Name ____ Date ______________________

1. Kyle wishes to expand his business and is entertaining four possible options. If he builds a new store he expects to make a profit of 9 million dollars if the market remains strong; however, if market growth declines, he could incur a loss of 5 million dollars. If Kyle invests in a franchise, he could profit 4 million dollars in a strong market but lose 3 million dollars in a declining market. If he modernizes his current facilities, he could profit 4 million dollars in a strong market but lose 2 million dollars in a declining one. If he sells his business, he will make a profit of 2 million dollars irrespective of the state of the market.

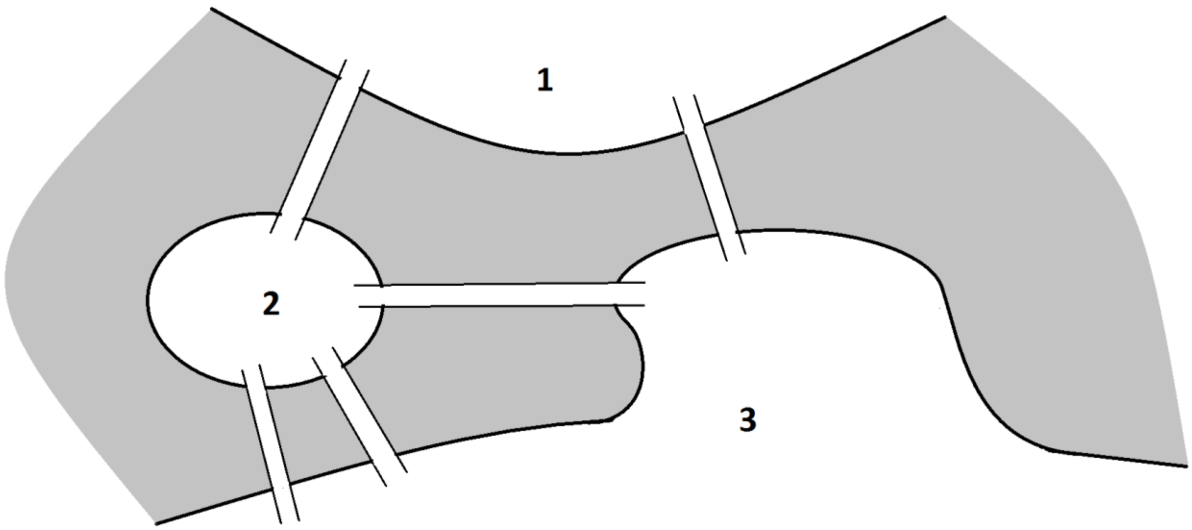
   a. Write down a 4×2 payoff matrix P summarizing the profits and losses Kyle could expect to see with all possible scenarios. (Record a loss as a profit in a negative amount.) Explain how to interpret your matrix.

   b. Kyle realized that all his figures need to be adjusted by 10% in magnitude due to inflation costs. What is the appropriate value of a real number λ so that the matrix λP represents a correctly adjusted payoff matrix? Explain your reasoning. Write down the new payoff matrix λP.

   c. Kyle is hoping to receive a cash donation of 1 million dollars. If he does, all the figures in his payoff matrix will increase by 1 million dollars.

      Write down a matrix Q so that if Kyle does receive this donation, his new payoff matrix is given by $Q + \lambda P$. Explain your thinking.

2. The following diagram shows a map of three land masses, numbered region 1, region 2, and region 3, connected via bridges over water. Each bridge can be traversed in either direction.



a. Write down a 3×3 matrix A with a_{ij}, for $i = 1, 2$, or 3 and $j = 1, 2$, or 3, equal to the number of ways to walk from region i to region j by crossing exactly one bridge. Notice that there are no paths that start and end in the same region crossing exactly one bridge.

b. Compute the matrix product A^2.

c. Show that there are 10 walking routes that start and end in region 2, crossing over water exactly twice. Assume each bridge, when crossed, is fully traversed to the next land mass.

d. How many walking routes are there from region 3 to region 2 that cross over water exactly three times? Again, assume each bridge is fully traversed to the next land mass.

e. If the number of bridges between each pair of land masses is doubled, how does the answer to part (d) change? That is, what would be the new count of routes from region 3 to region 2 that cross over water exactly three times?

3. Let $P = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$.

   a. Show the work needed and compute $2P - 3Q$.

   b. Show the work needed and compute PQ.

   c. Show that $P^2Q = PQP$.

4.

a. Show that if the matrix equation $(A+B)^2 = A^2 + 2AB + B^2$ holds for two square matrices A and B of the same dimension, then these two matrices commute under multiplication.

b. Give an example of a pair of 2×2 matrices A and B for which $(A+B)^2 \neq A^2 + 2AB + B^2$.

c. In general, does $AB = BA$? Explain.

d. In general, does $A(B + C) = AB + AC$? Explain.

e. In general, does $A(BC) = (AB)C$? Explain.

5. Let I be the 3×3 identity matrix and A the 3×3 zero matrix. Let the 3×1 column $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ represent a point in three dimensional space. Also, set $P = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

   a. Use examples to illustrate how matrix A plays the same role in matrix addition that the number 0 plays in real number addition. Include an explanation of this role in your response.

   b. Use examples to illustrate how matrix I plays the same role in matrix multiplication that the number 1 plays in real number multiplication. Include an explanation of this role in your response.

   c. What is the row 3, column 3 entry of $(AP + I)^2$? Explain how you obtain your answer.

d. Show that $(P-1)(P+1)$equals $P^2 - I$.

e. Show that Px is sure to be a point in the xz-plane in three-dimensional space.

f. Is there a 3×3 matrix Q, not necessarily the matrix inverse for P, for which $QPx = x$ for every 3×1 column x representing a point? Explain your answer.

g. Does the matrix P have a matrix inverse? Explain your answer.

h. What is the determinant of the matrix P?

6. What is the image of the point given by the 3×1 column matrix $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ when it is rotated $45°$ about the z-axis in the counterclockwise direction (as according to the orientation of the xy-plane) and then $180°$ about the y-axis?

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	**STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	**STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	**STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a** N-VM.C.6	Student gives an incorrect answer and not as a matrix.	Student gives correct answer but not as a matrix.	Student gives correct matrix with no explanation of how to interpret it.	Student gives correct matrix with clear explanation of how to interpret it.
	b N-VM.C.6	Student gives an incorrect answer and not as a matrix.	Student gives correct answer but not as a matrix.	Student gives correct matrix with no explanation of reasoning supporting calculation.	Student gives correct matrix with clear explanation of reasoning supporting calculations.
	c N-VM.C.6	Student gives an incorrect answer, not as a matrix, and with no explanation.	Student gives correct answer but not as a matrix.	Student gives correct matrix with no explanation of reasoning.	Student gives correct matrix with clear explanation of reasoning.
2	**a** N-VM.C.6	Student show little or no evidence of writing a matrix.	Student writes a matrix, but it is not a 3×3 matrix. <u>OR</u> Student writes a 3×3 matrix with three or more mistakes.	Student writes a 3×3 matrix, but one or two entries are incorrect.	Student writes a correct 3×3 matrix.
	b N-VM.C.6 N-VM.C.8	Student shows little or no evidence of understanding matrix multiplication.	Student attempts matrix multiplication but has major errors.	Student multiplies the matrices but makes a small computational error.	Student multiplies matrices correctly.

	c N-VM.C.6 N-VM.C.8	Student shows little or no evidence of interpreting matrix entries.	Student answers question using matrix, but uses wrong entry.	Student answers correctly but does not fully explain answer.	Student answers correctly and fully explains answer.
	d N-VM.C.6 N-VM.C.8	Student shows little or no evidence of matrix operations needed to answer question.	Student multiplies to find A^3 but does not explain answer or explains wrong entry as answer.	Student multiplies to find A^3 but makes mistakes multiplying. The answer is incorrect, but reasoning and justification are correct.	Student multiplies to find A^3, identifies correct answer, and justifies it correctly.
	e N-VM.C.6 N-VM.C.7 N-VM.C.8	Student shows little or no evidence of matrix operations needed to answer question.	Student multiplies to find $8A^3$ but does not explain answer or explains wrong entry as answer.	Student multiplies to find $8A^3$ and explains answer, but not completely.	Student multiplies to find $8A^3$ and explains answer completely.
3	**a** N-VM.C.7 N-VM.C.8	Student shows little or no evidence of matrix operations.	Student shows some knowledge of matrix operations but makes mistakes on two or more entries in the final matrix.	Student shows knowledge of matrix operations but has one entry wrong in final matrix.	Student shows knowledge of matrix operations arriving at correct final matrix.
	b N-VM.C.8	Student shows little or no evidence of matrix multiplication.	Student shows some knowledge of matrix multiplication but makes mistakes on two or more entries in the final matrix.	Student shows knowledge of matrix multiplication but has one entry wrong in final matrix.	Student shows knowledge of matrix multiplication arriving at correct final matrix.
	c N-VM.C.8	Student shows little or no evidence of matrix multiplication.	Student shows some knowledge of matrix multiplication but makes mistakes leading to incorrect answer.	Student shows knowledge of matrix multiplication, finding correct matrices, but does not explain that they are equal.	Student shows knowledge of matrix multiplication, finding correct matrices, and explains that they are equal.
4	**a** N-VM.C.9	Student makes little or no attempt to answer question.	Student expands the binomial $(A+B)^2$ but does not continue with proof, or steps are not correct.	Student expands the binomial $(A+B)^2$ and shows that $BA = AB$ but does not explain reasoning for these matrices being commutative under multiplication.	Student expands the binomial $(A+B)^2$, shows that $BA = AB$, and explains reasoning that these matrices are commutative under multiplication.

	b **N-VM.C.9**	Student makes little or no attempt to find matrices.	Student lists two 2×2 matrices but does not support or prove answer.	Student lists two 2×2 matrices but makes mistakes in calculations or reasoning to support answer.	Student lists two 2×2 matrices and shows supporting evidence to verify answer.
	c **N-VM.C.9**	Student states that $AB = BA$ is true for matrices.	Student states that $AB \neq BA$ but does not support answer with reasoning.	Student states that $AB \neq BA$ and attempts to explain reasoning but does not use the term commute or commutative.	Student states that $AB \neq BA$, explains reasoning, and states that matrix multiplication is not generally commutative.
	d **N-VM.C.9**	Student states that $A(B + C) \neq AB + BC$.	Student states that $A(B + C) = AB + BC$ but does not support answer with reasoning.	Student states that $A(B + C) = AB + AC$ and attempts to explain but makes minor errors in reasoning.	Student states that $A(B + C) = AB + AC$ and explains reasoning correctly.
	e **N-VM.C.9**	Student states that $A(BC) \neq (AB)C$.	Student states that $A(BC) = (AB)C$ but does not support answer with reasoning.	Student states that $A(BC) = (AB)C$ and attempts to explain but makes minor errors in reasoning.	Student states that $A(BC) = (AB)C$ and explains reasoning correctly.
5	**a** **N-VM.C.10**	Student shows little of no understanding of the 3×3 zero matrix.	Student writes the 3×3 zero matrix but does not explain its role in matrix addition.	Student writes the 3×3 zero matrix, showing an example of its role in matrix addition but does not explain the connection to the number zero in real number addition.	Student writes the 3×3 zero matrix, shows and example of its role in matrix addition, and explains the connection to the number zero in real number addition.
	b **N-VM.C.10**	Student shows little of no understanding of the 3×3 identity matrix.	Student writes the 3×3 identity matrix but does not explain its role in matrix multiplication.	Student writes the 3×3 identity matrix, showing an example of its role in matrix multiplication but does not explain the connection to the number one in real number multiplication.	Student writes the 3×3 identity matrix, shows an example of its role in matrix multiplication, and explains the connection to the number one in real number multiplication.
	c **N-VM.C.8** **N-VM.C.10**	Student shows little or no understanding of matrix operations.	Student finds $(AP + I)^2$ but does not identify the entry in row 3 column 3.	Student finds $(AP + I)^2$ and identifies the entry in row 3 column 3 but does not explain answer.	Student finds $(AP + I)^2$, identifies the entry in row 3 column 3, and explains answer.

	d **N-VM.C.9** **N-VM.C.10**	Student shows little or no understanding of matrix operations.	Student calculates two of $P-1$, $P+1$, or P^2-I correctly.	Student calculates $(P-1)(P+1)$ and P^2-I but does not explain why the expressions are equal.	Student calculates $(P-1)(P+1)$ and P^2-I and explains why the expressions are equal.
	e **N-VM.C.11**	Student shows little or no understanding of matrix operations.	Student sets up Px but does not find the matrix representing the product.	Student sets up and finds the matrix representing Px but does not explain the meaning of the point in 3-dimensional space.	Students sets up and finds the matrix representing Px and explains the meaning of the point in 3-dimensional space.
	f **N-VM.C.11**	Student shows little or no understanding of matrix operations.	Student finds Px but does not find QPx or explain reasoning.	Student finds QPx and attempts to explain, but not clearly, why Q cannot exist.	Student finds QPx and clearly shows that matrix Q cannot exist.
	g **N-VM.C.10**	Student shows little or no understanding of inverse matrices.	Student shows some understanding of inverse matrices but cannot answer or explain question.	Student says that the inverse does not exist and attempts to explain but explanation has minor mistakes.	Student clearly explains why the inverse matrix does not exist.
	h **N-VM.C.10**	Student shows little or no understanding of the determinant of matrix P.	Student incorrectly attempts to find the determinant.	Student states that the determinant is zero but with no explanation.	Student explains clearly why the determinant is zero.
6	**N-VM.C.11**	Student shows little or no understanding of matrices producing rotations.	Student attempts to write the matrices producing rotations but with errors or with only one correct.	Student writes the correct matrices producing the rotations required but makes calculation errors leading to an incorrect final answer.	Student writes the correct matrices producing the rotations required and calculates the correct final image point.

Name ______________________________ Date ________________

1. Kyle wishes to expand his business and is entertaining four possible options. If he builds a new store he expects to make a profit of 9 million dollars if the market remains strong; however, if market growth declines, he could incur a loss of 5 million dollars. If Kyle invests in a franchise, he could profit 4 million dollars in a strong market but lose 3 million dollars in a declining market. If he modernizes his current facilities, he could profit 4 million dollars in a strong market but lose 2 million dollars in a declining one. If he sells his business, he will make a profit of 2 million dollars irrespective of the state of the market.

   a. Write down a 4×2 payoff matrix P summarizing the profits and losses Kyle could expect to see with all possible scenarios. (Record a loss as a profit in a negative amount.) Explain how to interpret your matrix.

   We have $P = \begin{bmatrix} 9 & -5 \\ 4 & -3 \\ 4 & -2 \\ 2 & 2 \end{bmatrix}$.

   Here the four rows correspond to, in turn, the options of building a new store, investing in a franchise, modernizing, and selling. The first column gives the payoffs in a strong market, and the second column gives the payoffs in a declining market.

   All entries are in units of millions of dollars.

   Note: Other presentations for the matrix P are possible.

   b. Kyle realized that all his figures need to be adjusted by 10% in magnitude due to inflation costs. What is the appropriate value of a real number λ so that the matrix λP represents a correctly adjusted payoff matrix? Explain your reasoning. Write down the new payoff matrix λP.

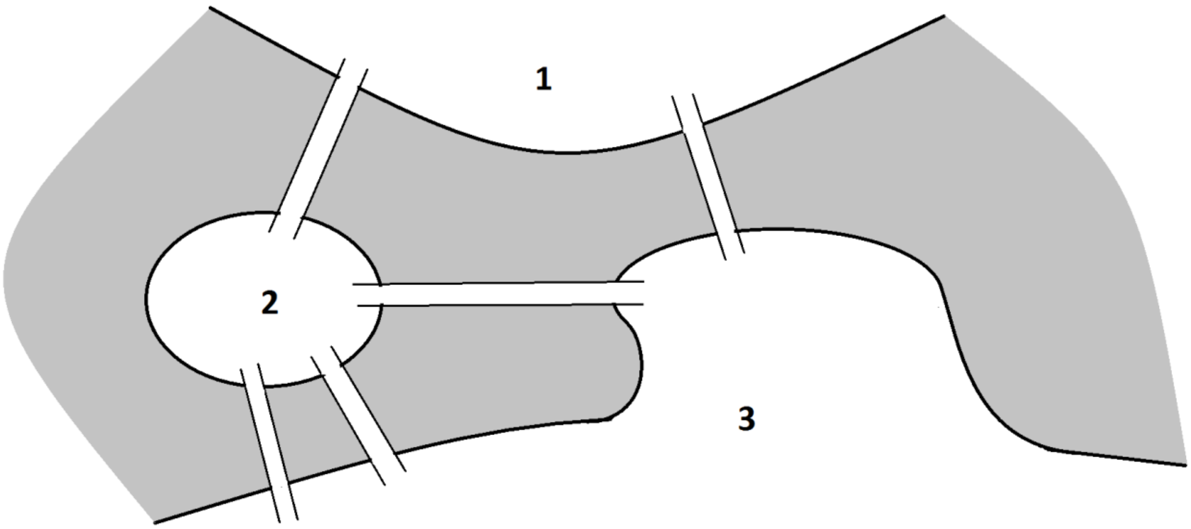
   Each entry in the matrix needs to increase 5% in magnitude. This can be accomplished by multiplying each entry by 1.10. If we set $\lambda = 1.1$, then $\lambda P = \begin{bmatrix} 9.9 & -5.5 \\ 4.4 & -3.3 \\ 4.4 & -2.2 \\ 2.2 & 2.2 \end{bmatrix}$ is the appropriate new payoff matrix.

c. Kyle is hoping to receive a cash donation of 1 million dollars. If he does, all the figures in his payoff matrix will increase by 1 million dollars.

Write down a matrix Q so that if Kyle does receive this donation, his new payoff matrix is given by $Q + \lambda P$. Explain your thinking.

Set $Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $Q + \lambda P$ is the matrix λP with 1 added to each entry. This is the effect we seek, increasing each expected payoff by 1 million dollars.

2. The following diagram shows a map of three land masses, numbered region 1, region 2, and region 3, connected via bridges over water. Each bridge can be traversed in either direction.



a. Write down a 3×3 matrix A with a_{ij}, for $i = 1, 2,$ or 3 and $j = 1, 2,$ or 3, equal to the number of ways to walk from region i to region j by crossing exactly one bridge. Notice that there are no paths that start and end in the same region crossing exactly one bridge.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

b. Compute the matrix product A^2.

$$A^2 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 10 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

c. Show that there are 10 walking routes that start and end in region 2, crossing over water exactly twice. Assume each bridge, when crossed, is fully traversed to the next land mass.

The entries of A^2 give the number of paths via two bridges between land regions. As the row 2, column 2 entry of A^2 is 10, this is the count of two-bridge journeys that start and end in region 2.

d. How many walking routes are there from region 3 to region 2 that cross over water exactly three times? Again, assume each bridge is fully traversed to the next land mass.

The entries of A^3 give the counts of three-bridge journeys between land masses. We seek the row 3, column 2 entry of the product.

$$\begin{bmatrix} 2 & 3 & 3 \\ 3 & 10 & 1 \\ 3 & 1 & 9 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

This entry is $(3 \cdot 1) + (1 \cdot 0) + (9 \cdot 3) = 30$. Therefore, there are 30 such routes.

e. If the number of bridges between each pair of land masses is doubled, how does the answer to part (d) change? That is, what would be the new count of routes from region 3 to region 2 that cross over water exactly three times?

We are now working with the matrix $2A$. The number of routes from region 3 to region 2 via three bridges is the row 3, column 2 entry of $(2A)^3 = 8A^3$. As all the entries are multiplied by eight, there are $8 \times 30 = 240$ routes of the particular type we seek.

3. Let $P = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$ for some fixed real numbers a, b, c, and d.

a. Show the work needed and compute $2P - 3Q$.

$$2P - 3Q = \begin{bmatrix} 6 & -10 \\ 10 & 6 \end{bmatrix} - \begin{bmatrix} -6 & 3 \\ -3 & -6 \end{bmatrix} = \begin{bmatrix} 12 & -13 \\ 13 & 12 \end{bmatrix}$$

b. Show the work needed and compute PQ.

$$PQ = \begin{bmatrix} -6+5 & 3+10 \\ -10-3 & 5-6 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ -13 & -1 \end{bmatrix}$$

c. Show that $P^2Q = PQP$.

$$PQ = \begin{bmatrix} -1 & 13 \\ -13 & -1 \end{bmatrix}$$

$$P^2Q = P(PQ) = \begin{bmatrix} 62 & 44 \\ -44 & 62 \end{bmatrix}$$

$$PQP = (PQ)P = \begin{bmatrix} -1 & 13 \\ -13 & -1 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 62 & 44 \\ -44 & 62 \end{bmatrix}$$

These are identical matrices.

4.

a. Show that if the matrix equation $(A+B)^2 = A^2 + 2AB + B^2$ holds for two square matrices A and B of the same dimension, then these two matrices commute under multiplication.

We have $(A+B)^2 = (A+B)(A+B)$.

By the distributive rule, which does hold for matrices, this equals $A(A+B) + B(A+B)$, which, again by the distributive rule, equals $A^2 + AB + BA + B^2$.

On the other hand, $A^2 + 2AB + B^2$ equals $A^2 + AB + AB + B^2$.

So, if $(A+B)^2 = A^2 + 2AB + B^2$, then we have $A^2 + AB + BA + B^2 = A^2 + AB + AB + B^2$.

Adding $-A^2$ and $-AB$ and $-B^2$ to each side of this equation gives $BA = AB$.

This shows that A and B commute under multiplication in this special case when $(A+B)^2 = A^2 + 2AB + B^2$, but in general, matrix multiplication is not commutative.

b. Give an example of a pair of 2×2 matrices A and B for which $(A+B)^2 \neq A^2 + 2AB + B^2$.

A pair of matrices that do not commute under multiplication, such as $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, should do the trick.

To check: $(A+B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$$\begin{aligned} A^2 + 2AB + B^2 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 + 2\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} \end{aligned}$$

These are indeed different.

c. In general, does $AB = BA$? Explain.

No. Since matrix multiplication can represent linear transformations, we know that they will not always commute since linear transformations do not always commute.

d. In general, does $A(B+C) = AB + AC$? Explain.

Yes. Consider the effect on the point x made by both sides of the equation. On the left-hand side, the transformation $B+C$ is applied to the point x, but we know that this is the same as $Bx+Cx$ from our work with linear transformations. Applying the transformation represented by A to either $Bx+Cx$ or $(B+C)x$ now is $ABx+ACx$ because they work like linear maps.

e. In general, does $A(BC) = (AB)C$? Explain.

Yes. If we consider the effect the matrices on both side make on a point x, the matrices are applied in the exact same order, C, then B, then A, regardless of whether AB is computed first or BC is computed first.

5. Let I be the 3×3 identity matrix and A the 3×3 zero matrix. Let the 3×1 column $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ represent a point in three dimensional space. Also, set $P = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

   a. Use examples to illustrate how matrix A plays the same role in matrix addition that the number 0 plays in real number addition. Include an explanation of this role in your response.

   The sum of two 3×3 matrices is determined by adding entries in corresponding positions of the two matrices to produce a new 3×3 matrix. Each and every entry of matrix A is zero, so a sum of the form A + P, where P is another 3×3 matrix, is given by adding zero to each entry of P. Thus, A + P = P. For example:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+4 & 0+7 \\ 0+2 & 0+5 & 0+8 \\ 0+3 & 0+6 & 0+9 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

   This is analogous to the role of zero in the real number system: $0 + p = p$ for every real number p.

   In the same way, $P + A = P$ for all 3×3 matrices P, analogous to $p + 0 = p$ for all real numbers p.

   b. Use examples to illustrate how matrix I plays the same role in matrix multiplication that the number 1 plays in real number multiplication. Include an explanation of this role in your response.

   We have $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By the definition of matrix multiplication we see, for example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} (1\cdot1)+(0\cdot2)+(0\cdot3) & (1\cdot4)+(0\cdot5)+(0\cdot6) & (1\cdot7)+(0\cdot8)+(0\cdot9) \\ (0\cdot1)+(1\cdot2)+(0\cdot3) & (0\cdot4)+(1\cdot5)+(0\cdot6) & (0\cdot7)+(1\cdot8)+(0\cdot9) \\ (0\cdot1)+(0\cdot2)+(1\cdot3) & (0\cdot4)+(0\cdot5)+(1\cdot6) & (0\cdot7)+(0\cdot8)+(1\cdot9) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

   We have, in general, that $I \times P = P$ for every 3×3 matrix P. We also have $P \times I = P$.

   Thus, the matrix I plays the role of the number I in real number arithmetic where $1 \times p = p$ and $p \times 1 = p$ for each real number p.

c. What is the row 3, column 3 entry of $(AP+I)^2$? Explain how you obtain your answer.

Since A is the zero matrix, AP equals the zero matrix. That is, AP = A.

Thus, $(AP+I)^2 = (A+I)^2 = I^2 = I$.

So, $(AP+I)^2$ is just the 3×3 identity matrix. The row 3, column 3 entry is thus 1.

d. Show that $(P-1)(P+1)$equals $P^2 - I$.

We have $(P-I)(P+I) = P^2 - IP + PI - I^2$ (using the distributive property which holds for matrices). Since $PI = P$, $IP = P$, and $I^2 = I$, this equals $P^2 - P + P + I = P^2 - I$.

e. Show that Px is sure to be a point in the xz-plane in three-dimensional space.

We have

$$Px = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+5z \\ 0 \\ 4z \end{bmatrix}.$$

The image is a point with y-coordinate zero and so is a point in the xz-plane in three-dimensional space.

f. Is there a 3×3 matrix Q, not necessarily the matrix inverse for P, for which $QPx = x$ for every 3×1 column x representing a point? Explain your answer.

If there were such a matrix Q, then QPx = x for $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. But $Px = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and so $QPx = Q\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, which is not $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ after all. There can be no such matrix Q.

g. Does the matrix P have a matrix inverse? Explain your answer.

If P had a matrix inverse P^{-1}, then we would have $P^{-1}P = I$ and so $P^{-1}Px = x$ for all 3×1 columns x representing a point. By part (d), there is no such matrix.

OR

By part (c), P takes all points in the three-dimensional space and collapses them to a plane. So there are points that are taken to the same image point by P. Thus, no inverse transformation, P^{-1}, can exist.

h. What is the determinant of the matrix P?

The unit cube is mapped onto a plane, and so the image of the unit cube under P has zero volume. The determinant of P is thus 0.

OR

By part (e), P has no multiplicative inverse, and so its determinant must be 0.

6. What is the image of the point given by the 3×1 column matrix $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ when it is rotated $45°$ about the z-axis in the counterclockwise direction (as according to the orientation of the xy-plane) and then $180°$ about the y-axis?

A rotation about the z-axis of 45° is effected by multiplication by the matrix:

$$R_1 = \begin{bmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the image of the point under the first rotation is

$$R_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ -1 \end{bmatrix}.$$

A rotation of 180° about the y-axis is effected by multiplication by

$$R_2 = \begin{bmatrix} \cos(180°) & 0 & -\sin(180°) \\ 0 & 1 & 0 \\ \sin(180°) & 0 & \cos(180°) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The final image point we seek is thus

$$R_2 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ 1 \end{bmatrix}.$$