Lesson 21: Margin of Error when Estimating a Population Mean

Student Outcomes

- Students use data from a random sample to estimate a population mean.
- Students calculate and interpret margin of error in context.
- Students know the relationship between sample size and margin of error in the context of estimating a population mean.

Lesson Notes

In the previous lesson, students estimated the population mean using the sample mean based on a random sample of size n. To determine how accurate their estimate was, they had to create a sampling distribution of the sample mean based on computing sample means for a large number of random samples. Finally, they computed the margin of error as twice the standard deviation of the sample means. Although the process was a lot of work, students developed a conceptual understanding of margin of error.

In this lesson, students use a formula for the standard deviation of the sample mean, $\frac{s}{\sqrt{n}}$, where s is the standard deviation of the sample and n is the size of the sample. The margin of error, $2\left(\frac{s}{\sqrt{n}}\right)$, is based on a single random sample, thus making the work much easier.

The formula $\frac{s}{\sqrt{n}}$ is used to calculate the standard deviation of the sample mean when the mean and the standard deviation of the population are stated. Previously, the formula $\sqrt{\frac{p(1-p)}{n}}$ was used to calculate the standard deviation of a sample *proportion* when the number of successes was known. In both formulas, as n gets larger, the standard deviation gets smaller. Both methods are applications of the central limit theorem, which says that regardless of the shape of the population from which samples are taken, the distributions of both the sample means and the sample proportions are approximately normal.

Classwork

This lesson continues to discuss using the sample mean as an estimate of the population mean and judging its accuracy based on the concept of margin of error. In the last lesson, the margin of error was defined as twice the standard deviation of the sampling distribution of the sample mean. In this lesson, a formula will be given for the margin of error that allows you to calculate the margin of error from a single random sample rather than having to create a sampling distribution of sample means.

Example 1 (5 minutes): Estimating a Population Mean Using a Random Sample

MP.2

Give students a few minutes to read the introductory material of this example, and remind them of the process they used in the previous lesson to get an estimate of margin of error. Then, write the formula for margin of error on the board, making sure that students understand that this will allow them to calculate an estimate of the margin of error using data from a single random sample.

Example 1: Estimating a Population Mean Using a Random Sample

Provide a one-sentence summary of our findings from the previous lesson.

Sample response: We took lots of random samples of computer game ratings, computed their means, displayed the distribution of their means, and, finally, computed a margin of error.

What were drawbacks of the calculation method?

Sample response: Many samples are required. If we had increased the sample size or the number of samples, the time required to take all those samples, calculate their means, and analyze the distribution would have increased significantly.

In practice, you do not have to use that process to find the margin of error. Fortunately, just as was the case with estimating a population proportion, there are some general results that lead to a formula that allows you to estimate the margin of error using a single sample. You can then gauge the accuracy of your estimate of the population mean by calculating the margin of error using the sample standard deviation.

The standard deviation of the distribution of sample means is approximated by $\frac{s}{\sqrt{n}}$, where s is the standard deviation of the sample, and n is the size of the sample.

Scaffolding:

- For struggling students – the bigger the value of n, the smaller the standard deviation. From a population where $s = 2$, if $n = 36$, the standard deviation is $\frac{1}{3}$; however, if the sample is larger, say 81, the standard deviation would be $\frac{2}{9}$.
- For advanced students – when a sample is taken from a population, the mean of the sample is the same as the mean of the population, but the *variance* (square of the standard deviation) is only $\frac{1}{n}$ as large. Regardless of the shape of the population, the distribution of the sample means will approach normal (central limit theorem). The variance is $\frac{s^2}{n}$; therefore, by applying a square root, the standard deviation is $\frac{s}{\sqrt{n}}$.

Exercises 1–5 (10 minutes)

Have students work independently on the calculations required to answer Exercises 1–3. Then, work through Exercises 4 and 5 as a class.

Exercises 1–5

1. **Suppose a random sample of size ten produced the following ratings in the computer games rating example in the last lesson: $12, 5, 2, 4, 1, 4, 18, 10, 1, 16$. Estimate the population mean rating based on these ten sampled ratings.**

   The sample mean estimate for the population mean rating is $\frac{73}{10} = 7.3$ rating points.

2. **Calculate the sample standard deviation. Round your answer to three decimal places.**

   The sample standard deviation is 6.2725 rating points.

3. **Use the formula given above to calculate the approximate standard deviation of the distribution of sample means. Round your answer to three decimal places.**

   The standard deviation of the distribution of sample means is $\frac{s}{\sqrt{n}} = \frac{6.273}{\sqrt{10}} = 1.984$ *rating points.*

4. **Recall that the margin of error is twice the standard deviation of the distribution of sample means. What is the value of the margin of error based on this sample? Write a sentence interpreting the value of the margin of error in the context of this problem on computer game ratings.**

   Margin of error is $2(1.984) = 3.968$ *rating points. The population mean rating for the* 100 *computer games is likely to be within* 3.968 *rating points of the sample mean estimate* 7.3*.*

5. **Based on the sample mean and the value of the margin of error, what is an interval of plausible values for the population mean?**

   Plausible values for the population mean rating are from $7.3 - 3.968 = 3.332$ *to* $7.3 + 3.968 = 11.268$ *rating points.*

Exercises 6–13 (20 minutes): The Gettysburg Address

Distribute a copy of the Gettysburg Address to each student in the class. (A copy is provided at the end of this lesson.) Have students work individually or in pairs to answer the questions in this set of exercises. Then, discuss the answers to the last question as a class. Consider challenging students to find the length of a typical word in the Gettysburg Address.

After students do this exercise "by hand" (using a calculator), you may want to show them an applet that displays three different estimates regarding the Gettysburg Address. One is the mean word length. The other two are estimating population proportions; one is the proportion of "long" words defined as words with more than four letters, and the other is the proportion of nouns. The applet can be found at the following site: http://www.rossmanchance.com/applets/GettysburgSample/GettysburgSample.html.

This applet may require an updated version of an operating system to work correctly. If the applet does not work for all students due to a computer's operating system or network settings, attempt to demonstrate it for the whole class, as it is an effective way to complement how students obtained their answers in the exercises. The applet allows the user to specify a sample size (ten in this exercise) and the number of samples desired. Note that only *one* sample is to be used to answer the questions in this exercise set.

To generate a sampling distribution for the sample mean (or proportion), enter a large number in the Num samples box, such as 500. The Animate box shows the observations for each sample taken and the resulting values of the statistics (mean or proportion) plotted on a histogram. (You may unclick the Animate box at any time to see the total results immediately.)

Students should begin work on Exercise 6. Exercises 7–13 are provided as scaffolding if necessary. Students should be able to clearly describe and fully implement a plan on their own. Sample responses are provided but will vary.

Exercises 6-13

The Gettysburg Address is considered one of history's greatest speeches. Some students noticed that the speech was very short (about 268 words, depending on the version) and wondered if the words were also relatively short. To estimate the mean length of words in the population of words in the Gettysburg Address, work with a partner on the following steps. Your teacher will give you a copy of the Gettysburg Address with words numbered from 001 to 268.

6. **Develop and describe a plan for collecting data from the Gettysburg Address and determining the typical length of a word. Then, implement your plan, and report your findings.**

   Many answers are possible. Every answer should include the following:

   - *A description of how a word sample is chosen, making sure to describe how randomization occurs*
   - *The actual sample chosen*
   - *Calculations of the sample mean, standard deviation, and margin of error*
   - *Interpretations in context of the sample mean, standard deviation, and margin of error*

MP.5

7. **Use a random-number table or a calculator with a random-number generator to obtain ten different random numbers from 001 to 268.**

   219 229 2 113 77 140 185 70 119 54

8. **Use the random numbers found in Exercise 7 as identification numbers for the words that will make up your random sample of words from the Gettysburg Address. Make a list of the ten words in your sample.**

   219 – *the*
   229 – *resolve*
   2 – *score*
   113 – *consecrate*
   77 – *final*
   140 – *The*
   185 – *nobly*
   70 – *a*
   119 – *The*
   54 – *endure*

9. **Count the number of letters in each of the ten words in your sample.**

   3 7 5 10 5 3 5 1 3 6

10. **Calculate the sample mean number of letters for the ten words in your sample.**

    The mean of the ten word lengths from Exercise 9 is $\frac{48}{10} = 4.8$ *letters.*

11. **Calculate the sample standard deviation of the number of letters for the ten words in your sample. Round your answer to three decimal places.**

    The standard deviation of the ten word lengths from Exercise 9 is 2.530 *letters.*

12. **Use the sample standard deviation from Exercise 11 to calculate the margin of error associated with using your sample mean as an estimate of the population mean. Round your answer to three decimal places.**

    The margin of error of this estimate is $2\left(\frac{2.530}{\sqrt{10}}\right) = 1.600$ *letters.*

MP.2

13. **Write a few sentences describing what you have learned about the mean length of the population of 268 words in the Gettysburg Address. Be sure to include an interpretation of the margin of error.**

    We estimate the mean word length of words in the Gettysburg Address to be 4.8 *letters. The margin of error of this estimate is* 1.6 *letters. So, plausible values for the population mean word length are from* 3.2 *to* 6.4 *letters.*

Closing (3 minutes)

- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

- **When using the sample mean to estimate a population mean, it is important to know something about how accurate that estimate might be.**
- **Accuracy can be described by the margin of error.**
- **The margin of error can be estimated using data from a single random sample (without the need to create a simulated sampling distribution) by using the formula $2\left(\frac{s}{\sqrt{n}}\right)$, where s is the standard deviation of a single sample, and n is the sample size.**

Exit Ticket (7 minutes)

Name _____ Date _____

Lesson 21: Margin of Error when Estimating a Population Mean

Exit Ticket

A Health Group study recommends that the total weight of a male student's backpack should not be more that 15% of his body weight. For example, if a student weighs 170 pounds, his backpack should not weigh more than 25.5 pounds. Suppose that ten randomly selected eleventh grade boys produced the following data:

Body weight	155	136	197	174	165	165	150	142	176	157
Backpack weight	29.8	27.2	32.5	34.8	31.8	28.8	31.1	26.0	28.3	31.4

a. For each student, calculate backpack weight as a percentage of body weight (round to one decimal place).

b. Based on the data in part (a), estimate the mean percentage of body weight that eleventh grade boys carry in their backpacks.

c. Find the margin of error for your estimate of part (b). Round your answer to three decimal places. Explain how you determined your answer.

d. Comment on the amount of weight eleventh grade boys at this school are carrying in their backpacks compared to the recommendation by the Health Group.

Exit Ticket Sample Solutions

A Health Group study recommends that the total weight of a male student's backpack should not be more that 15% of his body weight. For example, if a student weighs 170 pounds, his backpack should not weigh more than 25.5 pounds. Suppose that ten randomly selected eleventh grade boys produced the following data:

Body weight	155	136	197	174	165	165	150	142	176	157
Backpack weight	29.8	27.2	32.5	34.8	31.8	28.8	31.1	26.0	28.3	31.4

a. For each student, calculate backpack weight as a percentage of body weight (round to one decimal place).

Body weight	155	136	197	174	165	165	150	142	176	157
Backpack weight	29.8	27.2	32.5	34.8	31.8	28.8	31.1	26.0	28.3	21.4
Percentage	19.2	20.0	16.5	20.0	19.3	17.5	20.7	18.3	16.1	13.6

b. Based on the data in part (a), estimate the mean percentage of body weight that eleventh grade boys carry in their backpacks.

The sample mean percentage is 18.12 percentage points.

c. Find the margin of error for your estimate of part (b). Explain how you determined your answer.

The standard deviation of the percentages is 2.207 percent. So, the margin of error is $2\left(\frac{2.207}{\sqrt{10}}\right) = 1.396$ percentage points.

d. Comment on the amount of weight eleventh grade boys at this school are carrying in their backpacks compared to the recommendation by the Health Group.

Based on the data in this study, plausible percentages of mean body weight percentage that eleventh graders are carrying in their backpacks are $18.12 - 1.396 = 16.724$ to $18.12 + 1.396 = 19.516$ percentage points. The interval $(16.724\%, 19.516\%)$ is above the recommended 15% maximum. On average, eleventh grade boys at this school are carrying too much weight in their backpacks.

Problem Set Sample Solutions

1. A new brand of hot dog claims to have a lower sodium content than the leading brand.

a. A random sample of ten of these new hot dogs results in the following sodium measurements (mg).

$$370 \quad 326 \quad 322 \quad 297 \quad 326 \quad 289 \quad 293 \quad 264 \quad 327 \quad 331$$

Estimate the population mean sodium content of this new brand of hot dog based on the ten sampled measurements.

Based on the data, an estimate for the population mean sodium content of this new brand of hot dog is 314.5 mg of sodium.

b. Calculate the margin of error associated with your estimate of the population mean from part (a). Round your answer to three decimal places.

The margin of error is $2\left(\frac{29.436}{\sqrt{10}}\right) = 18.617$ mg.

c. The mean sodium content of the leading brand of hot dogs is known to be 350 mg. Based on the sample mean and the value of the margin of error for the new brand, is a mean sodium content of 350 mg a plausible value for the mean sodium content of the new brand? Comment on whether you think the new brand of hot dog has a lower sodium content on average than the leading brand.

Plausible values for population mean sodium content are between $314.5 - 18.617 = 295.883$ mg and $314.5 + 18.617 = 333.117$ mg. This interval is well below the 350 mg which is the mg content for the leading brand. So, the new hot dog brand has lower mean sodium content.

d. Another random sample of 40 new brand hot dogs is taken. Should this larger sample of hot dogs produce a more accurate estimate of the population mean sodium content than the sample of size 10? Explain your answer by appealing to the formula for margin of error.

The margin of error will be smaller. Sample size is in the denominator of formula for margin of error.

2. It is well known that astronauts increase their height in space missions because of the lack of gravity. A question is whether or not we increase height here on Earth when we are put into a situation where the effect of gravity is minimized. In particular, do people grow taller when confined to a bed? A study was done in which the heights of six men were taken before and after they were confined to bed for three full days.

a. The before-after differences in height measurements (mm) for the six men were:

$$12.6 \;\; 14.4 \;\; 14.7 \;\; 14.5 \;\; 15.2 \;\; 13.5.$$

Assuming that the men in this study are representative of the population of all men, what is an estimate of the population mean increase in height after three full days in bed?

Based on the given data, an estimate of the population mean increase in height after three full days in bed is 14.15 mm.

b. Calculate the margin of error associated with your estimate of the population mean from part (a). Round your answer to three decimal places.

The margin of error is approximated by $2\left(\frac{0.940}{\sqrt{6}}\right) = 0.768$ mm.

c. Based on your sample mean and the margin of error from parts (a) and (b), what are plausible values for the population mean height increase for all men who stay in bed for three full days?

Plausible values for the population mean height increase for all men who stay in bed for three full days are those between $14.15 - 0.768 = 13.382$ and $14.15 + 0.768 = 14.918$ mm.

Exercises 6-13: Gettysburg Address

001 Four	045 any	089 nation	133 our	177 they	221 full	265 perish
002 score	046 nation,	090 might	134 poor	178 who	222 measure	266 from
003 and	047 so	091 live.	135 power	179 fought	223 of	267 the
004 seven	048 conceived	092 It	136 to	180 here	224 devotion,	268 earth.
005 years	049 and	093 is	137 add	181 have	225 that	
006 ago,	050 so	094 altogether	138 or	182 thus	226 we	
007 our	051 dedicated,	095 fitting	139 detract.	183 far	227 here	
008 fathers	052 can	096 and	140 The	184 so	228 highly	
009 brought	053 long	097 proper	141 world	185 nobly	229 resolve	
010 forth	054 endure.	098 that	142 will	186 advanced.	230 that	
011 upon	055 We	099 we	143 little	187 It	231 these	
012 this	056 are	100 should	144 note,	188 is	232 dead	
013 continent	057 met	101 do	145 nor	189 rather	233 shall	
014 a	058 on	102 this.	146 long	190 for	234 not	
015 new	059 a	103 But,	147 remember,	191 us	235 have	
016 nation;	060 great	104 in	148 what	192 to	236 died	
017 conceived	061 battlefield	105 a	149 we	193 be	237 in	
018 in	062 of	106 larger	150 say	194 here	238 vain,	
019 liberty,	063 that	107 sense,	151 here,	195 dedicated	239 that	
020 and	064 war.	108 we	152 but	196 to	240 this	
021 dedicated	065 We	109 cannot	153 it	197 the	241 nation,	
022 to	066 have	110 dedicate,	154 can	198 great	242 under	
023 the	067 come	111 we	155 never	199 task	243 God,	
024 proposition	068 to	112 cannot	156 forget	200 remaining	244 shall	
025 that	069 dedicate	113 consecrate,	157 what	201 before	245 have	
026 all	070 a	114 we	158 they	202 us,	246 a	
027 men	071 portion	115 cannot	159 did	203 that	247 new	
028 are	072 of	116 hallow	160 here.	204 from	248 birth	
029 created	073 that	117 this	161 It	205 these	249 of	
030 equal.	074 field	118 ground.	162 is	206 honored	250 freedom,	
031 Now	075 as	119 The	163 for	207 dead	251 and	
032 we	076 a	120 brave	164 us	208 we	252 that	
033 are	077 final	121 men,	165 the	209 take	253 government	
034 engaged	078 resting	122 living	166 living,	210 increased	254 of	
035 in	079 place	123 and	167 rather,	211 devotion	255 the	
036 a	080 for	124 dead,	168 to	212 to	256 people,	
037 great	081 those	125 who	169 be	213 that	257 by	
038 civil	082 who	126 struggled	170 dedicated	214 cause	258 the	
039 war,	083 here	127 here	171 here	215 for	259 people,	
040 testing	084 gave	128 have	172 to	216 which	260 for	
041 whether	085 their	129 consecrated	173 the	217 they	261 the	
042 that	086 lives	130 it,	174 unfinished	218 gave	262 people,	
043 nation,	087 that	131 far	175 work	219 the	263 shall	
044 or	088 that	132 above	176 which	220 last	264 not	