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Vectors and Matrices

[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Precalculus and Advanced Topics • Module 2

Vectors and Matrices

OVERVIEW

In Module 1 students learned that throughout the 1800s, mathematicians encountered a number of disparate situations that seemed to call for displaying information via tables and performing arithmetic operations on those tables. One such context arose in Module 1, where students saw the utility of representing linear transformations in the two-dimensional coordinate plane via matrices. Students viewed matrices as representing transformations in the plane and developed an understanding of multiplication of a matrix by a vector as a transformation acting on a point in the plane. This module starts with a second context for matrix representation, networks.

In Topic A, students look at incidence relationships in networks and encode information about them via high-dimensional matrices (**N-VM.C.6**). Questions on counting routes, the results of combining networks, payoffs, and other applications, provide context and use for matrix manipulations: matrix addition and subtraction, matrix product, and multiplication of matrices by scalars (**N-VM.C.7**, **N-VM.C.8**).

The question naturally arises as to whether there is a geometric context for higher-dimensional matrices as there is for 2×2 matrices. Topic B explores this question, extending the concept of a linear transformation from Module 1 to linear transformations in three- (and higher-) dimensional space. The geometric effect of matrix operations—matrix product, matrix sum, and scalar multiplication—are examined, and students come to see, geometrically, that matrix multiplication for square matrices is not a commutative operation, but that it still satisfies the associative and distributive properties (**N-VM.C.9**). The geometric and arithmetic roles of the zero matrix and identity matrix are discussed, and students see that a multiplicative inverse to a square matrix exists precisely when the determinant of the matrix (given by the area of the image of the unit square in two-dimensional space, or the volume of the image of the unit cube in three-dimensional space) is non-zero (**N-VM.C.10**). This work is phrased in terms of matrix operations on vectors, seen as matrices with one column (**N-VM.C.11**).

Topic C provides a third context for the appearance of matrices via the study of systems of linear equations. Students see that a system of linear equations can be represented as a single matrix equation in a vector variable (**A-REI.C.8**), and that one can solve the system with the aid of the multiplicative inverse to a matrix if it exists (**A-REI.C.9**).

Topic D opens with a formal definition of a vector (the motivation and context for it is well in place at this point) and the arithmetical work for vector addition, subtraction, scalar multiplication, and vector magnitude is explored along with the geometrical frameworks for these operations (**N-VM.A.1**, **N-VM.A.2**, **N-VM.B.4**, **N-VM.B.5**). Students also solve problems involving velocity and other quantities that can be represented by vectors (**N-VM.A.3**). Parametric equations are introduced in Topic D allowing students to connect their prior work with functions to vectors.

The module ends with Topic E where students apply their knowledge developed in this module to understand how first-person video games use matrix operations to project three-dimensional objects onto two-dimensional screens and animate those images to give the illusion of motion (**N-VM.C.8**, **N-VM.C.9**, **N-VM.C.10**, **N-VM.C.11**).

Focus Standards

Represent and model with vector quantities.

N-VM.A.1 (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g. $\mathbf{v}$, $|\mathbf{v}|$, $||\mathbf{v}||$, v).

N-VM.A.2 (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM.A.3 (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors.

N-VM.B.4 (+) Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$, where $-\mathbf{w}$ is the additive inverse of $\mathbf{w}$, with the same magnitude as $\mathbf{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM.B.5 (+) Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.

b. Compute the magnitude of a scalar multiple $c\mathbf{v}$ using $||c\mathbf{v}|| = |c|v$. Compute the direction of $c\mathbf{v}$ knowing that when $|c|v \neq 0$, the direction of $c\mathbf{v}$ is either along $\mathbf{v}$ for $(c > 0)$ or against $\mathbf{v}$ (for $c < 0$).

Perform operations on matrices and use matrices in applications

N-VM.C.6 (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM.C.7 (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

N-VM.C.8 (+) Add, subtract, and multiply matrices of appropriate dimensions.

N-VM.C.9 (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

N-VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

Solve systems of equations

A-REI.C.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A-REI.C.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater.

Foundational Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.★

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)n$ as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal, and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t can be rewritten as $\left(1.15^{1/12}\right)^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Create equations that describe numbers or relationships.

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Experiment with transformations in the plane.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Translate between the geometric description and the equation for a conic section.

G-GPE.A.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

G-GPE.A.2 Derive the equation of a parabola given a focus and directrix.

Focus Standards for Mathematical Practice

MP.2 **Reason abstractly and quantitatively.** Students recognize matrices and justify the transformations that they represent. Students use 3×3 matrices to solve systems of equations and continue to calculate the determinant of matrices. Students also represent complex numbers as vectors and determine magnitude and direction. Students reason to determine the effect of scalar multiplication and the result of a zero vector.

MP.4 **Model with mathematics.** Students initially study matrix multiplication as a tool for modeling networks and create a model of a bus route. Later, students look at matrix transformations and their role in developing video games and create their own video game. The focus of the mathematics in the computer animation is such that the students come to see rotating and translating as dependent on matrix operations and the addition vectors.

MP.5 **Use appropriate tools strategically.** As students study 3×3 matrices, they begin to view matrices as a tool that can solve problems including networks, payoffs, velocity, and force. Students use calculators and computer software to solve systems of three equations and three unknowns using matrices. Computer software is also used to help students visualize three-dimensional changes on a two-dimensional screen and in the creation of their video games.

Terminology

New or Recently Introduced Terms

- **Argument** (The *argument of the complex number* z is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray (i.e., the ray corresponding to the positive real axis) to the ray from the origin through the complex number z in the complex plane. The argument of z is denoted $\arg(z)$.)
- **Complex Number** (A *complex number* is a number that can be represented by a point in the complex plane. A complex number can be expressed in two forms:
  1. The *rectangular form of a complex number* z is $a + bi$ where z corresponds to the point (a, b) in the complex plane, and i is the imaginary unit. The number a is called the *real part* of $a + bi$, and the number b is called the *imaginary part* of $a + bi$. Note that both the real and imaginary parts of a complex number are themselves real numbers.
  2. For $z \neq 0$, the *polar form of a complex number* z is $r(\cos(\theta) + \ i\sin(\theta))$ where $r = |z|$ and $\theta = \arg(z)$, and i is the imaginary unit.)
- **Complex Plane** (The *complex plane* is a Cartesian plane equipped with addition and multiplication operators defined on ordered pairs by
  - **Addition**: $(a, b) + (c, d) = (a + c, b + d)$
    When expressed in rectangular form, if $z = a + bi$ and $w = c + di$, then $z + w = (a + c) + (b + d)i$.

   - **Multiplication**: $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
     When expressed in rectangular form, if $z = a + bi$ and $w = c + di$, then $z \cdot w = (ac - bd) + (ad + bc)i$. The horizontal axis corresponding to points of the form $(x, 0)$ is called the real axis, and a vertical axis corresponding to points of the form $(0, y)$ is called the imaginary axis.)
- **Conjugate** (The *conjugate* of a complex number of the form $a + bi$ is $a - bi$. The conjugate of z is denoted $\overline{z}$.)
- **Determinant of 2×2 Matrix** (The *determinant* of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number computed by evaluating $ad - bc$ and is denoted by $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$.)
- **Determinant of 3×3 Matrix** (The determinant of the 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the number computed by evaluating the expression
$$a_{11} \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12} \det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + a_{13} \det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right)$$
and is denoted by $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$.)
- **Directed Graph** (A *directed graph* is an ordered pair $D = (V, E)$ with
   - V a set whose elements are called *vertices* or *nodes*, and
   - E a set of ordered pairs of vertices, called *arcs or directed edges*.)
- **Directed Segment** (A *directed segment* $\overrightarrow{AB}$ is the line segment $\overline{AB}$ together with a direction given by connecting an initial point A to a terminal point B.)
- **Identity Matrix** (The $n \times n$ *identity matrix* is the matrix whose entry in row i and column i for $1 \le i \le n$ is 1, and whose entries in row i and column j for $1 \le i, j \le n$ and $i \ne j$ are all zero. The identity matrix is denoted by I.)
- **Imaginary Axis** (See complex plane.)
- **Imaginary Number** (An *imaginary number* is a complex number that can be expressed in the form bi where b is a real number.)
- **Imaginary Part** (See complex number.)
- **Imaginary Unit** (The *imaginary unit*, denoted by i, is the number corresponding to the point $(0, 1)$ in the complex plane.)
- **Incidence Matrix** (The *incidence matrix of a network diagram* is the $n \times n$ matrix such that the entry in row i and column j is the number of edges that start at node i and end at node j.)
- **Inverse Matrix** (An $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix B so that $AB = BA = I$, where I is the $n \times n$ identity matrix. The matrix B, when it exists, is unique and is called the *inverse of* A and is denoted by A^{-1}.)
- **Linear Function** (A function $f: \mathbb{R} \to \mathbb{R}$ is called a *linear function* if it is a polynomial function of degree one, that is, a function with real number domain and range that can be put into the form $f(x) = mx + b$ for real numbers m and b. A linear function of the form $f(x) = mx + b$ is a linear transformation only if $b = 0$.)

- **Linear Transformation** (A function $L: \mathbb{R}^n \to \mathbb{R}^n$ for a positive integer n is a *linear transformation* if the following two properties hold
  - $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and
  - $L(k\mathbf{x}) = k \cdot L(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{R}$,

  where $\mathbf{x} \in \mathbb{R}^n$ means that $\mathbf{x}$ is a point in $\mathbb{R}^n$.)
- **Linear Transformation Induced by Matrix A** (Given a 2×2 matrix A, the *linear transformation induced by matrix A* is the linear transformation L given by the formula $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$. Given a 3×3 matrix A, the *linear transformation induced by matrix A* is the linear transformation L given by the formula $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.)
- **Matrix** (An $m \times n$ matrix is an ordered list of nm real numbers, $a_{11}, a_{12}, \dots, a_{1n}, a_{21}, a_{22}, \dots a_{2n}, \dots, a_{m1}, a_{m2}, \dots, a_{mn}$, organized in a rectangular array of m rows and n columns: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$. The number a_{ij} is called the entry in row i and column j.)
- **Matrix Difference** (Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $m \times n$ matrix whose entry in row i and column j is b_{ij}. Then the matrix difference $A - B$ is the $m \times n$ matrix whose entry in row i and column j is $a_{ij} - b_{ij}$.)
- **Matrix Product** (Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $n \times p$ matrix whose entry in row i and column j is b_{ij}. Then the matrix product AB is the $m \times p$ matrix whose entry in row i and column j is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.)
- **Matrix Scalar Multiplication** (Let k be a real number, and let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}. Then the scalar product $k \cdot A$ is the $m \times n$ matrix whose entry in row i and column j is $k \cdot a_{ij}$.)
- **Matrix Sum** (Let A be an $m \times n$ matrix whose entry in row i and column j is a_{ij}, and let B be an $m \times n$ matrix whose entry in row i and column j is b_{ij}. Then the matrix sum $A + B$ is the $m \times n$ matrix whose entry in row i and column j is $a_{ij} + b_{ij}$.)
- **Modulus** (The *modulus* of a complex number z, denoted $|z|$, is the distance from the origin to the point corresponding to z in the complex plane. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.)
- **Network Diagram** (A *network diagram* is a graphical representation of a directed graph where the n vertices are drawn as circles with each circle labeled by a number 1 through n, and the directed edges are drawn as segments or arcs with arrow pointing from the tail vertex to the head vertex.)
- **Polar Form of a Complex Number** (The *polar form of a complex number z* is $r(\cos(\theta) + \ i\sin(\theta))$ where $r = |z|$ and $\theta = \arg(z)$.)
- **Rectangular Form of a Complex Number** (The *rectangular form of a complex number z* is $a + bi$ where z corresponds to the point (a, b) in the complex plane, and i is the imaginary unit. The number a is called the real part of $a + bi$, and the number b is called the imaginary part of $a + bi$.)
- **Vector** (A *vector* is described as either a bound or free vector depending on the context. We refer to both bound and free vectors as vectors throughout this module.)

- **Bound Vector** (A *bound vector* is a directed line segment (an arrow). For example, the directed line segment $\overrightarrow{AB}$ is a bound vector whose initial point (or tail) is A and terminal point (or tip) is B.

  Bound vectors are bound to a particular location in space. A bound vector $\overrightarrow{AB}$ has a magnitude given by the length of segment $\overline{AB}$ and direction given by the ray $\overrightarrow{AB}$. Many times, only the magnitude and direction of a bound vector matters, not its position in space. In that case, we consider any translation of that bound vector to represent the same free vector.
  - A *bound vector* or a *free vector* (see below) is often referred to as just a *vector* throughout this module. The context in which the word *vector* is being used determines if the vector is free or bound.)
- **Free Vector** (A *free vector* is the equivalence class of all directed line segments (*arrows*) that are equivalent to each other by translation. For example, scientists often use free vectors to describe physical quantities that have magnitude and direction only, *freely* placing an arrow with the given magnitude and direction anywhere in a diagram where it is needed. For any directed line segment in the equivalence class defining a free vector, the directed line segment is said to be a *representation* of the free vector or is said to *represent* the free vector.
  - A *bound vector* (see above) or a *free vector* is often referred to as just a *vector* throughout this module. The context in which the word *vector* is being used determines if the vector is free or bound.)
- **Real Coordinate Space** (For a positive integer n, the n-*dimensional real coordinate space*, denoted $\mathbb{R}^n$, is the set of all n-tuple of real numbers equipped with a distance function d that satisfies
  $$d[(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)] = \sqrt{(y_1 - x_1)^2 + (y_2 - x_1)^2 + \cdots + (y_n - x_n)^2}$$
  for any two points in the space. One-dimensional real coordinate space is called a *number line,* and the two-dimensional real coordinate space is called the *Cartesian plane*.)
- **Position Vector** (For a point $P(v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, the *position vector* $\mathbf{v}$, denoted by $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ or $\langle v_1, v_2, \ldots, v_n \rangle$, is a free vector $\mathbf{v}$ that is represented by the directed line segment $\overrightarrow{OP}$ from the origin $O(0,0,0, \ldots, 0)$ to the point P. The real number v_i is called the i^{th} *component* of the vector $\mathbf{v}$.)
- **Vector Addition** (For vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, the *sum* $\mathbf{v} + \mathbf{w}$ is the vector whose i^{th} component is the sum of the i^{th} components of $\mathbf{v}$ and $\mathbf{w}$ for $1 \le i \le n$. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, then
  $$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}.)$$

- **Opposite Vector** (For a vector $\vec{v}$ represented by the directed line segment $\overrightarrow{AB}$, the opposite vector, denoted $-\mathbf{v}$, is the vector represented by the directed line segment $\overrightarrow{BA}$. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $-\mathbf{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}$.)
- **Vector Subtraction** (For vectors $\mathbf{v}$ and $\mathbf{w}$, the *difference* $\mathbf{v} - \mathbf{w}$ is the sum of $\mathbf{v}$ and the opposite of $\mathbf{w}$; that is, $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$, then $\mathbf{v} - \mathbf{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \\ \vdots \\ v_n - w_n \end{bmatrix}$.)
- **Vector Magnitude** (The *magnitude* or *length* of a vector $\mathbf{v}$, denoted $\|\mathbf{v}\|$, is the length of any directed line segment that represents the vector. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$, which is the distance from the origin to the associated point $P(v_1, v_2, \ldots, v_n)$.)
- **Vector Scalar Multiplication** (For a vector $\mathbf{v}$ in $\mathbb{R}^n$ and a real number k, the *scalar product* $k \cdot \mathbf{v}$ is the vector whose i^{th} component is the product of k and the i^{th} component of $\vec{v}$ for $1 \leq i \leq n$. If k is a real number and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $k \cdot \mathbf{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}$.)
- **Vector Representation of a Complex Number** (The *vector representation of a complex number* z is the position vector $\mathbf{z}$ associated to the point z in the complex plane. If $z = a + bi$ for two real numbers a and b, then $\mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix}$.)
- **Translation by a Vector in Real Coordinate Space** (A *translation by a vector* $\mathbf{v}$ in $\mathbb{R}^n$ is the translation transformation $T_{\mathbf{v}}: \mathbb{R}^n \to \mathbb{R}^n$ given by the map that takes $\mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, then $T_{\mathbf{v}}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \\ \vdots \\ x_n + v_n \end{bmatrix}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.)
- **Zero Matrix** (The $m \times n$ *zero matrix* is the $m \times n$ matrix in which all entries are equal to zero. For example, the 2×2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.)

- **Zero Vector** (The *zero vector* in $\mathbb{R}^n$ is the vector in which each component is equal to zero. For example, the zero vector in $\mathbb{R}^2$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and the zero vector in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.)

Familiar Terms and Symbols[2]

- Rotation
- Dilation
- Translation
- Rectangular Form

Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software
- ALICE 3.1
- Geogebra Software

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-VM.C.6, N-VM.C.7, N-VM.C.8, N-VM.C.9, N-VM.C.10, N-VM.C.11
End-of-Module Assessment Task	After Topic E	Constructed response with rubric	N-VM.A.1, N-VM.A.2, N-VM.A.3, N-VM.B.4, N-VM.B.5, N-VM.C.8, N-VM.C.11, A-REI.C.8, A-REI.C.9

[2] These are terms and symbols students have seen previously.