Lesson 29: The Mathematics Behind a Structured Savings Plan

Student Outcomes

- Students derive the sum of a finite geometric series formula.
- Students apply the sum of a finite geometric series formula to a structured savings plan.

Lesson Notes

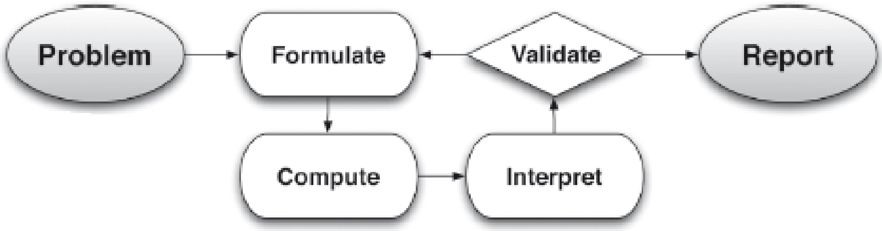
Module 3 ends with a series of lessons centered on finance. In prior lessons, students progressed through the mathematics of a structured savings plan, buying a car, borrowing on credit cards, and buying a house. The module ends with an investigation of how to save over one million dollars in assets by the time the students are 40 years old. Throughout these lessons, students engage in various parts of the modeling cycle: formulating, computing, interpreting, validating, etc.

In Lesson 29, students derive the formula for the sum of a finite geometric series (**A-SSE.B.4**). Once established, students work with and develop fluency with summation notation, sometimes called sigma notation. Students are then presented with the problem of a structured savings plan (known as a sinking fund, but this terminology is avoided). Students use the modeling cycle to identify essential features of structured savings plans and develop a model from the formula for the sum of a finite geometric series. By the end of the lesson, students will have both the formula for a sum of a finite geometric series (where the common ratio is not 1) and the formula for a structured savings plan. The structured savings plan will be modified in the remaining lessons to apply to other types of loans, and additional formulas will be developed based on the structured savings plan.

The formula $A_f = R \cdot \frac{(1+i^n)-1}{i}$ gives the future value of a structured savings plan A_f with recurring payment R, interest rate i per compounding period, and number of compounding periods n. In the context of loans, sometimes P is used to represent the payment instead of R, but R has been chosen because the formula is first presented as a structured savings plan and to avoid conflict with the common practice of using P for principal.

Recall that the definition of a sequence in high school is simply a function whose domain is the natural numbers or non-negative integers. Students have previously worked with arithmetic and geometric sequences. Sequences can be described both recursively and explicitly (**F-BF.A.2**). Although students will primarily be working with geometric sequences in Lessons 29 through 33, arithmetic sequences will be reviewed in the problem sets of the lessons.

A copy of the modeling cycle flowchart is included below to assist with the modeling portions of these lessons. Whenever students consider a modeling problem, they need to first identify variables representing essential features in the situation, formulate a model to describe the relationships between the variables, analyze and perform operations on the relationships, interpret their results in terms of the original situation, validate their conclusions, and either improve the model or report on their conclusions and their reasoning. Both descriptive and analytic modeling will be used. Suggestions are made for the teacher, but the full extent of the modeling portions of the first three lessons in this topic are left to the discretion of the teacher and should be completed as time permits.



Note: You might consider breaking this lesson up over two days.

Classwork

Opening Exercise (3 minutes)

This is a quick exercise designed to remind students of the function $F = P\left(1+\frac{r}{n}\right)^{nt}$ from Lesson 26.

> **Opening Exercise**
>
> **Suppose you invested $\$1000$ in an account that paid an annual interest rate of 3% compounded monthly. How much would you have after 1 year?**
>
> ***Since $F = 1000\left(1+\frac{0.03}{12}\right)^{12}$, we have $F = (1.0025)^{12} = \$1030.42$.***

After students have found the answer, have the following discussion *quickly*. It is important for setting up the notation for Topic E.

- To find the percent rate of change in the problem above, we took the annual interest rate of 3% compounded monthly (called the nominal APR) and divided it by 12. How do we express that in the future value formula from Lesson 26 using r and n?
  - *We can express in the future value formula by using the expression $\frac{r}{n}$, where r is the unit rate associated to the nominal APR and n is the number of compounding periods in a year.*
- In today's lesson and in the next set of lessons, we are going to replace $\frac{r}{n}$ with something simpler: the letter i for unit rate of the percent rate of change per time period (i.e., the interest rate per compounding period). What is the value of i in the problem above?
  - $i = 0.0025$
- Also, since we only need to make calculations based upon the number of time periods, we can use m to stand for the total number of time periods (i.e., total number of compounding periods). What is m in the problem above?
  - $m = 12$
- Thus, the formula we will work with is $F = P(1+i)^m$ where P is the present value and F is the future value. This formula will help make our calculations more transparent, but we need to always remember to find the interest rate per the compounding period first.

Discussion (7 minutes)

MP.4 & MP.8

This discussion sets up the reason for studying geometric series: Finding the future value of a structured savings plan. After this discussion, we will develop the formula for the sum of a series and then return to this discussion to show how to use the formula to find the future value of a structured savings plan. This sum of a geometric series will be brought up again and again throughout the remainder of the finance lessons to calculate information on structured savings plans, loans, annuities, and more. Look for ways you can use the discussion to help students formulate the problem.

- Let us consider the example of depositing $\$100$ at the end of every month into an account for 12 months. If there is no interest earned on these deposits, how much money will you have at the end of 12 months?
  - *We will have* $12 \cdot 100 = \$1{,}200$. *(Some students may be confused about the first and last payment. Have them imagine they start a job where they get paid at the end of each month, and they put* $\$100$ *from each paycheck into a special account. Then they would have* $\$0$ *in the account until the end of the first month at which time they deposit* $\$100$. *Similarly, they would have* $\$200$ *at the end of month* 2, $\$300$ *at the end of month* 3, *and so on.)*
- Now let us make an additional assumption: Suppose that our account is earning an annual interest rate of 3% compounded monthly. We would like to find how much will be in the account at the end of the year. The answer requires several calculations, not just one.
- What is the interest rate per month?
  - $i = 0.025$
- How much will the $\$100$ deposited at the end of month 1 and its interest be worth at the end of the year?
  - *Since the interest on the* $\$100$ *would be compounded* 11 *times, it and its interest would be worth* $100(1.025)^{11}$ *using the formula* $F = P(1+i)^m$. *(Some students may calculate this to be* $\$131.21$, *but tell them to leave their answers in exponential form for now.)*
- How much will the $\$100$ deposited at the end of month 2 and its interest be worth at the end of the year?
  - $100(1.025)^{10}$
- Continue: Find how much the $\$100$ deposited at the each of the remaining months plus its compounded interest will be worth at the end of the year by filling out the table:

Month deposited	Amount at the end of the year
1	$100(1.025)^{11}$
2	$100(1.025)^{10}$
3	$100(1.025)^{9}$
4	$100(1.025)^{8}$
5	$100(1.025)^{7}$
6	$100(1.025)^{6}$
7	$100(1.025)^{5}$
8	$100(1.025)^{4}$
9	$100(1.025)^{3}$
10	$100(1.025)^{2}$
11	$100(1.025)$
12	100

- Every deposit except for the final deposit will earn interest. If we list the calculations from the final deposit to the first deposit, we get the following sequence:

$$100, 100(1.025)^1, 100(1.025)^2, 100(1.025)^3, \dots, 1000(1.025)^{10}, 1000(1.02)^{11}$$

- What do you notice about this sequence?
  - *The sequence is geometric with initial term* 100 *and common ratio* 1.025*.*
- Describe how to calculate the amount of money we will have at the end of 12 months.
  - *Sum the* 12 *terms in the geometric sequence to get the total amount that will be in the account at the end of* 12 *months.*
- The answer,

$$100 + 100(1.025) + 100(1.025)^2 + \cdots + 100(1.025)^{11},$$

is an example of a geometric series. In the next example, we will find a formula to find the sum of this series quickly. For now, let's discuss the definition of series. When we add some of the terms of a sequence we get a series:

- **SERIES**: Let $a_1, a_2, a_3, a_4, \dots$ be a sequence of numbers. A sum of the form

$$a_1 + a_2 + a_3 + \cdots + a_n$$

for some positive integer n is called a *series* (or *finite series*) and is denoted S_n.

The a_i's are called the *terms* of the series. The number S_n that the series adds to is called the *sum* of the series.

- **GEOMETRIC SERIES:** A *geometric series* is a series whose terms form a geometric sequence.

Since a geometric sequence is of the form $a, ar, ar^2, ar^3, \dots$, the general form of a finite geometric sequence is of the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

- In the geometric series we generated from our structured savings plan, what is the first term a? What is the common ratio r? What is n?
  - $a = 100$, $r = 1.025$, *and* $n = 12$

Example 1 (10 minutes)

Work through the following example to establish the formula for finding the sum of a finite geometric series. In the example, we calculate the general form of a sum of a generic geometric series using letters a and r. We use letters instead of a numerical example for the following reasons:

- The use of letters in this situation actually makes the computations clearer. Students need only keep track of two letters instead of several numbers (as in $3, 6, 12, 24, 48, 96, \dots$).

- Students have already investigated series several times before in different forms without realizing it. For example, throughout Topic A of Module 1 of Algebra II, they worked with the identity $(1-r)(1+r^2+r^3+\cdots+r^{n-1}) = 1-r^n$ for any real number r and even used the identity to find the sum $1+2+2^2+2^3+\cdots+2^{31}$. In fact, students already derived the geometric series sum formula in the Problem Set of Lesson 1 of this module.

However, depending upon your class's strengths, you might consider doing a "mirror" calculation—on the left half of your board/screen do the calculation below, and on the right half "mirror" a numerical example; that is, for every algebraic line you write on the left, write the corresponding line using numbers on the right. A good numerical series to use is $2+6+18+54+162$.

Example 1

Let $a, ar, ar^2, ar^3, ar^4, \ldots$ be a geometric sequence with first term a and common ratio r. Show that the sum S_n of the first n terms of the geometric series

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \quad (r \neq 1)$$

is given by the equation

$$S_n = a\frac{1-r^n}{1-r}.$$

Give students 2–3 minutes to try the problem on their own (or in groups of two). It is completely okay if no one gets an answer—you are giving them structured time to persevere, test conjectures, and grapple with what they are expected to show. Give a hint as you walk around the class: What identity from Module 1 can we use? Or, just ask them to see if the formula works for $a=1$, $r=2$, $n=5$. If a student or group of students solves the problem, tell them to hold on to their solution for a couple of minutes. Then go through this discussion:

MP.7

- Multiply both sides of the equation $S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$ by r.
  - $r \cdot S_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$
- Compare the terms on the right-hand side of the old and new equations:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ r \cdot S_n &= \phantom{a + {}} ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

- What are the only terms on the right-hand side of the original equation and the new equation that are not found in both?
  - *The only terms that are not found in both are a and ar^n.*
- Therefore, when we subtract $S_n - rS_n$, all the common terms on the right-hand side of the equations subtract to zero leaving only $a - ar^n$ (after applying the associative and commutative properties repeatedly):

$$\begin{aligned} S_n - rS_n &= (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^{n-1} + ar^n) \\ &= a + (ar - ar) + (ar^2 - ar^2) + \cdots + (ar^{n-1} - ar^{n-1}) - ar^n \\ &= a - ar^n, \end{aligned}$$

or $S_n - rS_n = a - ar^n$.

- Isolate S_n in the equation $S_n - rS_n = a - ar^n$ to get the formula (when $r \neq 1$):

$$S_n(1-r) = a(1-r^n) \implies S_n = a\frac{1-r^n}{1-r}.$$

Scaffolding:

- For struggling students, do this calculation twice, first using a concrete sequence such as $a = 3$ and $r = 2$ before generalizing.

Any student who used the identity equation $(1-r)(1+r^2+r^3+\cdots+r^{n-1}) = 1-r^n$ or a different verifiable method (like proof by induction), should be sent to the board to explain the solution to the rest of class as an alternative explanation to your explanation. If no one was able to show the formula, you can wrap up the explanation by linking the formula back to the work they have done with this identity: Divide both sides of the identity equation by $(1-r)$ to isolate the sum $1+r+r^2+\cdots+r^{n-1}$ and then for $r \neq 1$,

$$a+ar+ar^2+\cdots+ar^{n-1} = a(1+r+r^2+\cdots+r^{n-1}) = a\frac{1-r^n}{1-r}.$$

Exercises 1–3 (5 minutes)

Exercises 1–3

1. **Find the sum of the geometric series $3+6+12+24+48+96+192$.**

   The first term is 3, so $a=3$. The common ratio is $6/3=2$, so $r=2$. Since there are 7 terms, $n=7$. Thus,

   $S_7 = 3\cdot\frac{1-2^7}{1-2}$, or $S_7 = 381$.

2. **Find the sum of the geometric series $40+40(1.005)+40(1.005)^2+\cdots+40(1.005)^{11}$.**

   Reading directly off the sum, $a=40$, $r=1.005$, and $n=12$.

   Thus, $S_{12} = 40\cdot\frac{1-1.005^{12}}{1-1.005}$, so $S_{12}\approx 493.42$.

3. **Describe a situation that might lead to calculating the sum of the geometric series in Exercise 2.**

   Investing $\$40$ a month into an account with an annual interest rate of 6% compounded monthly (or investing $\$40$ a month into an account with an interest rate of 0.05% per month) can lead to the geometric series in Exercise 2.

Scaffolding:

Ask advanced students to first find the sum
$a+ar^2+ar^4+\cdots+ar^{2n-2}$
and then the sum
$ar+ar^3+ar^5+\cdots+ar^{2n-1}$.

Example 2 (2 minutes)

Let's now return to the Opening Exercise and answer the problem we encountered there.

Example 2

A $\$100$ deposit is made at the end of every month for 12 months in an account that earns interest at an annual interest rate of 3% compounded monthly. How much will be in the account immediately after the last payment?

Answer: The total amount is the sum $100+100(1.025)+100(1.025)^2+\cdots+100(1.025)^{11}$. This is a geometric series with $a=100$, $r=1.025$, and $n=12$. Using the formula for the sum of a geometric series, $S_{12} = 100\cdot\frac{1-1.025^{12}}{1-1.025}$, so $S\approx 1{,}379.56$. The account will have $\$1{,}379.56$ in it immediately after the last payment.

Point out to your students that $1,379.56 is significantly more money than stuffing $100 in your mattress every month for 12 months. A structured savings plan like this is one way people can build wealth over time. Structured savings plans are examples of annuities. An annuity is commonly thought of as a type of retirement plan, but in this lesson, we are using the term in a much simpler way to refer to any situation where money is transferred into an account in equal amounts on a regular, recurring basis.

Discussion (5 minutes)

Explain the following text to your students and work with them to answer the questions below it.

Discussion

An *annuity* is a series of payments made at fixed intervals of time. Examples of annuities include structured savings plans, lease payments, loans, and monthly home mortgage payments. The term annuity sounds like it is only a yearly payment, but annuities are often monthly, quarterly, or semiannually. The *future amount of the annuity,* denoted A_f, is the sum of all the individual payments made plus all the interest generated from those payments over the specified period of time.

We can generalize the structured savings plan example above to get a generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of payment periods n. In the example above, we had a recurring payment of $R = 100$, an interest rate per time period of $i = 0.025$, and 12 payments, so $n = 12$. To make things simpler, we always assume that the payments and the time period in which interest is compounded are at the same time. That is, we do not consider plans where deposits are made halfway through the month with interest compounded at the end of the month.

In the example, the amount A_f of the structured savings plan annuity was the sum of all payments plus the interest accrued for each payment:

$$A_f = R + R(1+i)^1 + R(1+i)^2 + \cdots + R(1+i)^{n-1}.$$

This, of course, is a geometric series with n terms, $a = R$, and $r = 1 + i$, which after substituting into the formula for a geometric series and rearranging is

$$A_f = R \cdot \frac{(1+i)^n - 1}{i}.$$

- Explain how to get the formula above from the sum of a geometric series formula.
  - *Substitute R, $1 + i$, and n into the geometric series and rearrange:*

$$A_f = R \cdot \frac{1-(1+i)^n}{1-(1+i)} = R \cdot \frac{1-(1+i)^n}{-i} = R \cdot \frac{(1+i)^n - 1}{i}.$$

- (Optional depending on time.) How much money would need to be invested every month into an account with an annual interest rate of 12% compounded monthly in order to have $\$3{,}000$ after 18 months?
  - $3000 = R \cdot \frac{(1.01)^{18}-1}{0.01}$. *Solving for R yields $R \approx 152.95$. For such a savings plan, $\$152.95$ would need to be invested monthly.*

Example 3 (5 minutes)

Example 3 develops summation notation, sometimes called sigma notation. Including summation notation at this point is natural to reduce the amount of writing in sums and to develop fluency before relying on summation notation in precalculus and calculus. Plus, it "wraps up" all the important components of describing a series (starting value, index, explicit formula, and ending value) into one convenient notation. Summation notation will be further explored in the problem sets of Lessons 29 through 33 to develop fluency with the notation. Summation notation is specifically used to represent series. Students may be interested to know that a similar notation exists for products; product notation makes use of a capital pi, Π, instead of a capital sigma but is otherwise identical in form to summation notation.

Mathematicians use notation to reduce the amount of writing and to prevent ambiguity in mathematical sentences. Mathematicians use a special symbol with sequences to indicate that we would like to sum up the sequence. This symbol is a capital sigma, Σ. The sum of a sequence is called a *series*.

- The first letter of summation is "S," and the Greek letter for "S" is sigma. Capital sigma looks like this: Σ.
- There is no rigid way to use Σ to represent a summation, but all notations generally follow the same rules. We will discuss the most common way it used. Given a sequence $a_1, a_2, a_3, a_4, \ldots$, we can write the sum of the first n terms of the sequence using the expression:

$$\sum_{k=1}^{n} a_k.$$

- It is read, "The sum of a_k from $k = 1$ to $k = n$." The letter k is called the index of the summation. The notation acts like a for-next loop in computer programming: Replace k in the expression right after the sigma by the integers (in this case) $1, 2, 3, \ldots, n$, and add the resulting expressions together. Since

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n,$$

the notation can be used to greatly simplify writing out the sum of a series.

- When the terms in a sequence are given by an explicit function, like the geometric sequence given by $a_k = 2^k$ for example, we often use the expression defining the explicit function instead of sequence notation. For example, the sum of the first five powers of 2 can be written as

$$\sum_{k=1}^{5} 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5.$$

Exercises 4–5 (3 minutes)

Exercises 4–5

4. Write the sum without using summation notation, and find the sum.

a. $\displaystyle\sum_{k=0}^{5} k$

$$\sum_{k=0}^{5} k = 0 + 1 + 2 + 3 + 4 + 5 = 15$$

b. $\displaystyle\sum_{j=5}^{7} j^2$

$$\sum_{j=5}^{7} j^2 = 5^2 + 6^2 + 7^2 = 25 + 36 + 49 = 110$$

c. $\displaystyle\sum_{i=2}^{4} \frac{1}{i}$

$$\sum_{i=2}^{4} \frac{1}{i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

5. Write each sum using summation notation.

a. $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 + 7^4 + 8^4 + 9^4$

$$\sum_{k=1}^{9} k^4$$

b. $1 + \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) + \cos(5\pi)$

$$\sum_{k=0}^{5} \cos(k\pi)$$

c. $2 + 4 + 6 + \cdots + 1000$

$$\sum_{k=1}^{500} 2k$$

Closing (2 minutes)

- Have students summarize the lesson in writing or with a partner. Circulate and informally assess understanding. Ensure that every student has the formula for the sum of a finite geometric series and the formula for the future value of a structured savings plan. Ask questions to prompt student memory of the definitions and formulas below.

Lesson Summary

- Series: Let $a_1, a_2, a_3, a_4, \ldots$ be a sequence of numbers. A sum of the form
$$a_1 + a_2 + a_3 + \cdots + a_n$$
for some positive integer n is called a *series* (or finite series) and is denoted S_n. The a_i's are called the terms of the series. The number S_n that the series adds to is called the sum of the series.
- GEOMETRIC SERIES: A *geometric series* is a series whose terms form a geometric sequence.
- SUM OF A FINITE GEOMETRIC SERIES: The sum S_n of the first n terms of the geometric series $S_n = a + ar + \cdots + ar^{n-1}$ (when $r \neq 1$) is given by
$$S_n = a\frac{1-r^n}{1-r}.$$
- The sum of a finite geometric series can be written in summation notation as
$$\sum_{k=0}^{n-1} ar^k = a \cdot \frac{1-r^n}{1-r}.$$
- The generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of periods n is given by
$$A_f = R \cdot \frac{(1+i)^n - 1}{i}.$$

Exit Ticket (3 minutes)

Name ____ Date ____________________

Lesson 29: The Mathematics Behind a Structured Savings Plan

Exit Ticket

Martin attends a financial planning conference and creates a budget for himself, realizing that he can afford to put away $\$200$ every month in savings and that he should be able to keep this up for two years. If Martin has the choice between an account earning an interest rate of 2.3% yearly versus an account earning an annual interest rate of 2.125% compounded monthly, which account will give Martin the largest return in two years?

Exit Ticket Sample Solutions

MP.2

Martin attends a financial planning conference and creates a budget for himself, realizing that he can afford to put away $\$200$ every month in savings and that he should be able to keep this up for two years. If Martin has the choice between an account earning an interest rate of 2.3% yearly versus an account earning an annual interest rate of 2.125% compounded monthly, which account will give Martin the largest return in two years?

$$A_f = R \cdot \frac{(1+i)^n - 1}{i}$$

$$A_f = 2400 \cdot \frac{(1.023)^2 - 1}{0.023} = 4855.20$$

$$A_f = 200 \cdot \frac{\left(1 + \frac{0.0215}{12}\right)^{24} - 1}{\frac{0.0215}{12}} \approx 4900.21$$

The account earning an interest rate of 2.125% compounded monthly will return more than the yearly account.

Problem Set Sample Solutions

The Problem Set in this lesson begins with performing the research necessary to personalize Lesson 30 before continuing with more work in sequences and sums and practice using the future value of a structured savings plan formula.

1. **A car loan is one of the first secured loans most Americans obtain. Research used car prices and specifications in your area to find a reasonable used car that you would like to own (under $\$10,000$). If possible, print out a picture of the car you selected.**

   a. **What is the year, make, and model of your vehicle?**

      Answers will vary. For instance, 2006 Pontiac G6 GT.

   b. **What is the selling price for your vehicle?**

      Answers will vary. For instance, $\$7500$.

c. **The following table gives the monthly cost per $\$1000$ financed on a 5-year auto loan. Assume you can get a 5% annual interest rate. What is the monthly cost of financing the vehicle you selected? (A formula will be developed to find the monthly payment of a loan in Lesson 30.)**

Five-Year (60-month) Loan	
Interest Rate	Amount per $1000 Financed
1.0%	$\$17.09$
1.5%	$\$17.31$
2.0%	$\$17.53$
2.5%	$\$17.75$
3.0%	$\$17.97$
3.5%	$\$18.19$
4.0%	$\$18.41$
4.5%	$\$18.64$
5.0%	$\$18.87$
5.5%	$\$19.10$
6.0%	$\$19.33$
6.5%	$\$19.56$
7.0%	$\$19.80$
7.5%	$\$20.04$
8.0%	$\$20.28$
8.5%	$\$20.52$
9.0%	$\$20.76$

Answers will vary. At 5% interest, financing a $\$7,500$ car for 60 months will cost $18.87 \cdot 7.5 = 141.525$, which is approximately $\$141.53$ per month.

d. **What is the gas mileage for your vehicle?**

Answers will vary. For instance, the gas mileage might be 29 miles per gallon.

e. **If you drive 120 miles per week and gas is $\$4$ per gallon, then how much will gas cost per month?**

Answers will vary but should be based on the answer to part (d). For instance, with gas mileage of 29 miles per gallon, the gas cost would be $\frac{120}{29} \cdot 4.3 \cdot 4 \approx \$71.17$ per month.

2. **Write the sum without using summation notation, and find the sum.**

a. $\sum_{k=1}^{8} k$

$$1+2+3+4+5+6+7+8 = 4 \cdot 9 = 36$$

b. $\sum_{k=-8}^{8} k$

$$-8 + -7 + -6 + -5 + -4 + -3 + -2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 0$$

c. $\sum_{k=1}^{4} k^3$

$$1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100$$

d. $\sum_{m=0}^{6} 2m$

$$\begin{aligned} 2\cdot 0 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5 + 2\cdot 6 &= 0 + 2 + 4 + 6 + 8 + 10 + 12 \\ &= 3.5 \cdot 12 \\ &= 42 \end{aligned}$$

e. $\sum_{m=0}^{6} 2m + 1$

$$\begin{aligned} (2\cdot 0+1) + (2\cdot 1+1) + (2\cdot 2+1) + (2\cdot 3+1) + (2\cdot 4+1) + (2\cdot 5+1) + (2\cdot 6+1) &= 42 + 7 \\ &= 49 \end{aligned}$$

f. $\sum_{k=2}^{5} \frac{1}{k}$

$$\begin{aligned} \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} &= \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} \\ &= \frac{77}{60} \end{aligned}$$

g. $\sum_{j=0}^{3} (-4)^{j-2}$

$$\begin{aligned} (-4)^{-2} + (-4)^{-1} + (-4)^{0} + (-4)^{1} &= \frac{1}{16} + -\frac{1}{4} + 1 + -4 \\ &= \frac{1}{16} - \frac{4}{16} + \frac{16}{16} - \frac{64}{16} \\ &= -\frac{51}{16} \end{aligned}$$

h. $\sum_{m=1}^{4} 16\left(\frac{3}{2}\right)^m$

$$\begin{aligned} \left(16\left(\frac{3}{2}\right)^1\right) + \left(16\left(\frac{3}{2}\right)^2\right) + \left(16\left(\frac{3}{2}\right)^3\right) + \left(16\left(\frac{3}{2}\right)^4\right) &= 16\left(\frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16}\right) \\ &= 16\left(\frac{24}{16} + \frac{36}{16} + \frac{54}{16} + \frac{81}{16}\right) \\ &= 24 + 36 + 54 + 81 \\ &= 195 \end{aligned}$$

i. $\displaystyle\sum_{j=0}^{3} \frac{105}{2j+1}$

$$\begin{aligned}\frac{105}{2\cdot 0+1}+\frac{105}{2\cdot 1+1}+\frac{105}{2\cdot 2+1}+\frac{105}{2\cdot 3+1} &= \frac{105}{1}+\frac{105}{3}+\frac{105}{5}+\frac{105}{7}\\ &= 105+35+21+15\\ &= 176\end{aligned}$$

j. $\displaystyle\sum_{p=1}^{3} p\cdot 3^p$

$$\begin{aligned}1\cdot 3^1+2\cdot 3^2+3\cdot 3^3 &= 3+18+81\\ &= 102\end{aligned}$$

k. $\displaystyle\sum_{j=1}^{6} 100$

$$100+100+100+100+100+100=600$$

l. $\displaystyle\sum_{k=0}^{4} \sin\left(\frac{k\pi}{2}\right)$

$$\begin{aligned}\sin\left(\frac{0\pi}{2}\right)+\sin\left(\frac{1\pi}{2}\right)+\sin\left(\frac{2\pi}{2}\right)+\sin\left(\frac{3\pi}{2}\right)+\sin\left(\frac{4\pi}{2}\right) &= 0+1+0+\ -1+0\\ &= 0\end{aligned}$$

m. $\displaystyle\sum_{k=1}^{9} \log\left(\frac{k}{k+1}\right)$

(Hint: You do not need a calculator to find the sum.)

$$\log\left(\frac{1}{2}\right)+\log\left(\frac{2}{3}\right)+\log\left(\frac{3}{4}\right)+\log\left(\frac{4}{5}\right)+\log\left(\frac{5}{6}\right)+\log\left(\frac{6}{7}\right)+\log\left(\frac{7}{8}\right)+\log\left(\frac{8}{9}\right)+\log\left(\frac{9}{10}\right)$$

$$\begin{aligned}&= \log(1)-\log(2)+\log(2)-\log(3)+\cdots-\log(10)\\ &= \log(1)-\log(10)\\ &= 0-1\\ &= -1\end{aligned}$$

3. Write the sum without using sigma notation (you do not need to find the sum).

a. $\displaystyle\sum_{k=0}^{4} \sqrt{k+3}$

$$\sqrt{0+3}+\sqrt{1+3}+\sqrt{2+3}+\sqrt{3+3}+\sqrt{4+3}=\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}+\sqrt{7}$$

b. $\displaystyle\sum_{i=0}^{8} x^i$

$$1+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8$$

c. $\sum_{j=1}^{6} jx^{j-1}$

$1x^{1-1} + 2x^{2-1} + 3x^{3-1} + 4x^{4-1} + 5x^{5-1} + 6x^{6-1} = 1 + 2x^1 + 3x^2 + 4x^3 + 5x^4 + 6x^5$

d. $\sum_{k=0}^{9} (-1)^k x^k$

$$(-1)^0x^0 + (-1)^1x^1 + (-1)^2x^2 + (-1)^3x^3 + (-1)^4x^4 + (-1)^5x^5 + (-1)^6x^6 + (-1)^7x^7 + (-1)^8x^8 + (-1)^9x^9$$
$$= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9$$

4. Write each sum using summation notation.

a. $1 + 2 + 3 + 4 + \cdots + 1000$

$\sum_{k=1}^{1000} k$

b. $2 + 4 + 6 + 8 + \cdots + 100$

$\sum_{k=1}^{50} 2k$

c. $1 + 3 + 5 + 7 + \cdots + 99$

$\sum_{k=1}^{50} 2k - 1$

d. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{99}{100}$

$\sum_{k=1}^{99} \frac{k}{k+1}$

e. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 10,000^2$

$\sum_{k=1}^{10000} k^2$

f. $1 + x + x^2 + x^3 + \cdots + x^{200}$

$\sum_{k=0}^{200} x^k$

g. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{49\cdot 50}$

$\sum_{k=1}^{49} \frac{1}{k(k+1)}$

h. $1\ln(1) + 2\ln(2) + 3\ln(3) + \cdots + 10\ln(10)$

$$\sum_{k=1}^{10} k\ln(k)$$

5. **Find the sum of the geometric series.**

a. $1 + 3 + 9 + \cdots + 2187$

$$\frac{1-3^8}{1-3} = \frac{6560}{2} = 3280$$

b. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{512}$

$$\frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}} = \frac{\frac{1023}{1024}}{\frac{1}{2}} = \frac{1023}{1024}\cdot 2 = \frac{1023}{512}$$

c. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

$$\frac{1-\left(-\frac{1}{2}\right)^{10}}{1+\frac{1}{2}} = \frac{1023}{1024}\cdot\frac{2}{3} = \frac{341}{512}$$

d. $0.8 + 0.64 + 0.512 + \cdots + 0.32768$

$$0.8\cdot\frac{1-0.8^5}{1-0.8} = 0.8\cdot\frac{0.67232}{0.2} = 2.68928$$

e. $1 + \sqrt{3} + 3 + 3\sqrt{3} + \cdots + 243$

$$\frac{1-\sqrt{3}^{11}}{1-\sqrt{3}} \approx 573.5781477$$

f. $\sum_{k=0}^{5} 2^k$

$$\frac{1-2^6}{1-2} = 63$$

g. $\sum_{m=1}^{4} 5\left(\frac{3}{2}\right)^m$

$$\begin{aligned}\left(5\left(\frac{3}{2}\right)\right)\left(\frac{1-\left(\frac{3}{2}\right)^4}{1-\frac{3}{2}}\right) &= \left(\frac{15}{2}\right)\left(\frac{\frac{81}{16}-1}{\frac{1}{2}}\right)\\ &= \frac{15}{2}\left(\frac{65}{16}\right)2\\ &= \frac{975}{16}\\ &= 60.9375\end{aligned}$$

h. $1 - x + x^2 - x^3 + \cdots + x^{30}$ in terms of x

$$\frac{1-(-x)^{31}}{1-(-x)} = \frac{x^{31}+1}{x+1}$$

i. $\sum_{m=0}^{11} 4^{\frac{m}{3}}$

$$\frac{1-\left(4^{\frac{1}{3}}\right)^{12}}{1-4^{\frac{1}{3}}} \approx 434.1156679$$

j. $\sum_{n=0}^{14} \left(\sqrt[5]{6}\right)^n$

$$\frac{1-\left(\sqrt[5]{6}\right)^{15}}{1-\sqrt[5]{6}} \approx 498.8756953$$

k. $\sum_{k=0}^{6} 2 \cdot \left(\sqrt{3}\right)^k$

$$2 \cdot \frac{1-\sqrt{3}^7}{1-\sqrt{3}} \approx 125.033321$$

6. Let a_i represent the sequence of even natural numbers $\{2, 4, 6, 8, \ldots\}$, and evaluate the following expressions.

a. $\sum_{i=1}^{5} a_i$

$$2 + 4 + 6 + 8 + 10 = 30$$

b. $\sum_{i=1}^{4} a_{2i}$

$$\begin{aligned}a_2 + a_4 + a_6 + a_8 &= 4 + 8 + 12 + 16\\ &= 40\end{aligned}$$

c. $\sum_{i=1}^{5} (a_i - 1)$

$$\begin{aligned}(2-1) + (4-1) + (6-1) + (8-1) + (10-1) &= 1 + 3 + 5 + 7 + 9\\ &= 25\end{aligned}$$

7. Let a_i represent the sequence of integers giving the yardage gained per rush in a high school football game $\{3, -2, 17, 4, -8, 19, 2, 3, 3, 4, 0, 1, -7\}$.

a. Evaluate $\sum_{i=1}^{13} a_i$. What does this sum represent in the context of the situation?

$$\sum_{i=1}^{13} a_i = 3 + -2 + 17 + 4 + -8 + 19 + 2 + 3 + 3 + 4 + 0 + 1 + -7$$
$$= 56 + -17$$
$$= 39$$

This sum is the total rushing yards.

b. Evaluate $\frac{\sum_{i=1}^{13} a_i}{13}$. What does this expression represent in the context of the situation?

$$\frac{39}{13} = 3$$

The average yardage per rush is 3.

c. In general, if a_n describes any sequence of numbers, what does $\frac{\sum_{i=1}^{n} a_i}{n}$ represent?

The total divided by the number of numbers is the arithmetic mean or average of the set.

8. Let b_n represent the sequence given by the following recursive formula: $b_1 = 10$, $b_n = b_{n-1} \cdot 5$.

a. Write the first 4 terms of this sequence.

$$10, 50, 250, 1250$$

b. Expand the sum $\sum_{i=1}^{4} b_i$. Is it easier to add this series, or is it easier to use the formula for the sum of a finite geometric sequence? Explain your answer. Evaluate $\sum_{i=1}^{4} b_i$.

$$\sum_{i=1}^{4} b_i = 10 + 50 + 250 + 1250$$

Answers may vary based on personal opinion. Since this series consists of only four terms, it may be easier to simply add the terms together to find the sum. The sum is 1560.

c. Write an explicit form for b_n.

$b_n = 10 \cdot 5^{n-1}$, *where n is a positive integer.*

d. Evaluate $\sum_{i=1}^{10} b_i$.

$$(10) \cdot \left(\frac{1 - 5^{10}}{1 - 5}\right) = 10 \cdot \frac{9765624}{4} = 24{,}414{,}060$$

9. Consider the sequence given by $a_1 = 20$, $a_n = \frac{1}{2} \cdot a_{n-1}$.

a. Evaluate $\sum_{i=1}^{10} a_i$, $\sum_{i=1}^{100} a_i$, and $\sum_{i=1}^{1000} a_i$.

$$\sum_{i=1}^{10} a_i = 20 \cdot \frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2}} = 20 \cdot \frac{\frac{1023}{1024}}{\frac{1}{2}} = 39.9609375$$

$$\sum_{i=1}^{100} a_i = 20 \cdot \frac{1-\left(\frac{1}{2}\right)^{100}}{1-\frac{1}{2}} \approx 40$$
$$\sum_{i=1}^{1000} a_i = 20 \cdot \frac{1-\left(\frac{1}{2}\right)^{1000}}{1-\frac{1}{2}} \approx 40$$

b. What value does it appear this series is approaching as n continues to increase? Why might it seem like the series is bounded?

The series is almost exactly 40. In the numerator we are subtracting a number that is incredibly small and gets even smaller the farther we go in the sequence. So, as $n \to \infty$, the sum approaches $\frac{20}{\frac{1}{2}} = 40$.

10. The sum of a geometric series with 4 terms is 60, and the common ratio is $r = \frac{1}{2}$. Find the first term.

$$60 = a\left(\frac{1-\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}}\right)$$
$$60 = a\left(\frac{1-\frac{1}{16}}{\frac{1}{2}}\right)$$
$$60 = a\left(\frac{15}{16} \cdot 2\right)$$
$$60 = a\left(\frac{15}{8}\right)$$
$$a = 4 \cdot 8 = 32$$

11. The sum of the first 4 terms of a geometric series is 203, and the common ratio is 0.4. Find the first term.

$$203 = a\left(\frac{1-0.4^4}{1-0.4}\right)$$
$$a = 203\left(\frac{0.6}{1-0.4^4}\right) = 125$$

12. The third term in a geometric series is $\frac{27}{2}$, and the sixth term is $\frac{729}{16}$. Find the common ratio.

$$ar^2 = \frac{27}{2}$$
$$ar^5 = \frac{729}{16}$$
$$r^3 = \frac{729}{16} \cdot \frac{2}{27} = \frac{27}{8}$$
$$r = \frac{3}{2}$$

13. The second term in a geometric series is 10, and the seventh term is 10240. Find the sum of the first six terms.

$$ar = 10$$
$$ar^6 = 10240$$
$$r^5 = 1024$$
$$r = 4$$
$$a = \frac{10}{4} = \frac{5}{2}$$

$$\begin{aligned} S_6 &= \frac{5}{2}\left(\frac{1-4^6}{1-4}\right) \\ &= \frac{5}{2}\left(\frac{4095}{3}\right) \\ &= 3412.5 \end{aligned}$$

14. **Find the interest earned and the future value of an annuity with monthly payments of $\$200$ for two years into an account that pays 6% interest per year compounded monthly.**

$$A_f = 200\left(\frac{\left(1+\frac{0.06}{12}\right)^{24}-1}{\frac{0.06}{12}}\right)$$
$$\approx 5086.39$$

The future value is $\$5086.39$, and the interest earned is $\$286.39$.

15. **Find the interest earned and the future value of an annuity with annual payments of $\$1200$ for 15 years into an account that pays 4% interest per year.**

$$A_f = 1200\left(\frac{(1+0.04)^{15}-1}{0.04}\right)$$
$$\approx 24028.31$$

The future value is $\$24028.31$, and the interest earned is $\$6028.31$.

16. **Find the interest earned and the future value of an annuity with semiannual payments of $\$1000$ for 20 years into an account that pays 7% interest per year compounded semiannually.**

$$A_f = 1000\left(\frac{\left(1+\frac{0.07}{2}\right)^{40}-1}{\frac{0.07}{2}}\right)$$
$$\approx 84550.28$$

The future value is $\$84550.28$, and the interest earned is $\$44550.28$.

17. **Find the interest earned and the future value of an annuity with weekly payments of $\$100$ for three years into an account that pays 5% interest per year compounded weekly.**

$$A_f = 100\left(\frac{\left(1+\frac{0.05}{52}\right)^{156}-1}{\frac{0.05}{52}}\right)$$
$$\approx 16822.05$$

The future value is $\$16822.05$, and the interest earned is $\$1222.05$.

18. **Find the interest earned and the future value of an annuity with quarterly payments of $\$500$ for 12 years into an account that pays 3% interest per year compounded quarterly.**

$$A_f = 500\left(\frac{\left(1+\frac{0.03}{4}\right)^{48}-1}{\frac{0.03}{4}}\right)$$
$$\approx 28760.36$$

The future value is $\$28760.36$, and the interest earned is $\$3760.36$.

19. How much money should be invested every month with 8% interest per year compounded monthly in order to save up $\$10,000$ in 15 months?

$$10000 = R\left(\frac{\left(1+\frac{0.08}{12}\right)^{15}-1}{\frac{0.08}{12}}\right)$$
$$R = 10000\left(\frac{\frac{0.08}{12}}{\left(1+\frac{0.08}{12}\right)^{15}-1}\right)$$
$$\approx 636.11$$

Invest $\$636.11$ every month for 15 months at this interest rate to save up $\$10000$.

20. How much money should be invested every year with 4% interest per year in order to save up $\$40,000$ in 18 years?

$$40000 = R\left(\frac{(1+0.04)^{18}-1}{0.04}\right)$$
$$R = 40000\left(\frac{0.04}{(1.04)^{15}-1}\right)$$
$$\approx 1559.733$$

Invest $\$1559.74$ every year for 18 years at 4% interest per year to save up $\$40000$.

21. Julian wants to save up to buy a car. He is told that a loan for a car will cost $\$274$ a month for five years, but Julian does not need a car presently. He decides to invest in a structured savings plan for the next three years. Every month Julian invests $\$274$ at an annual interest rate of 2% compounded monthly.

a. How much will Julian have at the end of three years?

$$A_f = 274\left(\frac{\left(1+\frac{0.02}{12}\right)^{36}-1}{\frac{0.02}{12}}\right) \approx 10157.21$$

Julian will have $\$10157.21$ at the end of the three years.

b. What are the benefits of investing in a structured savings plan instead of taking a loan out? What are the drawbacks?

The biggest benefit is that instead of paying interest on a loan, you earn interest on your savings. The drawbacks include that you have to wait to get what you want.

22. An *arithmetic series* is a series whose terms form an arithmetic sequence. For example, $2+4+6+\cdots+100$ is an arithmetic series since $2, 4, 6, 8, \ldots, 100$ is an arithmetic sequence with constant difference 2.

The most famous arithmetic series is $1+2+3+4+\cdots+n$ for some positive integer n. We studied this series in Algebra I and showed that its sum is $S_n = \frac{n(n+1)}{2}$. It can be shown that the general formula for the sum of an arithmetic series $a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]$ is

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

where a is the first term and d is the constant difference.

a. Use the general formula to show that the sum of $1+2+3+\cdots+n$ is $S_n = \frac{n(n+1)}{2}$.

$$S_n = \frac{n}{2}(2\cdot 1+(n-1)\cdot 1) = \frac{n}{2}(2+n-1) = \frac{n}{2}(n+1)$$

b. Use the general formula to find the sum of $2 + 4 + 6 + 8 + 10 + \cdots + 100$.

$$S_n = \frac{50}{2}(4 + (50 - 1)2) = 25(102) = 2550$$

23. The sum of the first five terms of an arithmetic series is 25, and the first term is 2. Find the constant difference.

$$25 = \frac{5}{2}(2 + a_5)$$
$$10 = 2 + a_5$$
$$a_5 = 8$$

$$8 = 2 + d(4)$$
$$6 = d(4)$$
$$d = \frac{3}{2}$$

24. The sum of the first nine terms of an arithmetic series is 135, and the first term is 17. Find the ninth term.

$$135 = \frac{9}{2}(17 + a_9)$$
$$30 = 17 + a_9$$
$$13 = a_9$$

$$13 = 17 + d(8)$$
$$-4 = d(8)$$
$$d = -\frac{1}{2}$$

25. The sum of the first and 100^{th} terms of an arithmetic series is 101. Find the sum of the first 100 terms.

$$S_{100} = \frac{100}{2}(101) = 5050$$