test your

conjecture.

b. For each geometric transformation below, find a matrix A so that the geometric effect of $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is

the specified transformation.

i. Rotation by $-45°$ about the x-axis.

ii. Rotation by $45°$ about the y-axis.

iii. Rotation by $45°$ about the z-axis.

iv. Rotation by $90°$ about the x-axis.

v. Rotation by $90°$ about the y-axis.

vi. Rotation by $90°$ about the z-axis.

vii. Rotation by θ about the x-axis.

viii. Rotation by θ about the y-axis.

ix. Rotation by θ about the z-axis.

Exploratory Challenge 3 (Optional)

Describe the geometric effect of each transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given matrices A. Be as specific as you can.

a. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Lesson Summary

For a matrix A, the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a function from points in space to points in space.

Different matrices induce transformations such as rotation, dilation, and reflection.

The transformation induced by a diagonal matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ will scale by a in the direction parallel to the x-axis, by b in the direction parallel to the y-axis, and by c in the direction parallel to the z-axis.
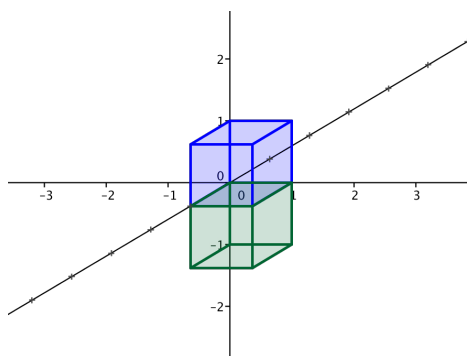
The matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$, $\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$, and $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ induce rotation by θ about the x, y, and z axes, respectively.

Problem Set

1. Suppose that we have a linear transformation $\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, for some matrix $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$.

    a. Evaluate $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. How does the result relate to the matrix A?

    b. Evaluate $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$. How does the result relate to the matrix A?

    c. Evaluate $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$. How does the result relate to the matrix A?

    d. James correctly said that if you know what a linear transformation does to the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, you can find the matrix of the transformation. Explain how you can find the matrix of the transformation given the image of these three points.

2. Use the result from Problem 1(d) to answer the following questions.

    a. Suppose a transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, and $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

        i. What is the matrix A that represents the transformation L?

        ii. What is the geometric effect of the transformation L?

        iii. Sketch the image of the unit cube after the transformation by L.

b. Suppose a transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}$.

    i.   What is the matrix A that represents the transformation L?

    ii.   What is the geometric effect of the transformation L?

    iii.   Sketch the image of the unit cube after the transformation by L.

c. Suppose a transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$.

    i.   What is the matrix A that represents the transformation L?

    ii.   What is the geometric effect of the transformation L?

    iii.   Sketch the image of the unit cube after transformation by L.

3. Find the matrix of the transformation that will produce the following images of the unit cube. Describe the geometric effect of the transformation.
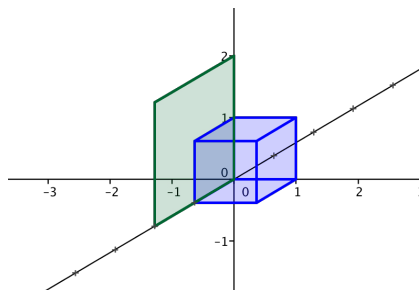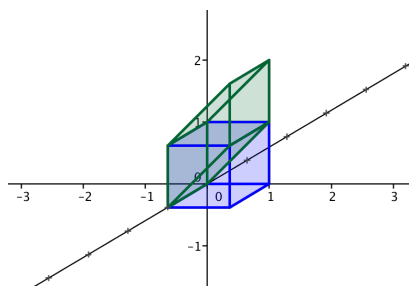
a.

b.
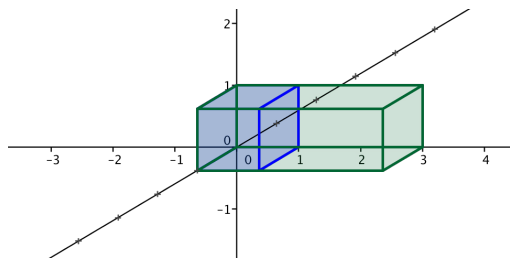
c.

d.

Lesson 8: Composition of Linear Transformations

Classwork

Opening Exercise

Compute the product AB for the following pairs of matrices.

a. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

e. $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Exploratory Challenge

1. For each pair of matrices A and B given below:

   i. Describe the geometric effect of the transformation $L_B \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = B \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.

   ii. Describe the geometric effect of the transformation $L_A \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.

   iii. Draw the image of the unit square under the transformation $L_B \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = B \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.

   iv. Draw the image of the transformed square under the transformation $L_A \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.

   v. Describe the geometric effect on the unit square of performing first L_B then L_A.

   vi. Compute the matrix product AB.

   vii. Describe the geometric effect of the transformation $L_{AB} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = AB \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.

   a. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b.　$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c.　$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

e. $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

2. Make a conjecture about the geometric effect of the linear transformation produced by the matrix AB. Justify your answer.

> **Lesson Summary**
>
> The linear transformation produced by matrix AB has the same geometric effect as the sequence of the linear transformation produced by matrix B followed by the linear transformation produced by matrix A.
>
> That is, if matrices A and B produce linear transformations L_A and L_B in the plane, respectively, then the linear transformation L_{AB} produced by the matrix AB satisfies
>
> $$L_{AB}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = L_A\left(L_B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right).$$

Problem Set

1. Let A be the matrix representing a dilation of $\frac{1}{2}$, and let B be the matrix representing a reflection across the y-axis.

   a. Write A and B.

   b. Evaluate AB. What does this matrix represent?

   c. Let $x = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and $z = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$. Find $(AB)x$, $(AB)y$, and $(AB)z$.

2. Let A be the matrix representing a rotation of $30°$, and let B be the matrix representing a dilation of 5.

   a. Write A and B.

   b. Evaluate AB. What does this matrix represent?

   c. Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $(AB)x$.

3. Let A be the matrix representing a dilation of 3, and let B be the matrix representing a reflection across the line $y = x$.

   a. Write A and B.

   b. Evaluate AB. What does this matrix represent?

   c. Let $x = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$. Find $(AB)x$.

4. Let $A = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   a. Evaluate AB.

   b. Let $x = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Find $(AB)x$.

   c. Graph x and $(AB)x$.

5. Let $A = \begin{bmatrix} \frac{1}{3} & 0 \\ 2 & \frac{1}{3} \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$.

    a. Evaluate AB.

    b. Let $x = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Find $(AB)x$.

    c. Graph x and $(AB)x$.

6. Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$.

    a. Evaluate AB.

    b. Let $x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Find $(AB)x$.

    c. Graph x and (AB).

7. Let A, B, C be 2×2 matrices representing linear transformations.

    a. What does $A(BC)$ represent?

    b. Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?

    c. Does $(AB)C$ represent something different from $A(BC)$? Explain.

8. Let AB represent any composition of linear transformations in $\mathbb{R}^2$. What is the value of $(AB)x$ where $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

Lesson 9: Composition of Linear Transformations

Classwork

Opening Exercise

Recall from Problem 1, part (d) of the Problem Set of Lesson 7 that if you know what a linear transformation does to the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, you can find the matrix of the transformation. How do the images of these three points lead to the matrix of the transformation?

    a.   Suppose that a linear transformation L_1 rotates the unit cube by $90°$ counterclockwise about the z-axis. Find the matrix A_1 of the transformation L_1.

    b.   Suppose that a linear transformation L_2 rotates the unit cube by $90°$ counterclockwise about the y-axis. Find the matrix A_2 of the transformation L_2.

    c.   Suppose that a linear transformation L_3 scales by 2 in the x-direction, scales by 3 in the y-direction, and scales by 4 in the z-direction. Find the matrix A_3 of the transformation L_3.

    d.   Suppose that a linear transformation L_4 projects onto the xy-plane. Find the matrix A_4 of the transformation L_4.

e.   Suppose that a linear transformation L_5 projects onto the xz-plane. Find the matrix A_5 of the transformation L_5.

f.   Suppose that a linear transformation L_6 reflects across the plane with equation $y = x$. Find the matrix A_6 of the transformation L_6.

g.   Suppose that a linear transformation L_7 satisfies $L_7\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, L_7\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $L_7\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$. Find the matrix A_7 of the transformation L_7. What is the geometric effect of this transformation?

h.   Suppose that a linear transformation L_8 satisfies $L_8\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, L_8\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $L_8\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the matrix of the transformation L_8. What is the geometric effect of this transformation?

Exploratory Challenge 1

Transformations L_1-L_8 refer to the linear transformations from the Opening Exercise. For each pair,

    i.   Make a conjecture to predict the geometric effect of performing the two transformations in the order specified.

    ii. Find the product of the corresponding matrices, in the order that corresponds to the indicated order of composition. Remember that if we perform a transformation L_B with matrix B and then L_A with matrix A, the matrix that corresponds to the composition $L_A \circ L_B$ is AB. That is, L_B is applied first, but matrix B is written second.

    iii. Use the GeoGebra applet TransformCubes.ggb to draw the image of the unit cube under the transformation induced by the matrix product in part (ii). Was your conjecture in part (i) correct?

a.   Perform L_6 and then L_6.

b.   Perform L_1 and then L_2.

c.   Perform L_4 and then L_5.

d.   Perform L_4 and then L_3.

e.   Perform L_3 and then L_7.

f.   Perform L_8 and then L_4.

g.   Perform L_4 and then L_6.

h.   Perform L_2 and then L_7.

i. Perform L_8 and then L_8.

Exploratory Challenge 2 (Optional)

Transformations L_1-L_8 refer to the transformations from the Opening Exercise. For each of the following pairs of matrices A and B below, compare the transformations $L_A \circ L_B$ and $L_B \circ L_A$.

a. L_4 and L_5

b. L_2 and L_5

c. L_3 and L_7

d. L_3 and L_6

e. L_7 and L_1

f.   What can you conclude about the order in which you compose two linear transformations?

> **Lesson Summary**
>
> - The linear transformation induced by a 3×3 matrix AB has the same geometric effect as the sequence of the linear transformation induced by the 3×3 matrix B followed by the linear transformation induced by the 3×3 matrix A.
>
> - That is, if matrices A and B induce linear transformations L_A and L_B in $\mathbb{R}^3$, respectively, then the linear transformation L_{AB} induced by the matrix AB satisfies $L_{AB}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = L_A\left(L_B\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$.

Problem Set

1. Let A be the matrix representing a dilation of $\frac{1}{2}$, and let B be the matrix representing a reflection across the yz-plane.

   a. Write A and B.

   b. Evaluate AB. What does this matrix represent?

   c. Let $x = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, and $z = \begin{bmatrix} 8 \\ -2 \\ -4 \end{bmatrix}$. Find $(AB)x$, $(AB)y$, and $(AB)z$.

2. Let A be the matrix representing a rotation of $30°$ about the x-axis, and let B be the matrix representing a dilation of 5.

   a. Write A and B.

   b. Evaluate AB. What does this matrix represent?

   c. Let $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find $(AB)x$, $(AB)y$, and $(AB)z$.

3. Let A be the matrix representing a dilation of 3, and let B be the matrix representing a reflection across the plane $y = x$.

   a. Write A and B.

   b. Evaluate AB. What does this matrix represent?

   c. Let $x = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}$. Find $(AB)x$.

4. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   a. Evaluate AB.

   b. Let $x = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$. Find $(AB)x$.

   c. Graph x and $(AB)x$.

5. Let $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & \frac{1}{3} \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   a. Evaluate AB.

   b. Let $x = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$. Find $(AB)x$.

   c. Graph x and $(AB)x$.

6. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   a. Evaluate AB.

   b. Let $x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. Find $(AB)x$.

   c. Graph x and $(AB)x$.

   d. What does AB represent geometrically?

7. Let A, B, C be 3×3 matrices representing linear transformations.

   a. What does $A(BC)$ represent?

   b. Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?

   c. Does $(AB)C$ represent something different from $A(BC)$? Explain.

8. Let AB represent any composition of linear transformations in $\mathbb{R}^3$. What is the value of $(AB)x$ where $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

Lesson 10: Matrix Multiplication Is Not Commutative

Classwork

Opening Exercise

Consider the vector $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If $\mathbf{v}$ is rotated $45°$ counterclockwise and then reflected across the y-axis, what is the resulting vector?

If $\mathbf{v}$ is reflected across the y-axis and then rotated $45°$ counterclockwise about the origin, what is the resulting vector?

Did these linear transformations commute? Explain.

Exercises 1–4

1. Let A equal the matrix that corresponds to a $45°$ rotation counterclockwise and B equal the matrix that corresponds to a reflection across the y-axis. Verify that matrix multiplication does not commute by finding the products AB and BA.

2. Let $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Verify that matrix multiplication does not commute by finding the products AB and BA.

3. Consider the vector $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

   a. If $\mathbf{v}$ is rotated $45°$ counterclockwise about the z-axis and then reflected across the xy-plane, what is the resulting vector?

b. If **v** is reflected across the xy-plane and then rotated 45° counterclockwise about the z-axis, what is the resulting vector?

c. Verify algebraically that the product of the two corresponding matrices is the same regardless of the order in which they are multiplied.

4. Write two matrices in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

   a. Verify algebraically that the products of these two matrices are equal.

b. Write each of your matrices as a complex number. Find the product of the two complex numbers.

c. Why is it the case that any two matrices in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ have products that are equal regardless of the order in which they are multiplied?

Problem Set

1. Let A be the matrix representing a dilation of 2, B the matrix representing a rotation of $30°$, and $x = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

    a. Evaluate AB.
    b. Evaluate BA.
    c. Find ABx.
    d. Find BAx.

2. Let A be the matrix representing a reflection across the line $y = x$, B the matrix representing a rotation of $90°$, and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

    a. Evaluate AB.
    b. Evaluate BA.
    c. Find Ax.
    d. Find Bx.
    e. Find ABx.
    f. Find BAx.
    g. Describe the linear transformation represented by AB.
    h. Describe the linear transformation represented by BA.

3. Let matrices A, B represent scalars. Answer the following questions.

    a. Would you expect $AB = BA$? Explain why or why not.
    b. Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Show $AB = BA$ through matrix multiplication and explain why.

4. Let matrices A, B represent complex numbers. Answer the following questions.

    a. Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$. Show $AB = BA$ through matrix multiplication.
    b. Would you expect $AC = CA$ for any matrix C? Explain.
    c. Let C be any 2×2 matrix, $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Show $AC \neq CA$ through matrix multiplication.
    d. Summarize your results from Problems 4 and 5.

5. Quaternions are a number system that extends to complex numbers discovered by William Hamilton in 1843. Multiplication of quaternions is not commutative and is defined as the quotient of two vectors. They are useful in 3-dimensional rotation calculations for computer graphics. Quaternions are formed the following way:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Multiplication by -1 and 1 works normally. We can represent all possible multiplications of quaternions through a table:

$\times$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

a. Is multiplication of quaternions commutative? Explain why or why not.

b. Is multiplication of quaternions associative? Explain why or why not.

Lesson 11: Matrix Addition Is Commutative

Classwork

Opening Exercise

Kiamba thinks $A + B = B + A$ for all 2×2 matrices. Rachel thinks it is not always true. Who is correct? Explain.

Exercises 1–6

1. In two-dimensional space, let A be the matrix representing a rotation about the origin through an angle of $45°$, and let B be the matrix representing a reflection about the x-axis. Let x be the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

    a.   Write down the matrices A, B, and $A + B$.

b. Write down the image points of Ax, Bx, and $(A + B)x$, and plot them on graph paper.

c. What do you notice about $(A + B)x$ compared to Ax and Bx?

2. For three matrices of equal size, A, B, and C, does it follow that $A + (B + C) = (A + B) + C$?

a. Determine if the statement is true geometrically. Let A be the matrix representing a reflection across the y-axis. Let B be the matrix representing a counterclockwise rotation of $30°$. Let C be the matrix representing a reflection about the x-axis. Let x be the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b. Confirm your results algebraically.

c. What do your results say about matrix addition?

3. If $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, what are the coordinates of a point y with the property $x + y$ is the origin $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

4. Suppose $A = \begin{pmatrix} 11 & -5 & 2 \\ -34 & 6 & 19 \\ 8 & -542 & 0 \end{pmatrix}$, and matrix B has the property that $Ax + Bx$ is the origin. What is the matrix B?

5. For three matrices of equal size, $A, B,$ and C, where A represents a reflection across the line $y = x$, B represents a counterclockwise rotation of $45°$, C represents a reflection across the y-axis, and $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$:

   a. Show that matrix addition is commutative: $Ax + Bx = Bx + Cx$.

b. Show that matrix addition is associative: $Ax + (Bx + Cx) = (Ax + Bx) + Cx$.

6. Let A, B, C, and D be matrices of the same dimensions. Use the commutative property of addition of two matrices to prove $A + B + C = C + B + A$.

Problem Set

1. Let A be matrix transformation representing a rotation of $45°$ about the origin and B be a reflection across the y-axis. Let $x = (3,4)$.

    a. Represent A and B as matrices, and find $A + B$.

    b. Represent Ax and Bx as matrices, and find $(A + B)x$.

    c. Graph your answer to part (b).

    d. Draw the parallelogram containing Ax, Bx, and $(A + B)x$.

2. Let A be matrix transformation representing a rotation of $300°$ about the origin and B be a reflection across the x-axis. Let $x = (2, -5)$.

    a. Represent A and B as matrices, and find $A + B$.

    b. Represent Ax and Bx as matrices, and find $(A + B)x$.

    c. Graph your answer to part (b).

    d. Draw the parallelogram containing Ax, Bx, and $(A + B)x$.

3. Let A, B, C, and D be matrices of the same dimensions.

    a. Use the associative property of addition for three matrices to prove $(A + B) + (C + D) = A + (B + C) + D$.

    b. Make an argument for the associative and commutative properties of addition of matrices to be true for finitely many matrices being added.

4. Let A be an $m \times n$ matrix with element in the ith row, j^{th} column a_{ij}, and B be an $m \times n$ matrix with element in the ith row, j^{th} column b_{ij}. Present an argument that $A + B = B + A$.

5. For integers x, y, define $x \oplus y = x \cdot y + 1$, read "$x$ plus y" where $x \cdot y$ is defined normally.

    a. Is this form of addition commutative? Explain why or why not.

    b. Is this form of addition associative? Explain why or why not.

6. For integers x, y, define $x \oplus y = x$.

    a. Is this form of addition commutative? Explain why or why not.

    b. Is this form of addition associative? Explain why or why not.

Lesson 12: Matrix Multiplication Is Distributive and Associative

Classwork

Opening Exercise

Write the 3×3 matrix that would represent the transformation listed.

  a. No change when multiplying (the multiplicative identity matrix)

  b. No change when adding (the additive identity matrix)

  c. A rotation about the x-axis of θ degrees

  d. A rotation about the y-axis of θ degrees

  e. A rotation about the z-axis of θ degrees

  f. A reflection over the xy-plane

g.   A reflection over the yz-plane

h.   A reflection over the xz-plane

i.   A reflection over $y = x$ in the xy-plane

Example 1

In three-dimensional space, let A represent a rotation of 90° about the x-axis, B represent a reflection about the yz-plane, and C represent a rotation of 180° about the z-axis. Let $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a.   As best you can, sketch a three-dimensional set of axes and the location of the point X.

b.   Using only your geometric intuition, what are the coordinates of BX? CX? Explain your thinking.

c. Write down matrices B and C, and verify or disprove your answers to part (b).

d. What is the sum of $BX + CX$?

e. Write down matrix A, and compute $A(BX + CX)$.

f. Compute AB and AC.

g. Compute $(AB)X$, $(AC)X$, and their sum. Compare your result to your answer to part (e). What do you notice?

h. In general, must $A(B + C)$ and $AB + AC$ have the same geometric effect on point, no matter what matrices A, B, and C are? Explain.

Exercises 1–2

1. Let $A = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$, $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, and $C = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$.

    a. Write down the products AB, AC, and $A(B + C)$.

b. Verify that $A(B + C) = AB + AC$.

2. Suppose $A, B,$ and C are 3×3 matrices, and X is a point in three-dimensional space.

a. Explain why the point $(A(BC))X$ must be the same point as $((AB)C)X$.

b. Explain why matrix multiplication must be associative.

c. Verify using the matrices from Exercise 1 that $A(BC) = (AB)C$.

Problem Set

1. Let matrix $A = \begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}$, matrix $B = \begin{pmatrix} 4 & 4 \\ 3 & 9 \end{pmatrix}$, and matrix $C = \begin{pmatrix} 8 & 2 \\ 7 & -5 \end{pmatrix}$. Calculate the following:

   a. AB

   b. AC

   c. $A(B + C)$

   d. $AB + AC$

   e. $(A + B)C$

   f. $A(BC)$

2. Apply each of the transformations you found in Problem 1 to the points $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, and $x + y$.

3. Let $A, B, C,$ and D be any four square matrices of the same dimensions. Use the distributive property to evaluate the following:

   a. $(A + B)(C + D)$

   b. $(A + B)(A + B)$

   c. What conditions need to be true for part (b) to equal $AA + 2AB + BB$?

4. Let A be a 2×2 matrix and B, C be the scalar matrices $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and $C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Answer the following questions.

   a. Evaluate the following:

      i. BC

      ii. CB

      iii. $B + C$

      iv. $B - C$

   b. Are your answers to part (a) what you expected? Why or why not?

   c. Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$; does $AB = BA$? Does $AC = CA$?

   d. What is $(A + B)(A + C)$? Write the matrix A with the letter and not in matrix form. How does this compare to $(x + 2)(x + 3)$?

   e. With B and C given as above, is it possible to factor $AA - A - BC$?

5. Define the sum of any two functions with the same domain to be the function $f + g$ such that for each x in the domain of f and g, $(f + g)(x) = f(x) + g(x)$. Define the product of any two functions to be the function fg, such that for each x in the domain of f and g, $(fg)(x) = \big(f(x)\big)\big(g(x)\big)$.

Let f, g, and h be real-valued functions defined by the equations $f(x) = 3x + 1$, $g(x) = -\frac{1}{2}x + 2$, and $h(x) = x^2 - 4$.

a. Does $f(g + h) = fg + fh$?

b. Show that this is true for any three functions with the same domains.

c. Does $f \circ (g + h) = f \circ g + f \circ h$ for the functions described above?

Lesson 13: Using Matrix Operations for Encryption

Classwork

Opening

A common way to send coded messages is to assign each letter of the alphabet to a number 1–26 and send the message as a string of integers. For example, if we encode the message "THE CROW FLIES AT MIDNIGHT" according to the chart below, we get the string of numbers

$$20, 8, 5, 0, 3, 18, 15, 23, 0, 6, 12, 9, 5, 19, 0, 1, 20, 0, 13, 9, 4, 14, 9, 7, 8, 20.$$

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z	SPACE
14	15	16	17	18	19	20	21	22	23	24	25	26	0

However, codes such as these are easily broken using an analysis of the frequency of numbers that appear in the coded messages.

We can instead encode a message using matrix multiplication. If a matrix E has an inverse, then we can encode a message as follows.

- First, convert the characters of the message to integers between 1 and 26 using the chart above.
- If the encoding matrix E is an $n \times n$ matrix, then break up the numerical message into n rows of the same length. If needed, add extra zeros to make the rows the same length.
- Place the rows into a matrix M.
- Compute the product EM to encode the message.
- The message is sent as the numbers in the rows of the matrix EM.

Exercises

1. You have received an encoded message: 34, 101, 13, 16, 23, 45, 10, 8, 15, 50, 8, 12. You know that the message was encoded using matrix $E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

    a. Store your message in a matrix C. What are the dimensions of C?

    b. You have forgotten whether the proper decoding matrix is matrix X, Y, or Z as shown below. Determine which of these is the correct matrix to use to decode this message.

$$X = \begin{bmatrix} -\dfrac{1}{3} & -\dfrac{1}{3} & \dfrac{4}{3} \\ -\dfrac{1}{3} & \dfrac{2}{3} & \dfrac{1}{3} \\ -\dfrac{2}{3} & \dfrac{1}{3} & -\dfrac{2}{3} \end{bmatrix}, Y = \begin{bmatrix} -\dfrac{1}{3} & -\dfrac{1}{3} & \dfrac{4}{3} \\ -\dfrac{1}{3} & \dfrac{2}{3} & -\dfrac{1}{3} \\ \dfrac{2}{3} & -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix}, Z = \begin{bmatrix} -\dfrac{1}{3} & -\dfrac{1}{3} & \dfrac{4}{3} \\ -\dfrac{1}{3} & \dfrac{2}{3} & \dfrac{1}{3} \\ \dfrac{2}{3} & -\dfrac{1}{3} & -\dfrac{2}{3} \end{bmatrix}$$

c. Decode the message

2. You have been assigned a group number. The message your group receives is listed below. This message is TOP SECRET! It is of such importance that it has been encoded four times.

Your group's portion of the coded message is listed below.

Group 1:

$1500, 3840, 0, 3444, 3420, 4350, 0, 4824, 3672, 3474, -2592, -6660, 0, -5976, -5940, -7560, 0, -8388, -6372, -6048$

Group 2:

$2424, 3024, -138, 396, -558, -1890, -1752, 1512, -2946, 1458, 438, 540, -24, 72, -90, -324, -300, 270, -510, 270$

Group 3:

$489, 1420, 606, 355, 1151, 33, 1002, 829, 99, 1121, 180, 520, 222, 130, 422, 12, 366, 304, 36, 410$

Group 4:

$-18, 10, -18, 44, -54, 42, -6, -74, -98, -124, 0, 10, -12, 46, -26, 42, -4, -36, -60, -82$

Group 5:

$-120, 0, -78, -54, -84, -30, 0, -6, -108, -30, -120, 114, 42, 0, -12, 42, 0, 36, 0, 0$

Group 6:

$126, 120, 60, 162, 84, 120, 192, 42, 84, 192, -18, -360, -90, -324, 0, -18, -216, -36, -90, -324$

a.   Store your message in a matrix C with two rows. How many columns does matrix C have?

b.   Begin at the station of your group number, and apply the decoding matrix at this first station.

c.   Proceed to the next station in numerical order; if you are at Station 6, proceed to Station 1. Apply the decoding matrix at this second station.

d.   Proceed to the next station in numerical order; if you are at Station 6, proceed to Station 1. Apply the decoding matrix at this third station.

e. Proceed to the next station in numerical order; if you are at Station 6, proceed to Station 1. Apply the decoding matrix at this fourth station.

f. Decode your message.

3. Sydnie was in Group 1 and tried to decode her message by calculating the matrix $(D_1 \cdot D_2 \cdot D_3 \cdot D_4)$ and then multiplying $(D_1 \cdot D_2 \cdot D_3 \cdot D_4) \cdot C$. This produced the matrix

$$M = \begin{bmatrix} \dfrac{10526}{3} & \dfrac{27020}{3} & 0 & \dfrac{24242}{3} & 8030 & \dfrac{30655}{3} & 0 & 11336 & 8616 & 8171 \\ -1455 & -3735 & 0 & -3351 & -3330 & -\dfrac{8475}{2} & 0 & -4701 & -3573 & -\dfrac{6177}{2} \end{bmatrix}.$$

a. How did she know that she made a mistake?

b. Matrix C was encoded using matrices E_1, E_2, E_3 and E_4, where D_1 decodes a message encoded by E_1, D_2 decodes a message encoded by E_2 and so on. What is the relationship between matrices E_1 and D_1, between E_2 and D_2, etc.?

c. The matrix that Sydnie received was encoded by $C = E_1 \cdot E_2 \cdot E_3 \cdot E_4 \cdot M$. Explain to Sydnie how the decoding process works to recover the original matrix M, and devise a correct method for decoding using multiplication by a single decoding matrix.

d. Apply the method you devised in part (c) to your group's message to verify that it works.

4. You received a coded message in the matrix $C = \begin{bmatrix} 30 & 30 & 69 \\ 2 & 1 & 15 \\ 9 & 14 & 20 \end{bmatrix}$. However, the matrix D that will decode this message has been corrupted, and you do not know the value of entry d_{12}. You know that all entries in matrix D are integers. Using x to represent this unknown entry, the decoding matrix D is given by $D = \begin{bmatrix} 2 & x & -4 \\ -1 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}$. Decode the message in matrix C.

Problem Set

1. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 7 \\ 3 & -4 \end{bmatrix}$, $C = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$, $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Evaluate the following.

    a. $A + B$

    b. $B + A$

    c. $A + (B + C)$

    d. $(A + B) + C$

    e. $A + I$

    f. $A + Z$

    g. $A \cdot Z$

    h. $Z \cdot A$

    i. $I \cdot A$

    j. $A \cdot B$

    k. $B \cdot A$

    l. $A \cdot C$

    m. $C \cdot A$

    n. $A \cdot B + A \cdot C$

    o. $A \cdot (B + C)$

    p. $A \cdot B \cdot C$

    q. $C \cdot B \cdot A$

    r. $A \cdot C \cdot B$

    s. $\det(A)$

    t. $\det(B)$

    u. $\det(C)$

    v. $\det(Z)$

    w. $\det(I)$

    x. $\det(A \cdot B \cdot C)$

    y. $\det(C \cdot B \cdot A)$

2. For any 2×2 matrix A and any real number k, show that if $kA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $k = 0$ or $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

3. Claire claims that she multiplied $A = \begin{bmatrix} -3 & 2 \\ 0 & 4 \end{bmatrix}$ by another matrix X and obtained $\begin{bmatrix} -3 & 2 \\ 0 & 4 \end{bmatrix}$ as her result. What matrix did she multiply by? How do you know?

4. Show that the only matrix B such that $A + B = A$ is the zero matrix.

5. A 2×2 matrix of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a *diagonal* matrix. Daniel calculated

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & -6 \end{bmatrix}$$

and concluded that if X is a diagonal matrix and A is any other matrix, then $X \cdot A = A \cdot X$.

a. Is there anything wrong with Daniel's reasoning? Prove or disprove that if X is a diagonal 2×2 matrix, then $X \cdot A = A \cdot X$ for any other matrix A.

b. For 3×3 matrices, Elda claims that only diagonal matrices of the form $X = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$ satisfy $X \cdot A = A \cdot X$ for any other 3×3 matrix A. Is her claim correct?

6. Calvin encoded a message using $E = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$, giving the coded message 4, 28, 42, 56, 2, −6, −1, 52. Decode the message, or explain why the original message cannot be recovered.

7. Decode the message below using the matrix $D = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$:

$$22, 17, 24, 9, -1, 14, -9, 34, 44, 64, 47, 77.$$

8. Brandon encoded his name with the matrix $E = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, producing the matrix $C = \begin{bmatrix} 6 & 33 & 15 & 14 \\ 12 & 66 & 30 & 28 \end{bmatrix}$. Decode the message, or explain why the original message cannot be recovered.

9. Janelle used the encoding matrix $E = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ to encode the message "FROG" by multiplying

$C = \begin{bmatrix} 6 & 18 \\ 15 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 24 & 30 \\ 22 & 37 \end{bmatrix}$. When Taylor decoded it, she computed

$M = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 24 & 30 \\ 22 & 37 \end{bmatrix} = \begin{bmatrix} 20 & 44 \\ 2 & -7 \end{bmatrix}$. What went wrong?

Lesson 14: Solving Equations Involving Linear Transformations of the Coordinate Plane

Classwork

Opening Exercise

Ahmad says the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ applied to the point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ will reflect the point to $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Randelle says that applying the matrix to the given point will produce a rotation of $180°$ about the origin. Who is correct? Explain your answer, and verify the result.

Example 1

a. Describe a transformation not already discussed that results in an image point of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and represent the transformation using a 2 x 2.

b. Determine whether any of the matrices listed represent linear transformations that can produce the image point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Justify your answers by describing the transformations represented by the matrices.

  i. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

  ii. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

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iii. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

c. Suppose a linear transformation L is represented by the matrix $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$. Find a point $L\begin{bmatrix} x \\ y \end{bmatrix}$ so that $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Exercises 1–4

1. Given the system of equations

$$2x + 5y = 4$$
$$3x - 8y = -25$$

a. Show how this system can be written as a statement about a linear transformation of the form $Lx = b$, with $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -25 \end{bmatrix}$.

b. Determine whether L has an inverse. If it does, compute $L^{-1}b$, and verify that the coordinates represent the solution to the system of equations.

2. The path of a piece of paper carried by the wind into a tree can be modeled with a linear transformation, where $L = \begin{bmatrix} 3 & -4 \\ 5 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$.

a. Write an equation that represents the linear transformation of the piece of paper.

b. Solve the equation from part (a).

c. Use your solution to provide a reasonable interpretation of the path of the piece of paper under the transformation by the wind.

3. For each system of equations, write the system as a linear transformation represented by a matrix and apply inverse matrix operations to find the solution, or explain why this procedure cannot be performed.

a. $6x + 2y = 1$
    $y = 3x + 1$

b. $4x - 6y = 10$
    $2x - 3y = 1$

4. In a two-dimensional plane, A represents a rotation of $30°$ counterclockwise about the origin, B represents a reflection over the line $y = x$, and C represents a rotation of $60°$ counterclockwise about the origin.

a. Write matrices A, B, and C.

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b. Transformations A, B, and C are applied to point $\begin{bmatrix} x \\ y \end{bmatrix}$ successively and produce the image point $\begin{bmatrix} 1 + 2\sqrt{3} \\ 2 - \sqrt{3} \end{bmatrix}$. Use inverse matrix operations to find $\begin{bmatrix} x \\ y \end{bmatrix}$.

Problem Set

1. In a two-dimensional plane, a transformation represented by $L = \begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix}$ is applied to point x, resulting in an image point $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Find the location of the point before it was transformed.

    a. Write an equation to represent the linear transformation of point x.

    b. Solve the equation to find the coordinates of the pre-image point.

2. Find the location of the point $\begin{bmatrix} x \\ y \end{bmatrix}$ before it was transformed when given:

    a. The transformation $L = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Verify your answer.

    b. The transformation $L = \begin{bmatrix} 4 & 7 \\ -1 & -2 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Verify your answer.

    c. The transformation $L = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Verify your answer.

    d. The transformation $L = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Verify your answer.

    e. The transformation $L = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and the resultant is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Verify your answer.

3. On a computer assembly line, a robot is placing a CPU onto a motherboard. The robot's arm is carried out by the transformation $L = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

    a. If the CPU is attached to the motherboard at point $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, at what location does the robot pick up the CPU?

    b. If the CPU is attached to the motherboard at point $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, at what location does the robot pick up the CPU?

    c. Find the transformation $L = \begin{bmatrix} -1 & c \\ b & 3 \end{bmatrix}$ that will place the CPU starting at $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ onto the motherboard at the location $\begin{bmatrix} -8 \\ 3 \end{bmatrix}$.

4. On a construction site, a crane is moving steel beams from a truck bed to workers. The crane is programed to perform the transformation $L = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

    a. If the workers are at location $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, where does the truck driver need to unload the steel beams so that the crane can pick them up and bring them to the workers?

    b. If the workers move to another location $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where does the truck driver need to unload the steel beams so that the crane can pick them up and bring them to the workers?

5. A video game soccer player is positioned at $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, where he kicks the ball. The ball goes into the goal, which is at point $\begin{bmatrix} 10 \\ 0 \end{bmatrix}$. When the player moves to point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and kicks the ball, he misses the goal. The ball lands at point $\begin{bmatrix} 10 \\ -1 \end{bmatrix}$. What is the program/transformation $L = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that this video soccer player uses?

6. Tim bought 5 shirts and 3 pair of pants, and it cost him $250. Scott bought 3 shirts and 2 pair of pants, and it cost him $160. All the shirts have the same cost, and all the pants have the same cost.
   a. Write a system of linear equations to find the cost of the shirts and pants.
   b. Show how this system can be written as a statement about a linear transformation of the form $Lx = b$ with $x = \begin{bmatrix} S \\ P \end{bmatrix}$ and $b = \begin{bmatrix} 250 \\ 160 \end{bmatrix}$.
   c. Determine whether L has an inverse. If it does, compute $L^{-1}b$, and verify your answer to the system of equations.

7. In a two-dimensional plane, A represents a reflection over the x-axis, B represents a reflection over the y-axis, and C represents a reflection over the line $y = x$.
   a. Write matrices A, B, and C.
   b. Write an equation for each linear transformation, assuming that each one produces an image point of $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$.
   c. Use inverse matrix operations to find the pre-image point for each equation. Explain how your solutions make sense based on your understanding of the effect of each geometric transformation on the coordinates of the pre-image points.

8. A system of equations is shown:
$$2x + 5y + z = 3$$
$$4x + y - z = 5$$
$$3x + 2y + 4z = 1$$
   a. Represent this system as a linear transformation in three-dimensional space represented by a matrix equation in the form of $Lx = b$.
   b. What assumption(s) need to be made to solve the equation in part (a) for x.
   c. Use algebraic methods to solve the system.

9. Assume
$$L^{-1} = \frac{1}{78} \begin{bmatrix} -6 & 18 & 6 \\ 19 & -5 & -6 \\ -5 & -11 & 18 \end{bmatrix}$$
Use inverse matrix operations to solve the equation from Problem 8, part (a) for x. Verify that your solution is the same as the one you found in Problem 8, part (c).

Lesson 15: Solving Equations Involving Linear Transformations of the Coordinate Space

Classwork

Opening Exercise

Mariah was studying currents in two mountain streams. She determined that five times the current in stream A was 8 feet per second stronger than twice the current in stream B. Another day she found that double the current in stream A plus ten times the current in stream B was 3 feet per second. She estimated the current in stream A to be 1.5 feet per second and stream B to be almost still (0 feet per second). Was her estimate reasonable? Explain your answer after completing parts (a)–(c).

    a.   Write a system of equations to model this problem.

    b.   Solve the system using algebra.

c. Solve the system by representing it as a linear transformation of the point x and then applying the inverse of the transformation matrix L to the equation. Verify that the solution is the same as that found in part (b).

Example 1

Dillon is designing a card game where different colored cards are assigned point values. Kryshna is trying to find the value of each colored card. Dillon gives him the following hints. If I have 3 green cards, 1 yellow card, and 2 blue cards in my hand, my total is 9. If I discard 1 blue card, my total changes to 7. If I have 1 card of each color (green, yellow, and blue), my cards total 1.

a. Write a system of equations for each hand of cards if $x = $ value of green cards, $y = $ value of yellow cards, and $z = $ value of blue cards.

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b. Solve the system using any method you choose.

c. Let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$. Find a matrix L so that the linear transformation equation $Lx = b$ would produce image coordinates that are the same as the solution to the system of equations.

d. Enter matrix L into a software program or app, and try to calculate its inverse. Does L have an inverse? If so, what is it?

e. Calculate $L^{-1} \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}$. Verify that the result is equivalent to the solution to the system you calculated in part (b). Why should the solutions be equivalent?

Exercises 1–3

1. The system of equations is given:

$$2x - 4y + 6z = 14$$
$$9x - 3y + z = 10$$
$$5x + 9z = 1$$

a. Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.

b. Verify your solution is correct.

c.   Justify your decision to use the method you selected to solve the system.

2.   An athletic director at an all-boys high school is trying to find out how many coaches to hire for the football, basketball, and soccer teams. To do this, he needs to know the number of boys that play each sport. He does not have names or numbers but finds a note with the following information listed:

The total number of boys on all three teams is 86.

The number of boys that play football is 7 less than double the total number of boys playing the other two sports.

The number of boys that play football is 5 times the number of boys playing basketball.

a.   Write a system of equations representing the number of boys playing each sport where x is the number of boys playing football, y basketball, and z soccer.

b.   Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.

c.   Verify that your solution is correct.

    d.   Justify your decision to use the method you selected to solve the system.

3.   Kyra had $20,000 to invest. She decided to put the money into three different accounts earning 3%, 5%, and 7% simple interest respectively and earned a total of $920.00 in interest. She invested half as much money at 7% as at 3%. How much did she invest in each account?

    a.   Write a system of equations that models this situation.

    b.   Find the amount invested in each account.

Problem Set

1. A small town has received funding to design and open a small airport. The airport plans to operate flights from three airlines. The total number of flights scheduled is 100. The airline with the greatest number of flights is planned to have double the sum of the flights of the other two airlines. The plan also states that the airline with the greatest number of flights will have 40 more flights than the airline with the least number of flights.

   a. Represent the situation described with a system of equations. Define all variables.

   b. Represent the system as a linear transformation using the matrix equation $Ax = b$. Define matrices A, x, and b.

   Equation:

   A:

   x:

   b:

   c. Explain how you can determine if the matrix equation has a solution without solving it.

   d. Solve the matrix equation for x.

   e. Discuss the solution in context.

2. A new blockbuster movie opens tonight, and several groups are trying to buy tickets. Three types of tickets are sold: adult, senior (over 65), and youth (under 10). A groups of 3 adults, 2 youths, and 1 senior pays $54.50 for their tickets. Another group of 6 adults and 12 youths pays $151.50. A final group of 1 adult, 4 youths, and 1 senior pays $49.00. What is the price for each type of ticket?

   a. Represent the situation described with a system of equations. Define all variables.

   b. Represent the system as a linear transformation using the matrix equation $Ax = b$.

   c. Explain how you can determine if the matrix equation has a solution without solving it.

   d. Solve the matrix equation for x.

   e. Discuss the solution in context.

   f. How much would it cost your family to attend the movie?

3. The system of equations is given:

$$5w - 2x + y + 3z = 2$$
$$4w - x + 6y + 2z = 0$$
$$w - x - y - z = 3$$
$$2w + 7x - 3y + 5z = 12$$

   a. Write the system using a matrix equation in the form $Ax = b$.

   b. Write the matrix equation that could be used to solve for x. Then use technology to solve for x.

   c. Verify your solution using back substitution.

   d. Based on your experience solving this problem and others like it in this lesson, what conclusions can you draw about the efficiency of using technology to solve systems of equations compared to using algebraic methods?

4. In three-dimensional space, a point x is reflected over the xz plane resulting in an image point of $\begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$.

   a. Write the transformation as an equation in the form $Ax = b$, where A represents the transformation of point x resulting in image point b.

   b. Use technology to calculate A^{-1}.

   c. Calculate $A^{-1}b$ to solve the equation for x.

   d. Verify that this solution makes sense geometrically.

5. Jamie needed money and decided it was time to open her piggy bank. She had only nickels, dimes, and quarters. The total value of the coins was $85.50. The number of quarters was 39 less than the number of dimes. The total value of the nickels and dimes was equal to the value of the quarters. How many of each type of coin did Jamie have? Write a system of equations and solve.

Lesson 16: Solving General Systems of Linear Equations

Classwork

Example 1

A scientist measured the greatest linear dimension of several irregular metal objects. He then used water displacement to calculate the volume of each of the objects. The data he collected are $(1, 3)$, $(2, 5)$, $(4, 9)$, and $(6, 20)$, where the first coordinate represents the linear measurement of the object in centimeters, and the second coordinate represents the volume in cubic centimeters. Knowing that volume measures generally vary directly with the cubed value of linear measurements, he wants to try to fit this data to a curve in the form of $v(x) = ax^3 + bx^2 + cx + d$.

    a.   Represent the data using a system of equations.

    b.   Represent the system using a matrix equation in the form $Ax = b$.

    c.   Use technology to solve the system.

    d.   Based on your solution to the system, what cubic equation models the data?

    e.   What are some of the limitations of the model?

Exercises 1–3

1. An attendance officer in a small school district noticed a trend among the four elementary schools in the district. This district used an open enrollment policy, which means any student within the district could enroll at any school in the district. Each year, 10% of the students from Adams Elementary enrolled at Davis Elementary, and 10% of the students from Davis enrolled at Adams. In addition, 10% of the students from Brown Elementary enrolled at Carson Elementary, and 20% of the students from Brown enrolled at Davis. At Carson Elementary, about 10% of students enrolled at Brown, and 10% enrolled at Davis, while at Davis, 10% enrolled at Brown, and 20% enrolled at Carson. The officer noted that this year, the enrollment was 490, 250, 300, and 370 at Adams, Brown, Carson, and Davis, respectively.

   a. Represent the relationship that reflects the annual movement of students among the elementary schools using a matrix.

   b. Write an expression that could be used to calculate the attendance one year prior to the year cited by the attendance officer. Find the enrollment for that year.

   c. Assuming that the trend in attendance continues, write an expression that could be used to calculate the enrollment two years after the year cited by the attendance officer. Find the attendance for that year.

   d. Interpret the results to part (c) in context.

2. Mrs. Kenrick is teaching her class about different types of polynomials. They have just studied quartics, and she has offered 5 bonus points to anyone in the class who can determine the quartic that she has displayed on the board. The quartic has 5 points identified: $(-6, 25)$, $(-3, 1)$, $\left(-2, \frac{7}{3}\right)$, $(0, -5)$, and $(3, 169)$. Logan really needs those bonus points and remembers that the general form for a quartic is $y = ax^4 + bx^3 + cx^2 + dx + e$. Can you help Logan determine the equation of the quartic?

    a. Write the system of equations that would represent this quartic.

    b. Write a matrix that would represent the coefficients of this quartic.

    c. Write an expression that could be used to calculate coefficients of the equation.

    d. Explain the answer in the context of this problem.

3. The Fibonacci numbers are the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Each number beyond the second is the sum of the previous two.

Let $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, u_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, u_5 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, and so on.

a. Show that $u_{n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} u_n$.

b. How could you use matrices to find u_{30}? Use technology to find u_{30}.

c. If $u_n = \begin{bmatrix} 165580141 \\ 267914296 \end{bmatrix}$, find u_{n-1}. Show your work.

Problem Set

1. The system of equations is given:

$$1.2x + 3y - 5z + 4.2w + v = 0$$
$$6x = 5y + 2w$$
$$3y + 4.5z - 6w + 2v = 10$$
$$9x - y + z + 2v = -3$$
$$-4x + 2y - w + 3v = 1$$

    a. Represent this system using a matrix equation.

    b. Use technology to solve the system. Show your solution process, and round your entries to the tenths place.

2. A caterer was preparing a fruit salad for a party. She decided to use strawberries, blackberries, grapes, bananas, and kiwi. The total weight of the fruit was 10 pounds. Based on guidelines from a recipe, the weight of the grapes was equal to the sum of the weight of the strawberries and blackberries; the total weight of the blackberries and kiwi was 2 pounds; half the total weight of fruit consisted of kiwi, strawberries, and bananas; and the weight of the grapes was twice the weight of the blackberries.

    a. Write a system of equations to represent the constraints placed on the caterer when she made the fruit salad. Be sure to define your variables.

    b. Represent the system using a matrix equation.

    c. Solve the system using the matrix equation. Explain your solution in context.

    d. How helpful would the solution to this problem likely be to the caterer as she prepares to buy the fruit?

3. Consider the sequence 1, 1, 1, 3, 5, 9, 17, 31, 57, … where each number beyond the third is the sum of the previous three. Let w_n be the points with the nth, $(n + 1)^{th}$, and $(n + 2)^{th}$ terms of the sequence.

    a. Find a 3×3 matrix A so that $Aw_n = w_{n+1}$ for each n.

    b. What is the 30th term of the sequence?

    c. What is A^{-1}? Explain what A^{-1} represents in terms of the sequence. In other words, how can you find w_{n-1} if you know w_n?

    d. Could you find the -5th term in the sequence? If so, how? What is its value?

4. Mr. Johnson completed a survey on the number of hours he spends weekly watching different types of television programs. He determined that he spends 30 hours a week watching programs of the following types: comedy, drama, movies, competition, and sports. He spends half as much time watching competition shows as he does watching dramas. His time watching sports is double his time watching dramas. He spends an equal amount of time watching comedies and movies. The total amount of time spent watching comedies and movies is the same as the total amount of time spent watching dramas and competition shows.

    Write and solve a system of equations to determine how many hours Mr. Johnson watches each type of programming each week.

5. A copper alloy is a mixture of metals having copper as their main component. Copper alloys do not corrode easily and conduct heat. They are used in all types of applications including cookware and pipes. A scientist is studying different types of copper alloys and has found one containing copper, zinc, tin, aluminum, nickel, and silicon. The alloy weighs 3.2 kilograms. The percentage of aluminum is triple the percentage of zinc. The percentage of silicon is half that of zinc. The percentage of zinc is triple that of nickel. The percentage of copper is fifteen times the sum of the percentages of aluminum and zinc combined. The percentage of copper is nine times the combined percentages of all the other metals.

   a.   Write and solve a system of equations to determine the percentage of each metal in the alloy.

   b.   How many kilograms of each alloy are present in the sample?

Lesson 17: Vectors in the Coordinate Plane

Classwork

Opening Exercise

When an earthquake hits, the ground shifts abruptly due to forces created when the tectonic plates along fault lines rub together. As the tectonic plates shift and move, the intense shaking can even cause the physical movement of objects as large as buildings.

Suppose an earthquake causes all points in a town to shift 10 feet to the north and 5 feet to the east.
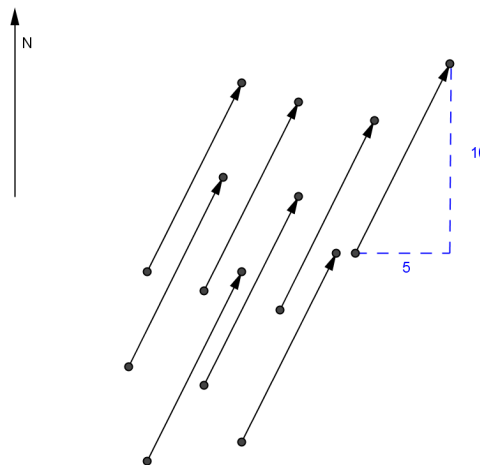
a. Explain how the diagram shown above could be said to represent the shifting caused by the earthquake.

b. Draw another arrow that shows the same shift. Explain how you drew your arrow.

Exercises 1–3

Several vectors are represented in the coordinate plane below using arrows.
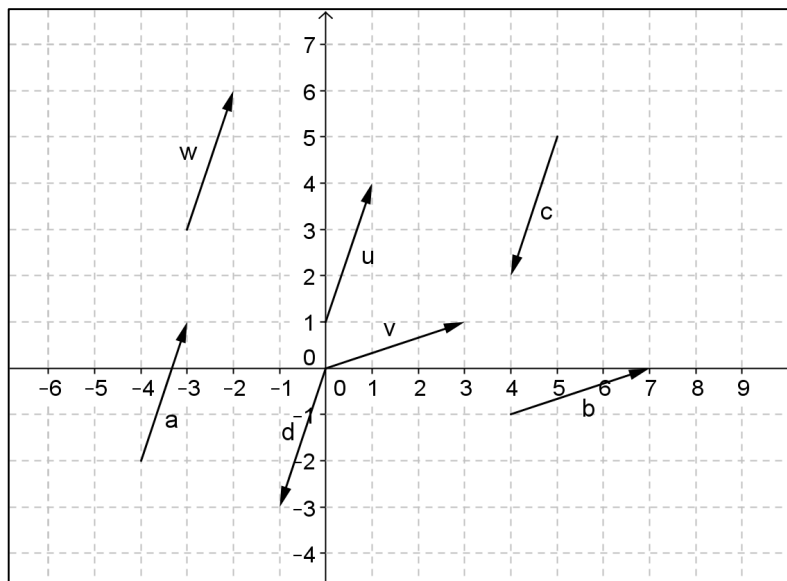
1. Which arrows represent the same vector? Explain how you know.

2. Why do arrows c and u not represent the same vector?

3. After the first earthquake shifted points 5 feet east and 10 feet north, suppose a second earthquake hits the town and all points shift 6 feet east and 9 feet south.

    a. Write and draw a vector **t** that represents this shift caused by the second earthquake.
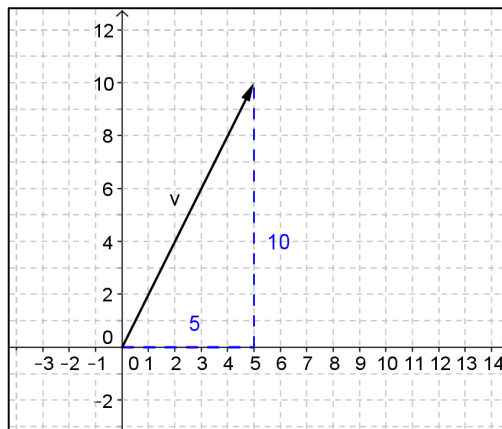
    b. Which earthquake, the first one or the second one, shifted all the points in the town further? Explain your reasoning.

Example 1: The Magnitude of a Vector

The magnitude of a vector $\mathbf{v} = \langle a, b \rangle$ is the length of the line segment from the origin to the point (a, b) in the coordinate plane, which we denote by $\|\mathbf{v}\|$. Using the language of translation, the magnitude of $\mathbf{v}$ is the distance between any point and its image under the translation a units horizontally and b units vertically. It is denoted $\|\mathbf{v}\|$.

    a. Find the magnitude of $\mathbf{v} = \langle 5, 10 \rangle$ and $\mathbf{t} = \langle 6, -9 \rangle$. Explain your reasoning.

b. Write the general formula for the magnitude of a vector.

Exercises 4–10

4. Given that $\mathbf{v} = \langle 3,7 \rangle$ and $\mathbf{t} = \langle -5,2 \rangle$.

   a. What is v + t*?*

   b. Draw a diagram that represents this addition and shows the resulting sum of the two vectors.

   c. What is t + v?

   d. Draw a diagram that represents this addition and shows the resulting sum of the two vectors.

5. Explain why vector addition is commutative.

6. Given $\mathbf{v} = \langle 3,7 \rangle$ and $\mathbf{t} = \langle -5,2 \rangle$.

    a. Show numerically that $\|\mathbf{v}\| + \|\mathbf{t}\| \neq \|\mathbf{v} + \mathbf{t}\|$.

    b. Provide a geometric argument to explain in general, why the sum of the magnitudes of two vectors is not equal to the magnitude of the sum of the vectors.

    c. Can you think of an example when the statement would be true? Justify your reasoning.

7. Why is the vector $\mathbf{o} = \langle 0,0 \rangle$ called the zero vector? Describe its geometric effect when added to another vector.

8. Given the vectors shown below.

$$\mathbf{v} = \langle 3, 6 \rangle$$
$$\mathbf{u} = \langle 9, 18 \rangle$$
$$\mathbf{w} = \langle -3, -6 \rangle$$
$$\mathbf{s} = \langle 1, 2 \rangle$$
$$\mathbf{t} = \langle -1.5, -3 \rangle$$
$$\mathbf{r} = \langle 6, 12 \rangle$$

a. Draw each vector with its initial point located at $(0,0)$. The vector $\mathbf{v}$ is already shown. How are all of these vectors related?
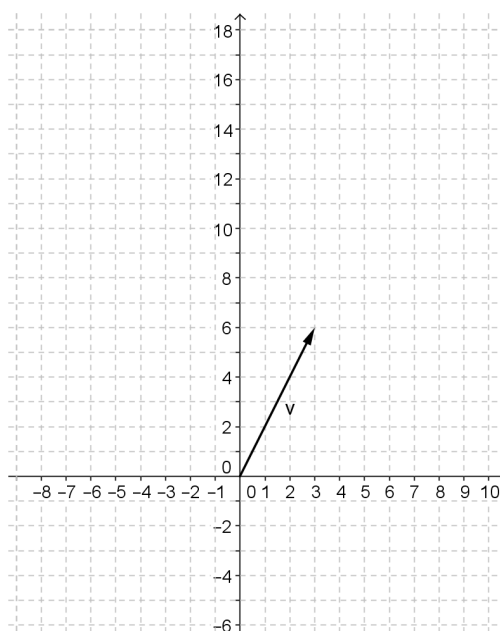
b. Which vector is $2\mathbf{v}$? Explain how you know.

c. Describe the remaining vectors as a scalar multiple of $\mathbf{v} = \langle 3, 6 \rangle$ and explain your reasoning.

　　d.　Is the vector $\mathbf{p} = \langle 3\sqrt{2}, 6\sqrt{2} \rangle$ a scalar multiple of $\mathbf{v}$? Explain.

9.　Which vector from Exercise 8 would it make sense to call the <u>opposite</u> of $\mathbf{v} = \langle 3,6 \rangle$?

10.　Describe a rule that defines vector subtraction. Use the vectors $\mathbf{v} = \langle 5,7 \rangle$ and $\mathbf{u} = \langle 6,3 \rangle$ to support your reasoning.

Lesson Summary

A vector can be used to describe a translation of an object. It has a magnitude and a direction based on its horizontal and vertical components. A vector $\mathbf{v} = \langle a, b \rangle$ can represent a translation of $\boldsymbol{a}$ units horizontally and $\boldsymbol{b}$ units vertically with magnitude given by $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$.

- To add two vectors, add their respective horizontal and vertical components.
- To subtract two vectors, subtract their respective horizontal and vertical components.
- Multiplication of a vector by a scalar multiplies the horizontal and vertical components of the vector by the value of the scalar.

Problem Set

1. Sasha says that a vector has a direction component in it; therefore, we cannot add two vectors or subtract one from the other. His argument is that we cannot add "east" to "north" nor subtract "east" from "north," for instance. Therefore, he claims, we cannot add or subtract vectors.

   a. Is he correct? Explain your reasons.

   b. What would you do if you need to add two vectors, $\mathbf{u}$ and $\mathbf{v}$, or subtract vector $\mathbf{v}$ from vector $\mathbf{u}$ arithmetically?

2. Given $\mathbf{u} = \langle 3, 1 \rangle$ and $\mathbf{v} = \langle -4, 2 \rangle$, write each vector in component form, graph it, and explain the geometric effect.

   a. $3\mathbf{u}$

   b. $\frac{1}{2}\mathbf{v}$

   c. $-2\mathbf{u}$

   d. $-\mathbf{v}$

   e. $\mathbf{u} + \mathbf{v}$

   f. $2\mathbf{u} + 3\mathbf{v}$

   g. $4\mathbf{u} - 3\mathbf{v}$

   h. $\frac{1}{2}\mathbf{u} - \frac{1}{3}\mathbf{v}$

3. Given $\mathbf{u} = \langle 3,1 \rangle$ and $\mathbf{v} = \langle -4,2 \rangle$, find the following.

   a. $\|\mathbf{u}\|$.

   b. $\|\mathbf{v}\|$.

   c. $\|2\mathbf{u}\|$ and $2\|\mathbf{u}\|$.

   d. $\left\|\frac{1}{2}\mathbf{v}\right\|$ and $\frac{1}{2}\|\mathbf{v}\|$

   e. Is $\|\mathbf{u} + \mathbf{u}\|$ equal to $\|\mathbf{u}\| + \|\mathbf{u}\|$? Explain how you know.

   f. Is $\|\mathbf{u} + \mathbf{v}\|$ equal to $\|\mathbf{u}\| + \|\mathbf{v}\|$? Explain how you know.

   g. Is $\|\mathbf{u} - \mathbf{v}\|$ equal to $\|\mathbf{u}\| - \|\mathbf{v}\|$? Explain how you know.

4. Given $\mathbf{u} = \langle 1,2 \rangle$, $\mathbf{v} = \langle 3,-4 \rangle$, and $\mathbf{w} = \langle -4,6 \rangle$, show that $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

5. Tyiesha says that if the magnitude of a vector $\mathbf{u}$ is zero, then $\mathbf{u}$ has to be a zero vector. Is she correct? Explain how you know.

6. Sergei experienced one of the biggest earthquakes when visiting Taiwan in 1999. He noticed that his refrigerator moved on the wooden floor and made marks on it. By measuring the marks he was able to trace how the refrigerator moved. The first move was northeast with a distance of 20 cm. The second move was northwest with a distance of 10 cm. The final move was northeast with a distance of 5 cm. Find the vectors that would re-create the refrigerator's movement on the floor and find the distance that the refrigerator moved from its original spot to its resting place. Draw a diagram of these vectors.

Lesson 18: Vectors and Translation Maps

Classwork

Opening Exercise

Write each vector described below in component form and find its magnitude. Draw an arrow originating from (0,0) to represent each vector's magnitude and direction.

    a.   Translate 3 units right and 4 units down.

    b.   Translate 6 units left.

    c.   Translate 2 units left and 2 units up.

    d.   Translate 5 units right and 7 units up.

Exercises 1–3

1.   Write a translation map defined by each vector from the opening.

    Consider the vector $\mathbf{v} = \langle -2, 5 \rangle$, and its associated translation map:

$$T_{\mathrm{v}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2 \\ y + 5 \end{bmatrix}$$

2. Suppose we apply the translation map T_v to each point on the circle $(x + 4)^2 + (y - 3)^2 = 25$.

    a. What is the radius and center of the original circle?

    b. Show that the image points satisfy the equation of another circle.

    c. What is center and radius of this image circle?

3. Suppose we apply the translation map T_v to each point on the line $2x - 3y = 10$.

    a. What are the slope and y-intercept of the original line?

    b. Show that the image points satisfy the equation of another line.

c.   What are the slope and y-intercept of this image line?

Example 1: Vectors and Translation Maps in $\mathbb{R}^3$

Translate by the vector $v = \langle 1,3,5 \rangle$ by applying the translation map T_v to the following objects in $\mathbb{R}^3$. A sketch of the original object and the vector is shown. Sketch the image.

$$T_v\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y+3 \\ z+5 \end{bmatrix}$$

a.   The point $A(2,-2,4)$
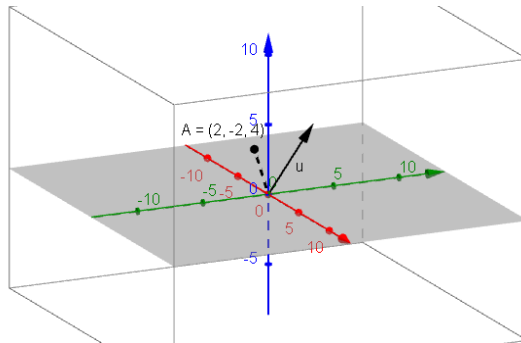
b.   The plane $2x + 3y - z = 0$
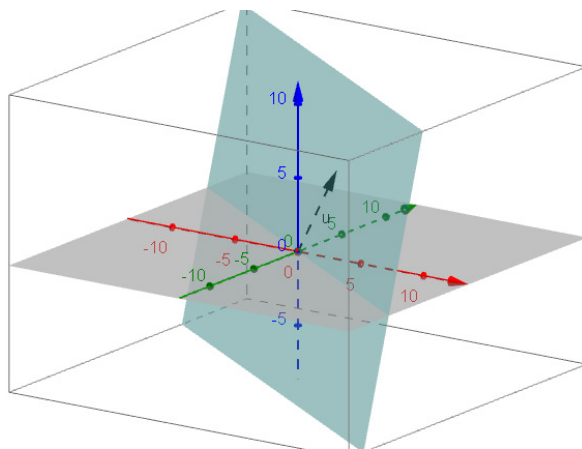
c.   The sphere $(x - 1)^2 + (y - 3)^2 + z^2 = 9$.
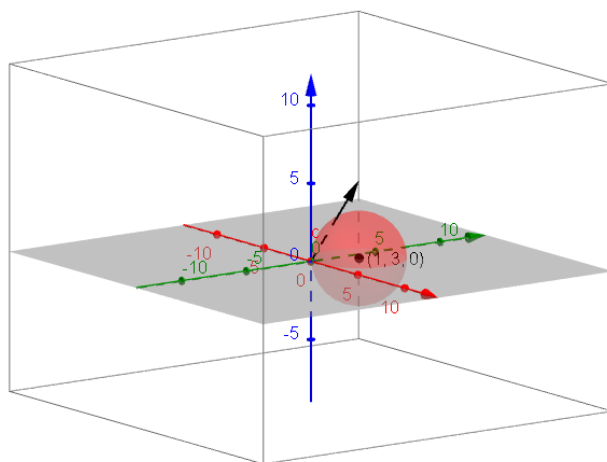
Exercise 4

4.   Given the sphere $(x + 3)^2 + (y - 1)^2 + (z - 3)^2 = 10$.

a.   What are its center and radius?

b.   Write a vector and its associated translation map that would take this sphere to its image centered at the origin.

Example 2: What is the Magnitude of a Vector in $\mathbb{R}^3$?

a.   Find a general formula for $\|\mathbf{v}\|^2$.

b.   Solve this equation for $\|\mathbf{v}\|$ to find the magnitude of the vector.

Exercises 5–8

5. Which vector has greater magnitude, $\mathbf{v} = \langle 0, 5, -4 \rangle$ or $\mathbf{u} = \langle 3, -4, 4 \rangle$? Show work to support your answer.

6. Explain why vectors can have equal magnitude but not be the same vector.

7. Vector arithmetic in $\mathbb{R}^3$ is analogous to vector arithmetic in $\mathbb{R}^2$. Complete the graphic organizer to illustrate these ideas.

	Vectors in $\mathbb{R}^2$	Vectors in $\mathbb{R}^3$
Component Form	$\langle a, b \rangle$	$\langle a, b, c \rangle$
Column Form	$\begin{bmatrix} a \\ b \end{bmatrix}$	
Magnitude	$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$	
Addition	If $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{u} = \langle c, d \rangle$, Then $\mathbf{v} + \mathbf{u} = \langle a + c, b + d \rangle$	
Subtraction	If $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{u} = \langle c, d \rangle$, Then $\mathbf{v} - \mathbf{u} = \langle a - c, b - d \rangle$	
Scalar Multiplication	If $\mathbf{v} = \langle a, b \rangle$ and k is a real number $k\mathbf{v} = \langle ka, kb \rangle$	

8. Given $\mathbf{v} = \langle 2, 0, -4 \rangle$ and $\mathbf{u} = \langle -1, 5, 3 \rangle$.

   a. Calculate the following.

      i. $\mathbf{v} + \mathbf{u}$

    ii. $2\mathbf{v} - \mathbf{u}$

    iii. $\|\mathbf{v}\|$

b.   Suppose the point $(1,3,5)$ is translated by $\mathbf{v}$ and then by $\mathbf{u}$. Determine a vector $\mathbf{w}$ that would return the point back to its original location $(1,3,5)$.

> **Lesson Summary**
>
> A vector **v** can define a translation map $T_\mathbf{v}$ that takes a point to its image under the translation. Applying the map to the set of all points that make up a geometric figure serves to translate the figure by the vector.

Problem Set

1. Myishia says that when applying the translation map $T_\mathbf{v}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 1 \\ y - 2 \end{bmatrix}$ to a set of points given by an equation relating x and y, we should replace every x that is in the equation by $x + 1$, and y by $y - 2$. For example, the equation of the parabola $y = x^2$ would become $y - 2 = (x + 1)^2$. Is she correct? Explain your answer.

2. Given the vector $\mathbf{v} = \langle -1, 3 \rangle$, find the image of the line $x + y = 1$ under the translation map $T_\mathbf{v}$. Graph the original line and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the line.

3. Given the vector $\mathbf{v} = \langle 2, 1 \rangle$, find the image of the parabola $y - 1 = x^2$ under the translation map $T_\mathbf{v}$. Draw a graph of the original parabola and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the parabola. Find the vertex and x-intercepts of the graph of the image.

4. Given the vector $\mathbf{v} = \langle 3, 2 \rangle$, find the image of the graph of $y + 1 = (x + 1)^3$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

5. Given the vector $\mathbf{v} = \langle 3, -3 \rangle$, find the image of the graph of $y + 2 = \sqrt{x + 1}$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

6. Given the vector $\mathbf{v} = \langle -1, -2 \rangle$, find the image of the graph of $y = \sqrt{9 - x^2}$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

7. Given the vector $\mathbf{v} = \langle 1, 3 \rangle$, find the image of the graph of $y = \frac{1}{x+2} + 1$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the equations of the asymptotes of the graph of the image.

8. Given the vector $\mathbf{v} = \langle -1, 2 \rangle$, find the image of the graph of $y = |x + 2| + 1$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

9. Given the vector $\mathbf{v} = \langle 1, -2 \rangle$, find the image of the graph of $y = 2^x$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

10. Given the vector $\mathbf{v} = \langle -1, 3 \rangle$, find the image of the graph of $y = \log_2 x$, under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the x-intercepts of the graph of the image.

11. Given the vector $\mathbf{v} = \langle 2, -3 \rangle$, find the image of the graph of $\frac{x^2}{4} + \frac{y^2}{16} = 1$ under the translation map $T_\mathbf{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_\mathbf{v}$ on the graph. Find the new center, major and minor axis of the graph of the image.

12. Given the vector $\mathbf{v}$, find the image of the given point P under the translation map $T_\mathbf{v}$. Graph P and its image.

    a. $\mathbf{v} = \langle 3, 2, 1 \rangle$, $P = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,

    b. $\mathbf{v} = \langle -2, 1, -1 \rangle$, $P = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$

13. Given the vector $\mathbf{v}$, find the image of the given plane under the translation map $T_\mathbf{v}$. Sketch the original vector and its image.

    a. $\mathbf{v} = \langle 2, -1, 3 \rangle$, $3x - 2y - z = 0$.

    b. $\mathbf{v} = \langle -1, 2, -1 \rangle$, $2x - y + z = 1$.

14. Given the vector $\mathbf{v}$, find the image of the given sphere under the translation map $T_\mathbf{v}$. Sketch the original sphere and its image.

    a. $\mathbf{v} = \langle -1, 2, 3 \rangle$, $x^2 + y^2 + z^2 = 9$.

    b. $\mathbf{v} = \langle -3, -2, 1 \rangle$, $(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 1$.

15. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the line $x - y = 1$ to the line $x - y = -3$. Sketch the original vector and its image.

16. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the parabola $y = x^2 + 4x + 1$ to the parabola $y = x^2$

17. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the circle with equation $x^2 + y^2 - 4x + 2y - 4 = 0$ to the circle with equation $(x + 3)^2 + (y - 4)^2 = 9$

18. Find a vector $\mathbf{v}$ and translation map $T_\mathbf{v}$ that will translate the graph of $y = \sqrt{x - 3} + 2$ to the graph of $y = \sqrt{x + 2} - 3$.

19. Find a vector **v** and translation map T_v that will translate the sphere $(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 1$ to the sphere $(x - 3)^2 + (y + 1)^2 + (z + 2)^2 = 1$

20. Given vectors $\mathbf{u} = \langle 2, -1, 3 \rangle$, $\mathbf{v} = \langle 2, 0, -2 \rangle$, and $\mathbf{w} = \langle -3, 6, 0 \rangle$, find the following.

    a. $3\mathbf{u} + \mathbf{v} + \mathbf{w}$

    b. $\mathbf{w} - 2\mathbf{v} - \mathbf{u}$

    c. $3\left(2\mathbf{u} - \frac{1}{2}\mathbf{v}\right) - \frac{1}{3}\mathbf{w}$

    d. $-2\mathbf{u} - 3(5\mathbf{v} - 3\mathbf{w})$.

    e. $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$.

    f. Show that $2\|\mathbf{v}\| = \|2\mathbf{v}\|$.

    g. Show that $\|\mathbf{u} + \mathbf{v}\| \neq \|\mathbf{u}\| + \|\mathbf{v}\|$.

    h. Show that $\|\mathbf{v} - \mathbf{w}\| \neq \|\mathbf{v}\| - \|\mathbf{w}\|$.

    i. $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$ and $\left\|\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right\|$.

Lesson 19: Directed Line Segments and Vectors

Classwork

A vector can be used to represent a translation that takes one point to an image point. The starting point is called the initial point, and the image point under the translation is called the terminal point.
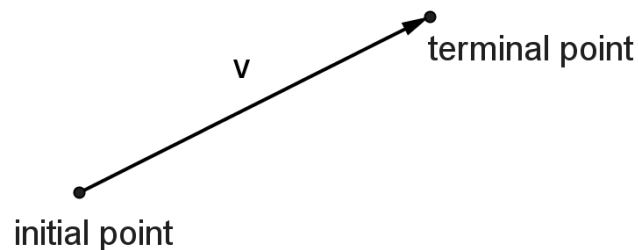
If we know the coordinates of both points, we can easily determine the horizontal and vertical components of the vector.
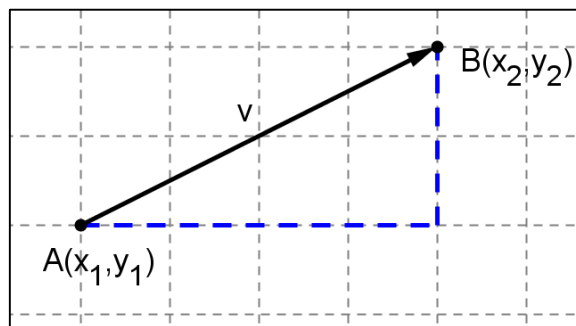
Exercises 1–3

1. Several vectors, represented by arrows, are shown below. For each vector, state the initial point, terminal point, component form of the vector and magnitude.
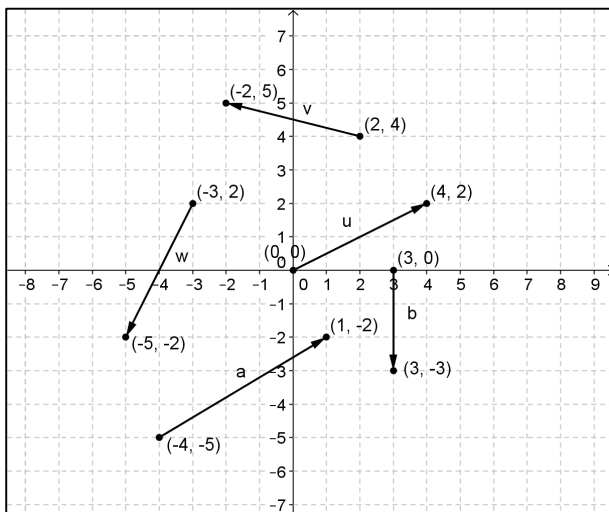
2. Several vectors, represented by arrows, are shown below. For each vector, state the initial point, terminal point, component form of the vector, and magnitude.
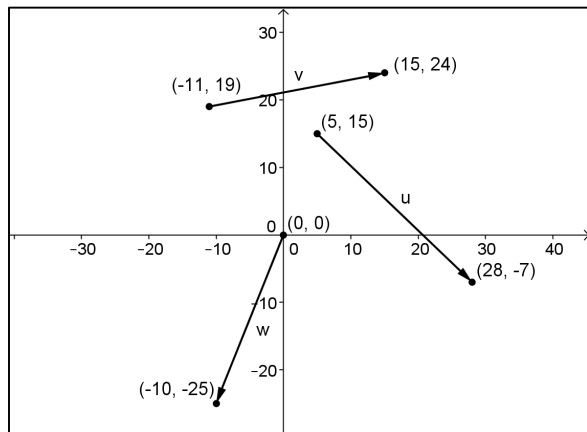
3. Write a rule for the component form of the vector **v** shown in the diagram. Explain how you got your answer.

When we use the initial and terminal points to describe a vector, we often refer to the vector as a <u>directed line segment</u>. A vector or directed line segment with initial point A and terminal point B is denoted $\overrightarrow{AB}$.

Exercises 4–7

4. Write each vector in component form.

    a. $\overrightarrow{AE}$

    b. $\overrightarrow{BH}$

    c. $\overrightarrow{DC}$

    d. $\overrightarrow{GF}$

    e. $\overrightarrow{IJ}$
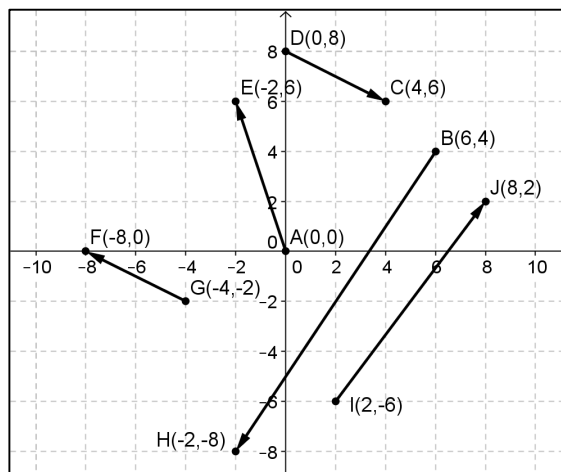
5. Consider points $P(2,1)$, $Q(-3,3)$ and $R(1,4)$.

    a. Compute $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ and show that $\overrightarrow{PQ} + \overrightarrow{QP}$ is the zero vector. Draw a diagram to show why this makes sense geometrically.

b.   Plot the points P, Q, and R. Use the diagram to explain why $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$ is the zero vector. Show that this is true by computing the sum $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$.

6.   Show for any two points A and B that $-\overrightarrow{AB} = \overrightarrow{BA}$.

7.   Given the vectors $\mathbf{v} = \langle 2, -3 \rangle$, $\mathbf{w} = \langle -5, 1 \rangle$, $\mathbf{u} = \langle 4, -2 \rangle$ and $\mathbf{t} = \langle -1, 4 \rangle$.

   a.   Verify that the sum of these four vectors is the zero vector.

   b.   Draw a diagram representing the vectors as arrows placed end-to-end to support why their sum would be the zero vector.

Example 1: The Parallelogram Rule for Vector Addition

When the initial point of a vector is the origin, then the coordinates of the terminal point will correspond to the horizontal and vertical components of the vector. This type of vector, with initial point at the origin, is often called a position vector.

    a.   Draw arrows to represent the vectors $\mathbf{v} = \langle 5,3 \rangle$ and $\mathbf{u} = \langle 1,7 \rangle$ with the initial point of each vector at $(0,0)$.

    b.   Add $\mathbf{v} + \mathbf{u}$ end-to-end. What is $\mathbf{v} + \mathbf{u}$? Draw the arrow that represents $\mathbf{v} + \mathbf{u}$ with initial point at the origin.

    c.   Add $\mathbf{u} + \mathbf{v}$ end-to-end. What is $\mathbf{u} + \mathbf{v}$? Draw the arrow that represents $\mathbf{u} + \mathbf{v}$ with an initial point at the origin.
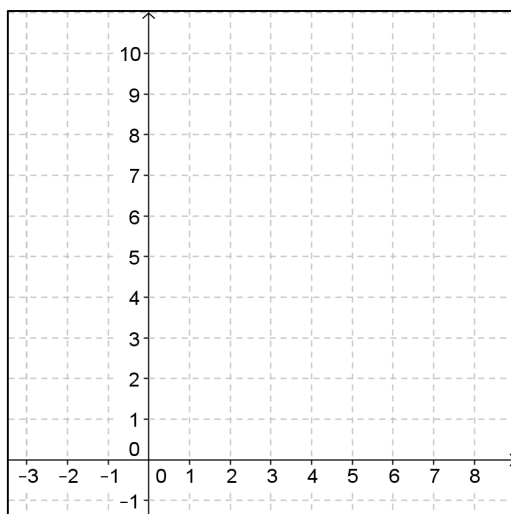
Exercises 8

8.   Let $\mathbf{u} = \langle -2,5 \rangle$ and $\mathbf{v} = \langle 4,3 \rangle$.

    a.   Draw a diagram to illustrate $\mathbf{v}$ and $\mathbf{u}$ and then find $\mathbf{v} + \mathbf{u}$ using the parallelogram rule.

    b.   Draw a diagram to illustrate $2\mathbf{v}$ and then find $2\mathbf{v} + \mathbf{u}$ using the parallelogram rule.

c. Draw a diagram to illustrate $-\mathbf{v}$ and then find $\mathbf{u} - \mathbf{v}$ using the parallelogram rule.

d. Draw a diagram to illustrate $3\mathbf{v}$ and $-3\mathbf{v}$.

i. How do the magnitudes of these vectors compare to one another and to that of $\mathbf{v}$?

ii. How do the directions of $3\mathbf{v}$ and $-3\mathbf{v}$ compare to the direction of $\mathbf{v}$?

Directed line segments can also be represented in $\mathbb{R}^3$. Instead of two coordinates like we use in $\mathbb{R}^2$, we simply use three to locate a point in space relative to the origin, denoted $(0,0,0)$. Thus the vector $\overrightarrow{AB}$ with initial point $A(x_1, y_1, z_1)$ and terminal point $B(x_2, y_2, z_2)$ would have component form

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Exercises 9–10

9. Consider points $A(1,0,-5)$ and $B(2,-3,6)$.

   a. What is the component form of $\overrightarrow{AB}$?

   b. What is the magnitude of $\overrightarrow{AB}$?

10. Consider points $A(1,0,-5)$, $B(2,-3,6)$, and $C(3,1,-2)$.

    a. Show that $\overrightarrow{AB} + \overrightarrow{BA} = 0$. Explain your answer using geometric reasoning.

    b. Show that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$. Explain your answer using geometric reasoning.

Lesson Summary

A vector **v** can be used to represent a directed line segment $\overrightarrow{AB}$. If the initial point is $A(x_1, y_1)$ and the terminal point is $B(x_2, y_2)$, then the component form of the vector is $\mathbf{v} = \overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$

Vectors can be added end-to-end or using the parallelogram rule.

Problem Set

1. Vectors **u,v**, **w**, **a**, and **b** are shown at right.

   a. Find the component form of **u**, **v**, **w**, **a**, and **b**.

   b. Find the magnitudes $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, $\|\mathbf{a}\|$, and $\|\mathbf{b}\|$.

   c. Find the component form *of* **u** + **v** and calculate $\|\mathbf{u} + \mathbf{v}\|$.

   d. Find the component form of **w** − **b** and calculate $\|\mathbf{w} - \mathbf{b}\|$.

   e. Find the component form of 3**u** − 2**v**.

   f. Find the component form of **v** − 2(**u** + **b**).

   g. Find the component form of 2(**u** − 3**v**) − **a**.

   h. Find the component form of **u** + **v** + **w** + **a** + **b**.

   i. Find the component form of **u** − **v** − **w** − **a** + **b**.

   j. Find the component form of 2(**u** + 4**v**) − 3(**w** − 3**a** + 2**b**).
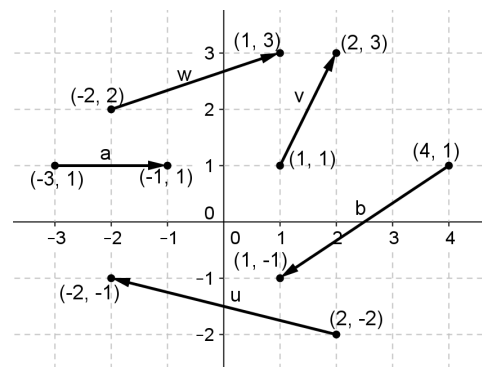
2. Given points $A(1,2,3), B(-3,2,-4), C(-2,1,5)$, find component forms of the following vectors.

   a. $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

   b. $\overrightarrow{BC}$ and $\overrightarrow{CB}$

   c. $\overrightarrow{CA}$ and $\overrightarrow{AC}$

   d. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC}$

   e. $-\overrightarrow{BC} + \overrightarrow{BA} + \overrightarrow{AC}$

   f. $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA}$

3. Given points $A(1,2,3), B(-3,2,-4), C(-2,1,5)$, find the following magnitudes.

   a. $\left\| \overrightarrow{AB} \right\|$

   b. $\left\| \overrightarrow{AB} + \overrightarrow{BC} \right\|$.

   c. $\left\| \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} \right\|$

4. Given vectors $\mathbf{u} = \langle -3, 2 \rangle$, $\mathbf{v} = \langle 2, 4 \rangle$, $\mathbf{w} = \langle 5, -3 \rangle$, use the parallelogram rule to graph the following vectors.

   a. $\mathbf{u} + \mathbf{v}$

   b. $\mathbf{v} + \mathbf{w}$

   c. $\mathbf{u} - \mathbf{v}$

   d. $\mathbf{v} - \mathbf{w}$

   e. $2\mathbf{w} + \mathbf{u}$

   f. $3\mathbf{u} - 2\mathbf{v}$

   g. $\mathbf{u} + \mathbf{v} + \mathbf{w}$

5. Points A, B, C, D, E, F, G and H and vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are shown below. Find the components of the following vectors.


   a. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

   b. $\mathbf{u} + \mathbf{v} + \mathbf{w}$
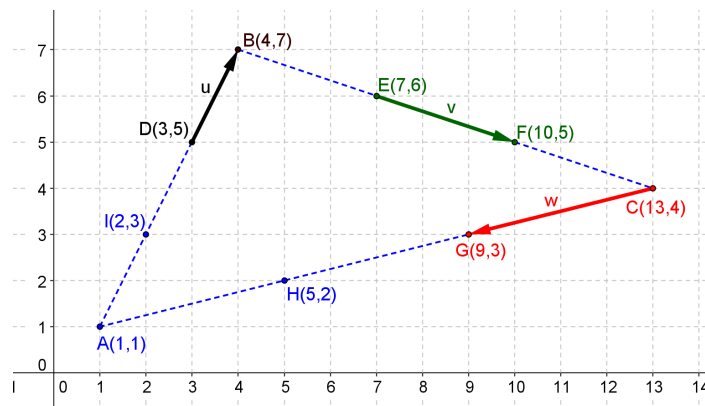
   c. $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CG}$

6. Consider Example 5, part (b) in the lesson and Problem 5, part (a) above. What can you conclude about three vectors that form a triangle when placed tip-to-tail? Explain by graphing.

7. Consider the vectors shown below.

   a. Find the components of vectors $\mathbf{u} = \overrightarrow{AC}$, $\mathbf{w} = \overrightarrow{AD}$, $\mathbf{v} = \overrightarrow{AB}$, and $\mathbf{c} = \overrightarrow{EF}$.

   b. Is vector $\mathbf{u}$ equal to vector $\mathbf{c}$?

   c. Jens says that if two vectors $\mathbf{u}$ and $\mathbf{v}$ have the same initial point A and lie on the same line, then one vector is a scalar multiple of the other. Do you agree with him? Explain how you know. Given an example to support your answer.

# Lesson 20: Vectors and Stone Arches

Classwork

Exploratory Challenge

1. For this Exploratory Challenge, we will consider an arch made with five trapezoidal stones on top of the base columns as shown. We will focus only on the stones labeled 1, 2 and 3.
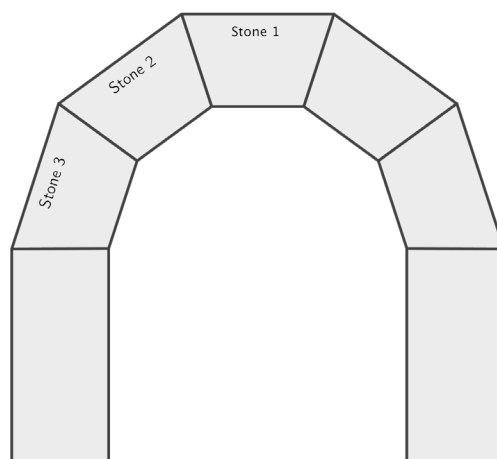
   a.   We will study the force vectors acting on the keystone (stone 1) and stones 2 and 3 on the left side of the arch. Why is it acceptable for us to disregard the forces on the right side of the arch?

b. We will first focus on the forces acting on the keystone. Stone 2 pushes on the left side of the keystone with force vector $\mathbf{p_{1L}}$. The stone to the right of the keystone pushes on the right of the keystone with force vector $\mathbf{p_{1R}}$. We know that these vectors push perpendicular to the sides of the stone, but we do not know their magnitude. All we know is that vectors $\mathbf{p_{1L}}$ and $\mathbf{p_{1R}}$ have the same magnitude.

   i. Find the measure of the acute angle formed by $\mathbf{p_{1L}}$ and the horizontal.
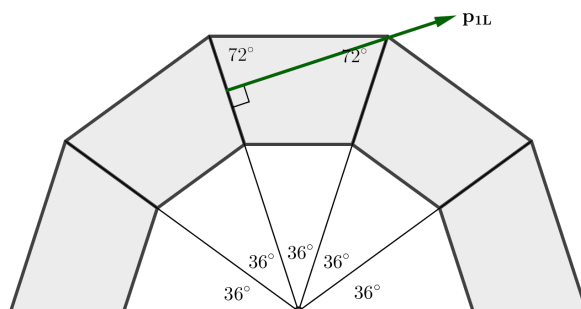
   ii. Find the measure of the acute angle formed by $\mathbf{p_{1R}}$ and the horizontal.

c. Move vectors $\mathbf{p_{1L}}$, $\mathbf{p_{1R}}$ and $\mathbf{g}$ tip-to-tail. Why must these vectors form a triangle?

d. Suppose that vector $\mathbf{g}$ has magnitude 1. Use triangle trigonometry together with the measure of the angles you found in part (b) to find the magnitudes of vectors $\mathbf{p_{1L}}$ and $\mathbf{p_{1R}}$ to the nearest tenth of a unit.

   i. Find the magnitude and direction form of $\mathbf{g}$.

    ii.   Find the magnitude and direction form of $\mathbf{p}_{1L}$.

    iii.   Find the magnitude and direction form of $\mathbf{p}_{1R}$.

e.   Vector $\mathbf{p}_{1L}$ represents the force of stone 1 pushing on the keystone, and by Newton's third law of motion, there is an equal and opposite reaction. Thus, there is a force of the keystone acting on stone 2 that has the same magnitude as $\mathbf{p}_{1L}$ and the opposite direction. Call this vector $\mathbf{v}_{1L}$.

   i.   Find the magnitude and direction form of $\mathbf{v}_{1L}$.

    ii.   Carefully draw vector $\mathbf{v}_{1L}$ on the arch below, with initial point at the point marked O, which is the center of mass of the keystone. Use a protractor measured in degrees and a ruler measured in centimeters.

f.   We will assume that the forces $\mathbf{v}_{2L}$ of stone 2 acting on stone 3 and $\mathbf{v}_{3L}$ of stone 3 acting on the base column have the same magnitude as each other, and twice the magnitude as the force $\mathbf{v}_{1L}$. Why does it make sense that the force vector $\mathbf{v}_{1L}$ is significantly shorter than the other two force vectors?

g.   Find the magnitude and direction form of vector $\mathbf{v}_{2L}$, the force of stone 2 pressing on stone 3. Carefully draw vector $\mathbf{v}_{2L}$ on the arch on page 152, placing its initial point at the terminal point of $\mathbf{v}_{1L}$.

h.   Find the magnitude and direction form of vector v_{3L}, the force of stone 3 pressing on the base column. Carefully draw vector v_{3L} on the arch on page 152, placing its initial point at the terminal point of v_{2L}.

i.   Use the parallelogram method to find the sum of the force vectors v_{1L}, v_{2L}, and v_{3L} on the left side of the arch.

j.   Will the arch stand or fall? Explain how you know.

Plot the force vectors acting on the arch on this diagram to determine whether or not this arch will be able to stand or if it will collapse.
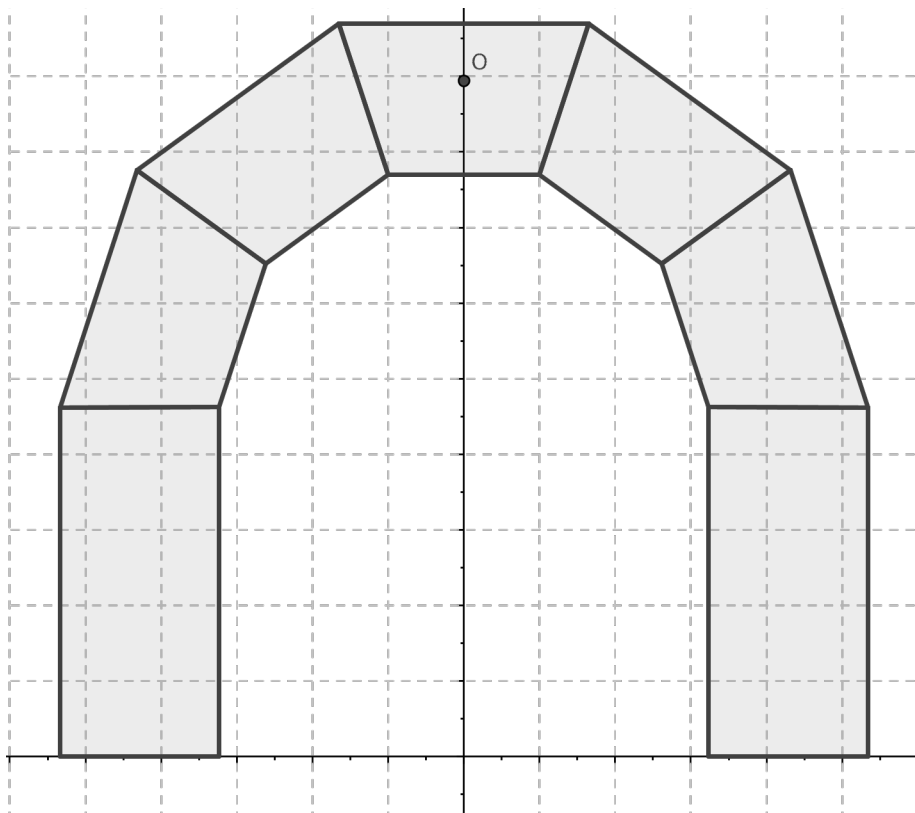

Lesson Summary

A vector can be described using its magnitude and direction.

The direction of a vector **v** can be described either using geographical description, such as 32° north of west, or by the amount of rotation the positive x-axis must undergo to align with the vector **v**, such as rotation by 148° from the positive x-axis.


Problem Set

1. Vectors **v** and **w** are given in magnitude and direction form. Find the coordinate representation of the sum **v** + **w** and the difference **v** − **w**. Give coordinates to the nearest tenth of a unit.
   a. **v**: magnitude 12, direction 50° east of north

      w: magnitude 8, direction 30° north of east
   b. **v**: magnitude 20, direction 54° south of east

      w: magnitude 30, direction 18° west of south

2. Vectors **v** and **w** are given by specifying the length r and the amount of rotation from the positive x-axis. Find the coordinate representation of the sum **v** + **w** and the difference **v** − **w**. Give coordinates to the nearest tenth of a unit.
   a. **v**: length $r = 3$, rotated 12° from the positive x-axis

      w: length $r = 4$, rotated 18° from the positive x-axis
   b. **v**: length $r = 16$, rotated 162° from the positive x-axis

      w: length $r = 44$, rotated $-18°$ from the positive x-axis

3. Vectors **v** and **w** are given in magnitude and direction form. Find the magnitude and direction of the sum **v** + **w** and the difference **v** − **w**. Give the magnitude to the nearest tenth of a unit and the direction to the nearest tenth of a degree.
   a. **v**: magnitude 20, direction 45° north of east

      w: magnitude 8, direction 45° west of north
   b. **v**: magnitude 12.4, direction 54° south of west

      w: magnitude 16.0, direction 36° west of south

4. Vectors **v** and **w** are given by specifying the length r and the amount of rotation from the positive x-axis. Find the length and direction of the sum **v** + **w** and the difference **v** − **w**. Give the magnitude to the nearest tenth of a unit and the direction to the nearest tenth of a degree.
   a. **v**: magnitude $r = 1$, rotated 102° from the positive x-axis

      w: magnitude $r = \frac{1}{2}$, rotated 18° from the positive x-axis

---

    b.   **v**: magnitude $r = 1000$, rotated $-126°$ from the positive x-axis

         w: magnitude $r = 500$, rotated $-18°$ from the positive x-axis

5.   You hear a rattlesnake while out on a hike. You abruptly stop hiking at point S and take eight steps. Then you take another six steps. For each distance below, draw a sketch to show how the sum of your two displacements might add so that you find yourself that distance from point S. Assume that your steps are a uniform size.

    a.   14 steps

    b.   10 steps

    c.   2 steps

6.   A delivery driver travels 2.6 km due north, then 5.0 km due west, and then 4.2 km 45° north of west. How far is he from his starting location? Include a sketch with your answer.

7.   Morgan wants to swim directly across a river, from the east to the west side. She swims at a rate of 1 m/s. The current in the river is flowing due north at a rate of 3 m/s. Which direction should she swim so that she travels due west across the river?

8.   A motorboat traveling at a speed of 4.0 m/s pointed east encounters a current flowing at a speed 3.0 m/s north.

    a.   What is the speed and direction that the motorboat travels?

    b.   What distance downstream does the boat reach the opposite shore?

9.   A ball with mass 0.5 kg experiences a force F due to gravity of 4.9 Newtons directed vertically downward. If this ball is rolling down a ramp that is 30° inclined from the horizontal, what is the magnitude of the force that is directed parallel to the ramp? Assume that the ball is small enough so that all forces are acting at the point of contact of the ball and the ramp.
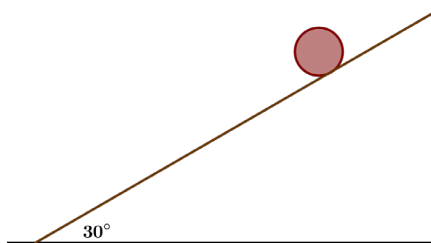
10. The stars in the Big Dipper may all appear to be the same distance from Earth, but they are, in fact, very far from each other. Distances between stars are measured in light years, the distance that light travels in one year. The star Alkaid at one end of the Big Dipper is 138 light years from Earth, and the star Dubhe at the other end of the Big Dipper is 105 light years from earth. From the Earth, it appears that Alkaid and Dubhe are 25.7° apart. Find the distance in light years between stars Alkaid and Dubhe.
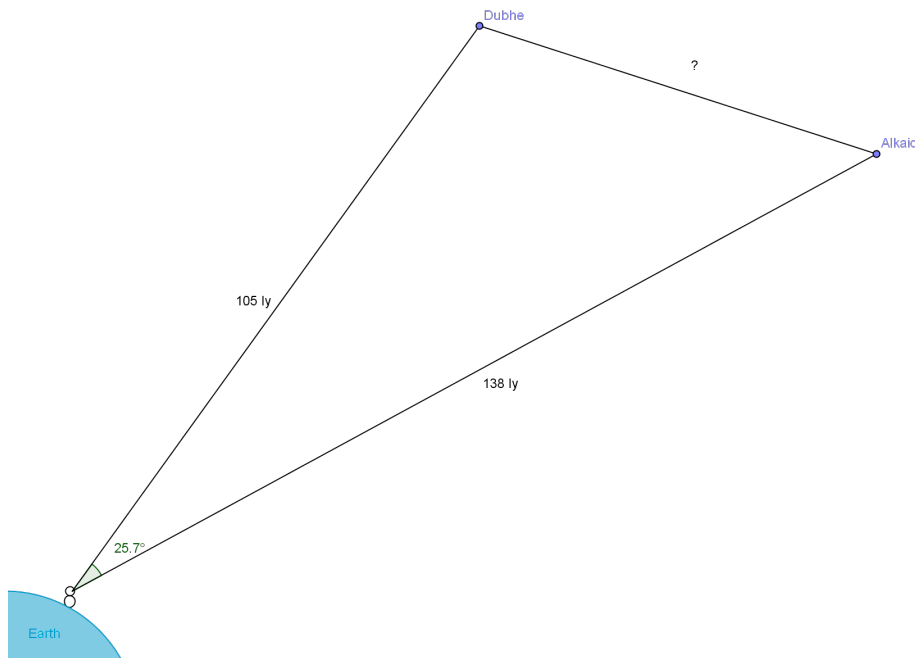
11. A radio station has selected three listeners to compete for a prize buried in a large, flat field. Starting in the center, the contestants were given a meter stick, a compass, a calculator, and a shovel. Each contestant was given the following three vectors, in a different order for each contestant.

    64.2 m, 36° east of north

    42.5 m, 20° south of west

    18.2 m due south.

    The three displacements led to the point where the prize was buried. The contestant that found the prize first won. Instead of measuring immediately, the winner began by doing calculations on paper. What did she calculate?
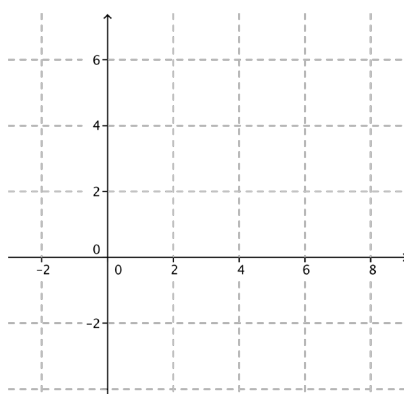
Lesson 21: Vectors and the Equation of a Line

Classwork

Opening Exercise

a. Find three different ways to write the equation that represents the line in the plane that passes through points $(1,2)$ and $(2,-1)$.

b. Graph the line through point $(1,1)$ with slope 2.

Exercises

1. Consider the line ℓ in the plane given by the equation $3x - 2y = 6$.

   a. Sketch a graph of line ℓ on the axes provided.

   b. Find a point on line ℓ and the slope of line ℓ.

c. Write a vector equation for line ℓ using the information you found in part (b).

d. Write parametric equations for line ℓ.

e. Verify algebraically that your parametric equations produce points on line ℓ.

2. Olivia wrote parametric equations $x(t) = 4 + 2t$ and $y(t) = 3 + 3t$. Are her equations correct? What did she do differently from you?

3. Convert the parametric equations $x(t) = 2 - 3t$ and $y(t) = 4 + t$ into slope-intercept form.

4. Find parametric equations to represent the line that passes through point (4,2,9) and has direction vector

$$\vec{v} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}.$$

5. Find a vector form of the equation of the line given by the parametric equations

$$x(t) = 3t$$
$$y(t) = -4 - 2t$$
$$z(t) = 3 - t.$$

Lesson Summary

Lines in the plane and lines in space can be described by either a vector equation or a set of parametric equations.

- Let ℓ be a line in the plane that contains point (x_1, y_1) and has direction vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. If the slope of line ℓ is defined, then $m = \frac{b}{a}$.

A vector form of the equation that represents line ℓ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} t.$$

Parametric equations that represent line ℓ are

$$x(t) = x_1 + at$$
$$y(t) = y_1 + bt.$$

- Let ℓ be a line in space that contains point (x_1, y_1, z_1) and has direction vector $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

A vector form of the equation that represents line ℓ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} t.$$

Parametric equations that represent line ℓ are

$$x(t) = x_1 + at$$
$$y(t) = y_1 + bt$$
$$z(t) = z_1 + ct.$$

Problem Set

1. Find three points on the line in the plane with parametric equations $x(t) = 4 - 3t$ and $y(t) = 1 + \frac{1}{3}t$.

2. Find vector and parametric equations to represent the line in the plane with the given equation.
   a. $y = 3x - 4$
   b. $2x - 5y = 10$
   c. $y = -x$
   d. $y - 2 = 3(x + 1)$

3. Find vector and parametric equations to represent the following lines in the plane.
   a. the x-axis
   b. the y-axis

c. the horizontal line with equation $y = 4$

d. the vertical line with equation $x = -2$

e. the horizontal line with equation $y = k$, for a real number k

f. the vertical line with equation $x = h$, for a real number h

4. Find the point-slope form of the line in the plane with the given parametric equations.

   a. $x(t) = 2 - 4t, y(t) = 3 - 7t$

   b. $x(t) = 2 - \frac{2}{3}t, y(t) = 6 + t$

   c. $x(t) = 3 - t, y(t) = 3$

   d. $x(t) = t, y(t) = t$

5. Find vector and parametric equations for the line in the plane through point P in the direction of vector $\mathbf{v}$.

   a. $P = (1,5), \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   b. $P = (0,0), \vec{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

   c. $P = (-3,-1), \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

6. Determine if the point A is on the line ℓ represented by the given parametric equations.

   a. $A = (3,1), x(t) = 1 + 2t$ and $y(t) = 3 - 2t$.

   b. $A = (0,0), x(t) = 3 + 6t$ and $y(t) = 2 + 4t$

   c. $A = (2,3), x(t) = 4 - 2t$ and $y(t) = 4 + t$

   d. $A = (2,5), x(t) = 12 + 2t$ and $y(t) = 15 + 2t$

7. Find three points on the line in space with parametric equations $x(t) = 4 + 2t, y(t) = 6 - t$, and $z(t) = t$.

8. Find vector and parametric equations to represent the following lines in space.

   a. the x-axis

   b. the y-axis

   c. the z-axis

9. Convert the equation given in vector form to a set of parametric equations for the line ℓ.

   a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} t$

   b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} t$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ -8 \end{bmatrix} t$

10. Find vector and parametric equations for the line in space through point P in the direction of vector $\vec{v}$.

   a. $P = (1,4,3), \vec{v} = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$

   b. $P = (2,2,2), \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   c. $P = (0,0,0), \vec{v} = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$

11. Determine if the point A is on the line ℓ represented by the given parametric equations.

   a. $A = (3,1,1), x(t) = 5 - t, y(t) = -5 + 3t$ and $z(t) = 9 - 4t$

   b. $A = (1,0,2), x(t) = 7 - 2t, y(t) = 3 - t$ and $z(t) = 4 - t$

   c. $A = (5,3,2), x(t) = 8 + t, y(t) = -t$ and $z(t) = -4 - 2t$

Lesson 22: Linear Transformations of Lines

Classwork

Opening Exercise

a. Find parametric equations of the line through point $P(1,1)$ in the direction of vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

b. Find parametric equations of the line through point $P(2,3,1)$ in the direction of vector $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$.

Exercises 1–3

1. Consider points $P(2,1,4)$ and $Q(3,-1,2)$, and define a linear transformation by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find parametric equations to describe the image of line $\overleftrightarrow{PQ}$ under the transformation L.

2. The process that we developed for images of lines in $\mathbb{R}^3$ also applies to lines in $\mathbb{R}^2$. Consider points $P(2,3)$ and $Q(-1,4)$. Define a linear transformation by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Find parametric equations to describe the image of line $\overleftrightarrow{PQ}$ under the transformation L.

3. Not only is the image of a line under a linear transformation another line, but the image of a line segment under a linear transformation is another line segment. Let P, Q, and L be as specified in Exercise 2. Find parametric equations to describe the image of segment $\overline{PQ}$ under the transformation L.

Lesson Summary

We can find vector and parametric equations of a line in the plane or in space if we know two points that the line passes through, and we can find parametric equations of a line segment in the plane or in space by restricting the values of t in the parametric equations for the line.

- Let ℓ be a line in the plane that contains points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then a direction vector is given by $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$, and an equation in vector form that represents line ℓ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} t, \text{ for all real numbers } t.$$

Parametric equations that represent line ℓ are

$$x(t) = x_1 + (x_2 - x_1)t$$
$$y(t) = y_1 + (y_2 - y_1)t \text{ for all real numbers } t.$$

Parametric equations that represent segment $\overline{PQ}$ are

$$x(t) = x_1 + (x_2 - x_1)t$$
$$y(t) = y_1 + (y_2 - y_1)t \text{ for } t \le t \le 1.$$

- Let ℓ be a line in space that contains points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Then a direction vector is given by $\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$, and an equation in vector form that represents line ℓ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} t, \text{ for all real numbers } t.$$

Parametric equations that represent line ℓ are

$$x(t) = x_1 + (x_2 - x_1)t$$
$$y(t) = y_1 + (y_2 - y_1)t$$
$$z(t) = z_1 + (z_2 - z_1)t \text{ for all real numbers } t.$$

Parametric equations that represent segment $\overline{PQ}$ are

$$x(t) = x_1 + (x_2 - x_1)t$$
$$y(t) = y_1 + (y_2 - y_1)t$$
$$z(t) = z_1 + (z_2 - z_1)t \text{ for } 0 \le t \le 1.$$

- The image of a line $\overleftrightarrow{PQ}$ in the plane under a linear transformation L is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = L(P) + \big(L(Q) - L(P)\big)t, \text{ for all real numbers } t.$$

- The image of a line $\overleftrightarrow{PQ}$ in space under a linear transformation L is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = L(P) + \big(L(Q) - L(P)\big)t, \text{ for all real numbers } t.$$

Problem Set

1. Find parametric equations of the line $\overleftrightarrow{PQ}$ through points P and Q in the plane.
   a. $P(1,3)$, $Q(2,-5)$
   b. $P(3,1)$, $Q(0,2)$
   c. $P(-2,2)$, $Q(-3,-4)$

2. Find parametric equations of the line $\overleftrightarrow{PQ}$ through points P and Q in space.
   a. $P(1,0,2)$, $Q(4,3,1)$
   b. $P(3,1,2)$, $Q(2,8,3)$
   c. $P(1,4,0)$, $Q(-2,1,-1)$

3. Find parametric equations of segment $\overline{PQ}$ through points P and Q in the plane.
   a. $P(2,0)$, $Q(2,10)$
   b. $P(1,6)$, $Q(-3,5)$
   c. $P(-2,4)$, $Q(6,9)$

4. Find parametric equations of segment $\overline{PQ}$ through points P and Q in space.
   a. $P(1,1,1)$, $Q(0,0,0)$
   b. $P(2,1,-3)$, $Q(1,1,4)$
   c. $P(3,2,1)$, $Q(1,2,3)$

5. Jeanine claims that the parametric equations $x(t) = 3 - t$ and $y(t) = 4 - 3t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is she correct? Explain how you know.

6. Kelvin claims that the parametric equations $x(t) = 3 + t$ and $y(t) = 4 + 3t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.

7. LeRoy claims that the parametric equations $x(t) = 1 + 3t$ and $y(t) = -2 + 9t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.

8. Miranda claims that the parametric equations $x(t) = -2 + 2t$ and $y(t) = 3 - t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is she correct? Explain how you know.

9. Find parametric equations of the image of the line $\overleftrightarrow{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ for the given points P, Q, and matrix A.
   a. $P(2,4)$, $Q(5,-1)$, $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

b. $P(1,-2), Q(0,0), A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

c. $P(2,3), Q(1,10), A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

10. Find parametric equations of the image of the line $\overleftrightarrow{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given points P, Q, and matrix A.

a. $P(1,-2,1), Q(-1,1,3), A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

b. $P(2,1,4), Q(1,-1,-3), A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

c. $P(0,0,1), Q(4,2,3), A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

11. Find parametric equations of the image of the segment $\overline{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \begin{bmatrix} x \\ y \end{bmatrix}$ for the given points P, Q, and matrix A.

a. $P(2,1), Q(-1,-1), A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

b. $P(0,0), Q(4,2), A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

c. $P(3,1), Q(1,-2), A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

12. Find parametric equations of the image of the segment $\overline{PQ}$ under the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given points P, Q and matrix A.

a. $P(0,1,1), Q(-1,1,2), A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

b. $P(2,1,1), Q(1,1,2), A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

c. $P(0,0,1), Q(1,0,0), A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

Lesson 23: Why Are Vectors Useful?

Classwork

Opening Exercise

Suppose a person walking through South Boston, MA travels from point A due east along E 3rd Street for 0.3 miles and then due south along K Street for 0.4 miles to end on point B (as shown on the map).

a. Find the magnitude and direction of vector $\overrightarrow{AB}$.

b. What information does vector $\overrightarrow{AB}$ provide?

Example 1

An airplane flying from Dallas-Fort Worth to Atlanta veers off course to avoid a storm. The plane leaves Dallas-Fort Worth traveling 50° east of north and flies for 450 miles before turning to travel 70° east of south for 350 miles. What is the resultant displacement of the airplane? Include both the magnitude and direction of the displacement.
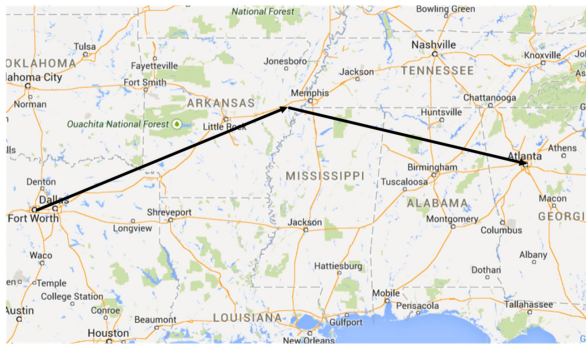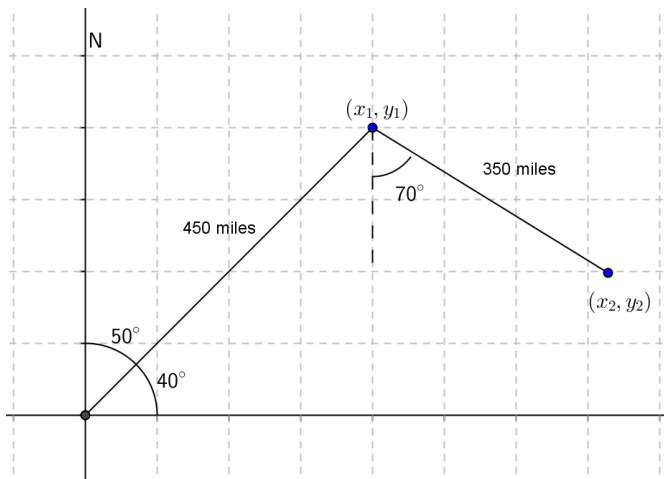
Exercises 1–4

1. A motorized robot moves across the coordinate plane. Its position $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ at time t seconds is given by $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a + tv$ where $a = \begin{pmatrix} 4 \\ -10 \end{pmatrix}$ and $v = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$. The units of distance are measured in meters.

    a. Where is the robot at time $t = 0$?

   b. Plot the path of the robot.

    c. Describe the path of the robot.

    d. Where is the robot 10 seconds after it starts moving?

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e.   Where is the robot when it is 10 meters from where it started?

f.   Is the robot traveling at a constant speed? Explain, and if the speed is constant, state the robot's speed.

2.   A row boat is crossing a river that is 500 m wide traveling due east at a speed of 2.2 m/s. The river's current is 0.8 m/s due south.

a.   What is the resultant velocity of the boat?

b.   How long does it take for the boat to cross the river?

c.   How far downstream is the boat when it reaches the other side?

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3. Consider the airplane from Example 1 that leaves Dallas-Fort Worth with a bearing of 50°. (Note that the bearing is the number of degrees east of north.) The plane is traveling at a speed of 550 mph. There is a crosswind of 40 mph due east. What is the resultant velocity of the airplane?

4. A raft floating in the water experiences an eastward force of 100 N due to the current of the water and a southeast force of 400 N due to wind.

    a. In what direction will the boat move?
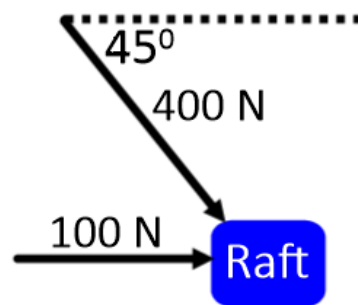
    b. What is the magnitude of the resultant force on the boat?

    c. If the force due to the wind doubles, does the resultant force on the boat double? Explain or show work that supports your answer.

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Problem Set

1. Suppose Madison is traveling due west for 0.5 miles and then due south for 1.2 miles.

    a. Draw a picture of this scenario with her starting point labeled A, ending point B, and include the vector $\overrightarrow{AB}$.

    b. State the value of $\overrightarrow{AB}$.

    c. What is the magnitude and direction of $\overrightarrow{AB}$?

2. An object's azimuth is the angle of rotation of its path measured clockwise from due north. For instance, an object traveling due north would have an azimuth of $0°$, and due east would have an azimuth of $90°$.

    a. What are the azimuths for due south and due west?

    b. Consider a craft on an azimuth of $215°$ traveling 30 knots.

        i. Draw a picture representing the situation.

        ii. Find the vector representing this craft's speed and direction.

3. Bearings can be given from any direction, not just due north. For bearings, like azimuths, clockwise angles are represented by positive degrees and counterclockwise angles are represented by negative degrees. A ship is traveling $30°$ east of north at 18 kn, then turns $20°$, maintaining its speed.

    a. Draw a picture representing the situation.

    b. Find vectors v and w representing the first and second bearing.

    c. Find the sum of v and w. What does v + w represent?

    d. If the ship travels for one hour along each bearing, then how far north of its starting position has it traveled? How far east has it traveled?

4. A turtle starts out on a grid with coordinates $\begin{pmatrix} 4 \\ -6.5 \end{pmatrix}$ where each unit is one furlong. Its horizontal location is given by the function $x(t) = 4 + -2t$, and its vertical location is given by $y(t) = -6.5 + 3t$ for t in hours.

    a. Write the turtle's location using vectors.

    b. What is the speed of the turtle?

    c. If a hare's location is given as $\begin{pmatrix} x_h(t) \\ y_h(t) \end{pmatrix} = a + tv$ where $a = \begin{pmatrix} 23 \\ -35 \end{pmatrix}$ and $v = \begin{pmatrix} -8 \\ 12 \end{pmatrix}$, then what is the speed of the hare? How much faster is the hare traveling than the turtle?

    d. Which creature will reach $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ first?

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5. A rocket is launched at an angle of $33°$ from the ground at a rate of 50 m/s.

   a. How fast is the rocket traveling up to the nearest m/s?

   b. How fast is the rocket traveling to the right to the nearest m/s?

   c. What is the rocket's velocity vector?

   d. Does the magnitude of the velocity vector agree with the set-up of the problem? Why or why not?

   e. If a laser is in the path of the rocket and would like to strike the rocket, in what direction does the laser need to be aimed? Express your answer as a vector.

6. A boat is drifting downriver at a rate of 5 nautical miles per hour. If the occupants of the boat want to travel to the shore, do they need to overcome the current downriver? Use vectors to explain why or why not.

7. A group of friends moored their boats together and fell asleep on the lake. Unfortunately, their lashings came undone in the night, and they have drifted apart. Gerald's boat traveled due west along with the current of the lake which moves at a rate of $\frac{1}{2}$ mi/hr and Helena's boat was pulled southeast by some pranksters and set drifting at a rate of 2 mi/hr.

   a. If the boats came untied three hours ago, how far apart are the boats?

   b. If Gerald drops anchor, then in what direction does Helena need to travel in order to reunite with Gerald?

8. Consider any two vectors in space, u and v with θ the angle between them.

   a. Use the law of cosines to find the value of $\|u - v\|$.

   b. Use the law of sines to find the value of ψ, the angle between $u - v$ and u. State any restrictions on the variables.

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Lesson 24: Why Are Vectors Useful?

Classwork

Opening Exercise

Two particles are moving in a coordinate plane. Particle 1 is at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and moving along the velocity vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Particle 2 is at the point $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and moving along the velocity vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Are the two particles going to collide? If so, at what point, and at what time? Assume that time is measured in seconds.

Exercise 1

Consider lines $\ell = \{(x, y) | \langle x, y \rangle = t\langle 1, -2 \rangle\}$, and $m = \{(x, y) | \langle x, y \rangle = t\langle -1, 3 \rangle\}$.

    a.   To what graph does each line correspond?

    b.   Describe what happens to the vectors defining these lines under the transformation $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

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c.   Show this transformation graphically.

Exercise 2

Consider lines $\ell = \{(x, y)|\langle x, y \rangle = \langle 1,1 \rangle + t\langle 1, -2 \rangle\}$, and $m = \{(x, y)|\langle x, y \rangle = \langle 1,1 \rangle + t\langle -1,3 \rangle\}$.

a.   What is the solution to the system of equations given by lines ℓ and m?

b.   Describe what happens to the lines under the transformation $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.

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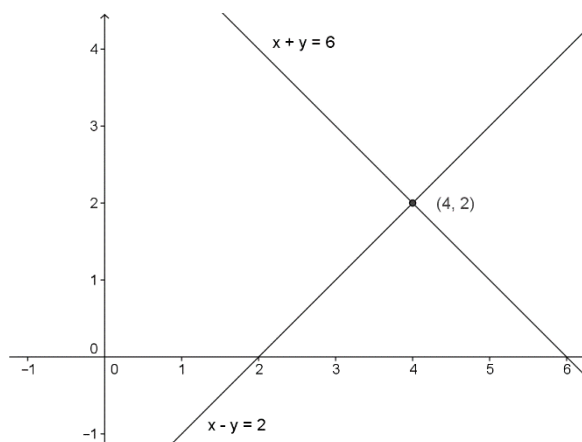
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c.   What is the solution to the system of equations after the transformation?

Exercise 3

The system of equations is given below. A graph of the equations and their intersection point is also shown.

$$x + y = 6$$
$$x - y = 2$$

a.   Write each line in the form $L(t) = \mathbf{p} + \mathbf{v}t$ where $\mathbf{p}$ is the position vector whose terminal point is the solution of the system, and $\mathbf{v}$ is the velocity vector that defines the path of a particle traveling along the line such that when $t = 0$, the solution to the system is $(x(0), y(0))$.

b.   Describe a translation that will take the point $(x(0), y(0))$ to the origin. What is the new system?

c. Describe a transformation matrix A that will rotate the lines to the x- and y-axes. What is the new system?

d. Describe a translation that will result in a system that has the same solution set as the original system. What is the new system of equations?

Problem Set

1. Consider the system of equations $\begin{cases} y = 3x + 2 \\ y = -x + 14 \end{cases}$.

   a. Solve the system of equations.

   b. Ilene wants to rotate the lines representing this system of equations about their solution and wishes to apply the matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ to any point A on either of the lines. If Ilene is correct, then applying a rotation to the solution will map the solution to itself. Let $\theta = 90°$, and find where Ilene's strategy maps the solution you found in part (a). What is wrong with Ilene's strategy?

   c. Jasmine thinks that in order to apply a rotation to some point on either of these two lines, the entire system needs to be shifted so that the pivot point is translated to the origin. For an arbitrary point A on either of the two lines, what transformation needs to be applied so that the pivot point is mapped to the origin?

   d. After applying your transformation in part (c), apply Ilene's rotation matrix for $\theta = 90°$. Show that the pivot point remains on the origin. What happens to the point $(0,2)$ after both of these transformations?

   e. Although Jasmine and Ilene were able to work together to rotate the points around the pivot point, now their lines are nowhere near the original lines. What transformation will bring the system of equations back so that the pivot point returns to where it started and all other points have been rotated? Find the final image of the point $(0,2)$.

   f. Summarize your results in parts (a)–(e).

Extension:

1. Let $b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then answer the following questions.

   a. Find $1 \cdot b_1 + 0 \cdot b_2$.

   b. Find $0 \cdot b_1 + 1 \cdot b_2$.

   c. Find $1 \cdot b_1 + 1 \cdot b_2$.

   d. Find $3 \cdot b_1 + 2 \cdot b_2$.

   e. Find $0 \cdot b_1 + 0 \cdot b_2$.

   f. Find $x \cdot b_1 + y \cdot b_2$ for x, y real numbers.

   g. Summarize your results from parts (a)–(f). Can you use b_1 and b_2 to define any point in $\mathbb{R}^2$?

2. Let $b_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $b_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Then answer the following questions.

   a. Find $1 \cdot b_1 + 1 \cdot b_2$.

   b. Find $0 \cdot b_1 + 1 \cdot b_2$.

   c. Find $1 \cdot b_1 + 0 \cdot b_2$.

   d. Find $-4 \cdot b_1 + 2 \cdot b_2$.

   e. Solve $r \cdot b_1 + s \cdot b_2 = 0$.

   f. Solve $r \cdot b_1 + s \cdot b_2 = \begin{pmatrix} 22 \\ -7 \end{pmatrix}$.

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g.   Is there any point $\begin{pmatrix} x \\ y \end{pmatrix}$ that cannot be expressed as a linear combination of b_1 and b_2 (i.e., where $r \cdot b_1 + s \cdot b_2 = \begin{pmatrix} x \\ y \end{pmatrix}$ has real solutions, for x, y real numbers)?

h.   Explain your response to part (g) geometrically.

Lesson 25: First-Person Computer Games

Classwork

Exploratory Challenge

In this drawing task, the "eye" or the "camera" is the point, and the shaded figure is the "TV screen." The cube is in the 3-D universe of the computer game.

By using lines drawn from each vertex of the cube to the point, draw the image of the 3-D cube on the screen.
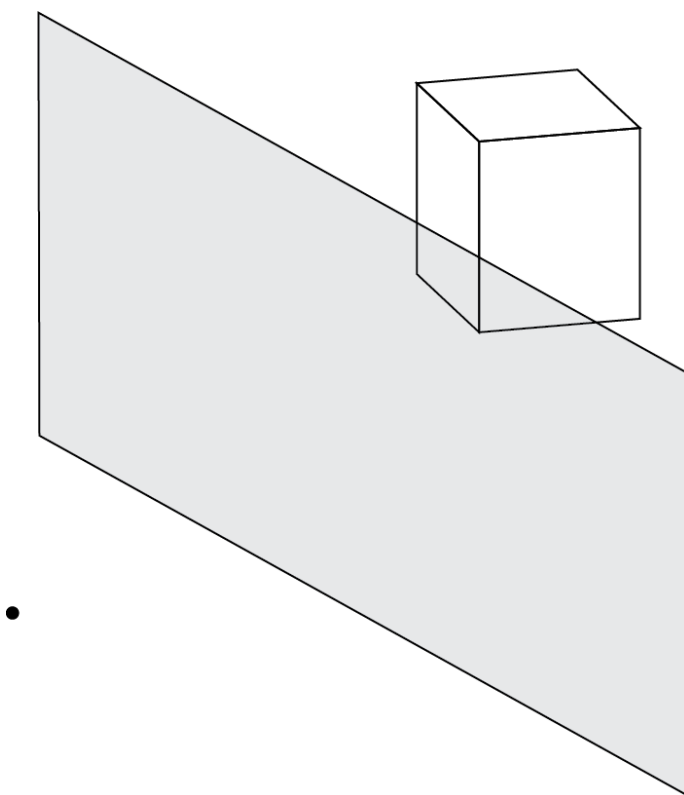
Example

1. When three-dimensional objects are projected onto screens with finite dimensions, it often limits the field of view (FOV), or the angle the scene represents. This limiting effect can vary based on the size of the screen and position of the observer.

    a. Sketch a diagram that could be used to calculate a viewer's field of view θ in relation to the horizontal width of the screen w and the distance the viewer is from the screen d.

    b. Assume that a person is sitting directly in front of a television screen whose width is 48 inches at a distance of 8 feet from the screen. Use your diagram and right-triangle trigonometry to find the viewer's horizontal field of view θ.

    c. How far would a viewer need to be from the middle of a computer screen with a width of 15 inches to produce the same field of view as the person in front of the television?

d.   Write a general statement about the relationship between screen size and field of view.

Exercise 1

In this drawing task, the "eye" or the "camera" is the point, and the shaded figure is the "TV screen." The cube is in the 3-D universe of the computer game.

By using lines drawn from each vertex of the cube to the point, draw the image of the 3-D cube on the screen. A horizon line and two additional vanishing points have been included to help you. The image point of the first vertex is shown.
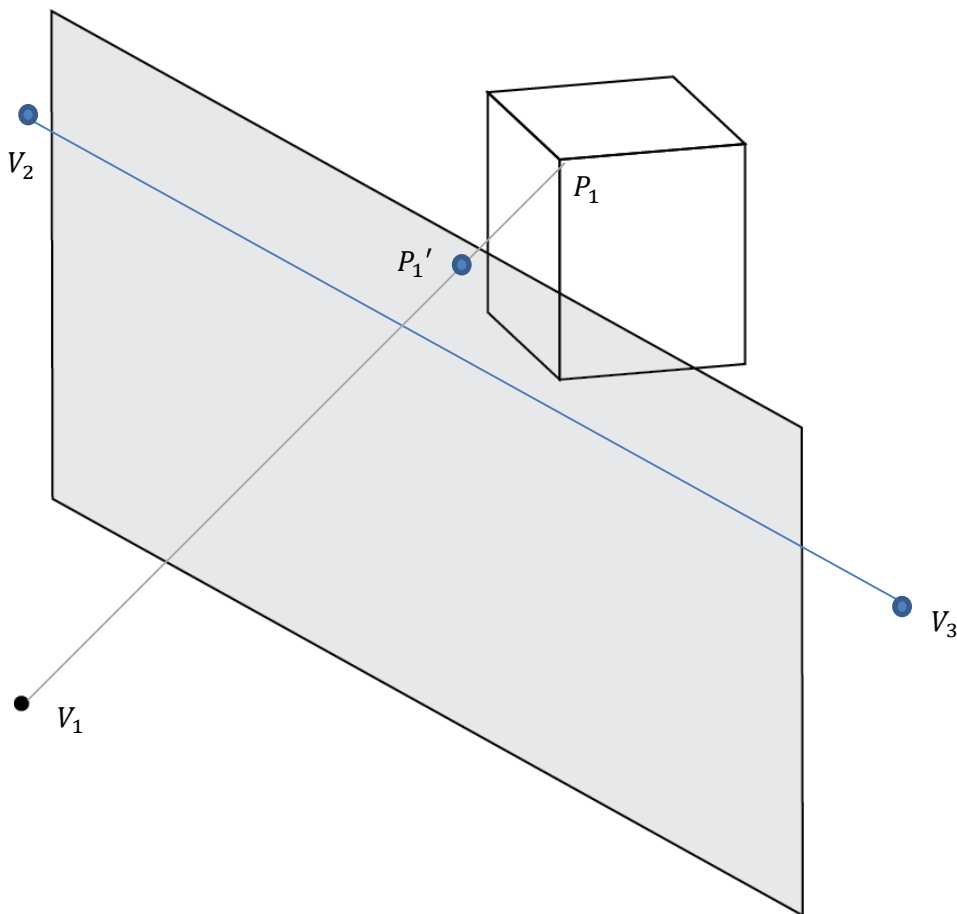
**Exercise 2**

Let's assume that the point V_1 in our projection diagram is at the origin and the upper right vertex of the cube is located at $\begin{pmatrix} 5 \\ 8 \\ 4 \end{pmatrix}$. If our screen represents the plane $y = 2$, use matrix multiplication to determine the vector that represents the line of sight from the observer to the projected point on the screen. Explain your thinking.

Problem Set

1. Projecting the image of a three-dimensional scene onto a computer screen has the added constraint of the screen size limiting our field of view, or FOV. When we speak of FOV, we wish to know what angle of view the scene represents. Humans have remarkably good peripheral vision. In New York State, the requirement for a driver's license is a horizontal FOV of no less than $140°$. There is no restriction placed on the vertical field of vision, but humans normally have a vertical FOV of greater than $120°$.

    a. Consider the (simulated) distance the camera is from the screen as d, the horizontal distance of the screen as w, and the horizontal FOV as θ, then use the diagram below and right-triangle trigonometry to help you find θ in terms of w and d.
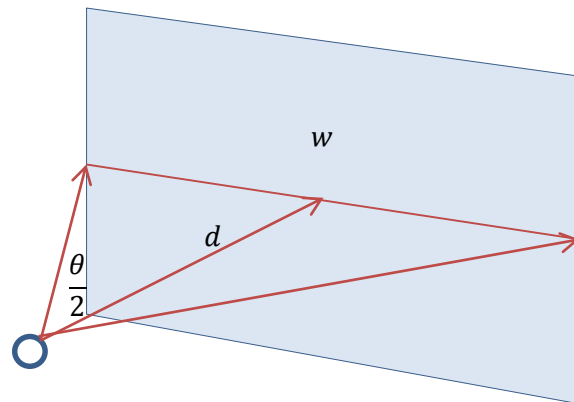
    b. Repeat procedures from part (a), but this time let h represent the height of the screen and ψ represent the vertical FOV.
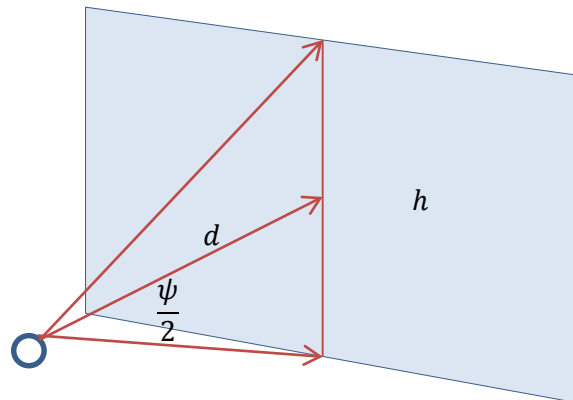
    c. If a particular game uses an aspect ratio of $16\colon 9$ as its standard view and treats the camera as though it were 8 units away, find the horizontal and vertical FOVs for this game. Round your answers to the nearest degree.

    d. When humans sit too close to monitors with FOVs less than what they are used to in real life or in other games, they may grow dizzy and feel sick. Does the game in part (c) run the risk of that? Would you recommend this game be played on a computer or on a television with these FOVs?

2. Computers regularly use polygon meshes to model three-dimensional objects. Most polygon meshes are a collection of triangles that approximate the shape of a three-dimensional object. If we define a *face* of a polygon mesh to be a triangle connecting three vertices of the shape, how many faces at minimum do the following shapes require?

   a. A cube.

   b. A pyramid with a square base.

   c. A tetrahedron.

   d. A rectangular prism.

   e. A triangular prism.

   f. An octahedron.

   g. A dodecahedron.

   h. An icosahedron.

   i. How many faces should a sphere have?

3. In the beginning of 3-D graphics, objects were created only using the wireframes from a polygon mesh without shading or textures. As processing capabilities increased, 3-D models became more advanced, and shading and textures were incorporated into 3-D models. One technique that helps viewers visualize how shading works on a 3-D figure is to include both an "eye" and a "light source." Vectors are drawn from the eye to the figure, and then reflected to the light (this technique is called ray tracing). See the diagram below.
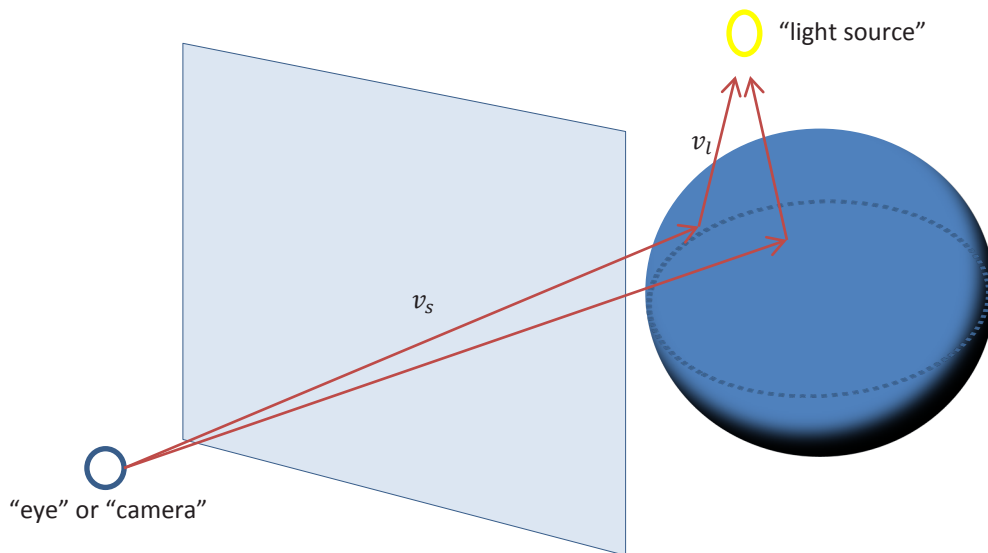
a. Using this technique, the hue of the object depends on the sum of the magnitudes of the vectors. Assume the eye in the picture above is located at the origin, v_s is the vector from the eye to the location $\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$, and the light source is located at $\begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}$. Then find v_l, the vector from v_s to the light source, and the sum of the magnitudes of the vectors.

b. What direction does light travel in real life, and how does this compare to the computer model portrayed above? Can you think of any reason why the computer only traces the path of vectors that start at the "eye"?

Lesson 26: Projecting a 3-D Object onto a 2-D Plane

Exercises

1. Describe the set of points $(8t, 3t)$, where t represents a real number.

2. Project the point $(8, 3)$ onto the line $x = 1$.

3. Project the point $(8, 3)$ onto the line $x = 5$.

4. Project the point $(-1, 4, \ 5)$ onto the plane $y = 1$.

5. Project the point $(9, 5, -8)$ onto the plane $z = 3$.

Problem Set

1. A cube in 3-D space has vertices $\begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}, \begin{pmatrix} 13 \\ 10 \\ 10 \end{pmatrix}, \begin{pmatrix} 10 \\ 13 \\ 10 \end{pmatrix}, \begin{pmatrix} 10 \\ 10 \\ 13 \end{pmatrix}, \begin{pmatrix} 13 \\ 13 \\ 10 \end{pmatrix}, \begin{pmatrix} 13 \\ 10 \\ 13 \end{pmatrix}, \begin{pmatrix} 10 \\ 13 \\ 13 \end{pmatrix}, \begin{pmatrix} 13 \\ 13 \\ 13 \end{pmatrix}.$

    a. How do we know that these vertices trace a cube?

    b. What is the volume of the cube?

    c. Let $z = 1$. Find the eight points on the screen that represent the vertices of this cube (some may be obscured).

    d. What do you notice about your result in part (c)?

2. An object in 3-D space has vertices $\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}.$

    a. What kind of shape is formed by these vertices?

    b. Let $y = 1$. Find the five points on the screen that represent the vertices of this shape.

3. Consider the shape formed by the vertices given in Problem 2.

    a. Write a transformation matrix that will rotate each point around the y-axis θ degrees.

    b. Project each rotated point onto the plane $y = 1$ if $\theta = 45°$.

    c. Is this the same as rotating the values you obtained in Problem 3 by $45°$?

4. In technical drawings, it is frequently important to preserve the scale of the objects being represented. In order to accomplish this, instead of a perspective projection, an orthographic projection is used. The idea behind the orthographic projection is that the points are translated at right angles to the screen (the word stem *ortho-* means *straight* or *right*). To project onto the xy-plane for instance, we can use the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

    a. Project the cube in Problem 1 onto the xy-plane by finding the 8 points that correspond to the vertices.

    b. What do you notice about the vertices of the cube after projecting?

    c. What shape is visible on the screen?

    d. Is the area of the shape that is visible on the screen what you expected from the original cube? Explain.

    e. Summarize your findings from parts (a)–(d).

    f. State the orthographic projection matrices for the xz-plane and the yz-plane.

    g. In regard to the dimensions of the orthographic projection matrices, what causes the outputs to be two-dimensional?

5. Consider the point $A = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ in the field of view from the origin through the plane $z = 1$.

   a. Find the projection of A onto the plane $z = 1$.

   b. Find a 3×3 matrix P such that PA finds the projection of A onto the plane $z = 1$.

   c. How does the matrix change if instead of projecting onto $z = 1$, we project onto $z = c$, for some real number $c \neq 0$?

   d. Find the scalars that will generate the image of A onto the planes $x = c$ and $y = c$, assuming the image exists. Describe the scalars in words.

Extension:

6. Instead of considering the rotation of a point about an axis, consider the rotation of the camera. Rotations of the camera will cause the screen to rotate along with it, so that to the viewer, the screen appears immobile.

   a. If the camera rotates θ_x around the x-axis, how does the computer world appear to move?

   b. State the rotation matrix we could use on a point A to simulate rotating the camera and computer screen by θ_x about the x-axis but in fact keeping the camera and screen fixed.

   c. If the camera rotates θ_y around the y-axis, how does the computer world appear to move?

   d. State the rotation matrix we could use on a point A to simulate rotating the camera and computer screen by θ_y about the y-axis but in fact keeping the camera and screen fixed.

   e. If the camera rotates θ_z around the z-axis, how does the computer world appear to move?

   f. State the rotation matrix we could use on a point A to simulate rotating the camera and computer screen by θ_z about the z-axis but in fact keeping the camera and screen fixed.

   g. What matrix multiplication could represent the camera starting at a relative angle $(\theta_x, \theta_y, \theta_z)$? Apply the transformations in the order z-y-x. Do not find the product.

Lesson 27: Designing Your Own Game

Classwork

Example 1

a.   Select the Sea Surface as your background scene, and select the setup scene mode. From the Gallery By Class Hierarchy, select Swimmer Classes, then Marine Mammals, and a Dolphin. Name your dolphin. Describe what you see.

b.   Click on the dolphin, and from the handle style buttons in the top right corner of the screen, select translation. By selecting the arrows on the dolphin, we can move it to different locations in the screen. Move the dolphin left and right, up and down, forward and backward. Then, move it so that its coordinates are $(0, 0, 0)$.

c.   Use a matrix to describe each of the movements of the dolphin from its location at the origin.

   i.   Move 2 units right.

   ii.   Move 4 units down.

   iii. Move 3 units forward.

d. Click on the rotation button from the handle style buttons, and practice rotating the dolphin about the three axes through its center. Use a matrix to represent the motion of the dolphin described. Assume the center of rotation of the dolphin is at the origin.

   i. Rotation counterclockwise one full turn about the z-axis

   ii. Rotation counterclockwise one half turn about the x-axis

e. Select Edit Code from the screen. Drag and drop this.dolphinturn from the procedures menu and drop it into the declare procedures region on the right. Select from the drop-down menus: this.dolphin turn LEFT 0.25 as seen by this duration 2.0 BEGIN_AND_END_ABRUPTLY. Then run the program. Describe what you see. Represent the motion using a matrix.

f. Drag and drop this.dolphinroll from the procedures menu, and drop it into the declare procedures region on the right beneath the turn procedure. Select from the drop-down menus: this.dolphin roll RIGHT 3.0 as seen by this duration 2.0 BEGIN_AND_END_ABRUPTLY. Then run the program. Describe what you see. Represent the motion using a matrix.

Exercises

1. Open ALICE 3.1. Select a background and 3 characters to create a scene. Describe the scene, including the coordinates of the pivot point for each character and the direction each character is facing.

2.

    a. Describe the location of a plane $x = 5$ in your scene from the perspective of the viewer.

    b. Determine the coordinates of the pivot points for each character if they were projected onto the plane $x = 5$.

3. Create a short scene that includes a one-step turn or roll procedure for each of the characters. The procedures should be unique. Write down the procedures in the space provided. After each procedure, describe what the character did in the context of the scene. Then describe the character's motion using transformational language. Finally, represent each procedure using matrix operations.

Problem Set

1. For the following commands, describe a matrix you can use to get the desired result. Assume the character is centered at the origin, facing in the negative z direction with positive x on its right for each command. Let c be a nonzero real number.

    a. move LEFT c

    b. move RIGHT c

    c. move UP c

    d. move DOWN c

    e. move FORWARD c

    f. move BACKWARD c

    g. turn LEFT c

    h. turn RIGHT c

    i. turn FORWARD c

    j. turn BACKWARD c

    k. roll LEFT c

    l. roll RIGHT c

    m. resize c

    n. resizeWidth c

    o. resizeHEIGHT c

    p. resizeDEPTH c

2. In each of the transformations above, we have assumed that each animation will take one second of time. If T is the number of seconds an animation takes and t is the current running time of the animation, then rewrite the following commands as a function of t.

    a. move RIGHT c duration T

    b. turn FORWARD c duration T

    c. resize c duration T

3. For computational simplicity, we have been assuming that the pivot points of our characters occur at the origin.

    a. If we apply a rotation matrix like we have been when the pivot point is at the origin, what will happen to characters that are not located around the origin?

    b. Let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the pivot point of any three-dimensional object and $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a point on the surface of the object. Moving the pivot point to the origin has what effect on A? Find A', the image of A after moving the object so that its pivot point is the origin.

    c. Apply a rotation of θ about the x-axis to A'. Does this transformation cause a pivot or what you described in part (a)?

    d. After applying the rotation, translate the object so that its pivot point returns to $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Was the x-coordinate affected by the rotation? Was the pivot point?

    e. Summarize what you found in (a)–(d).

Extension:

4. In first-person computer games, we think of the camera moving left-right, forward-backward, and up-down. For computational simplicity, the camera and screen stays fixed and the objects in the game world move in the opposite direction instead. Let the camera be located at $(0,0,0)$, the screen be located at $z = 1$, and the point $v = \begin{bmatrix} 10 \\ 6 \\ 5 \end{bmatrix}$ represent the center of an object in the game world. If a character in ALICE is the camera, then answer the following questions.

    a. What are the coordinates of the projection of v on the screen?

    b. What is the value of the image of v as the character moves 4 units closer to v in the x direction?

    c. What are the coordinates of the projection of v'?

    d. If the character jumps up 6 units, then where does the image of v move on the screen?

    e. In Lesson 26, you learned that if the camera is not in the standard orientation, that rotation matrices need to be applied to the camera first; in Lesson 27, you learned that rotation matrices only pivot an object if that object is located at the origin; if a camera is in a non-standard orientation and not located at the origin, then should the rotation matrices be applied first or a translation to the origin?