Lesson 29: The Mathematics Behind a Structured Savings Plan

Classwork

Opening Exercise

Suppose you invested in an account that paid an annual interest rate of compounded monthly. How much would you have after year?

**Example 1**

Let be a geometric sequence with first term and common ratio . Show that the sum of the first terms of the geometric series

is given by the equation

Exercises 1–3

1. Find the sum of the geometric series .
2. Find the sum of the geometric series .
3. Describe a situation that might lead to calculating the sum of the geometric series in Exercise 2.

**Example 2**

A deposit is made at the end of every month for months in an account that earns interest at an annual interest rate of compounded monthly. How much will be in the account immediately after the last payment?

Discussion

An *annuity* is a series of payments made at fixed intervals of time. Examples of annuities include structured savings plans, lease payments, loans, and monthly home mortgage payments. The term annuity sounds like it is only a yearly payment, but annuities are often monthly, quarterly, or semiannually. The *future amount of the annuity,* denoted , is the sum of all the individual payments made plus all the interest generated from those payments over the specified period of time.

We can generalize the structured savings plan example above to get a generic formula for calculating the future value of an annuity in terms of the recurring payment , interest rate , and number of payment periods . In the example above, we had a recurring payment of , an interest rate per time period of , and payments, so   
. To make things simpler, we always assume that the payments and the time period in which interest is compounded are at the same time. That is, we do not consider plans where deposits are made halfway through the month with interest compounded at the end of the month.

In the example, the amount of the structured savings plan annuity was the sum of all payments plus the interest accrued for each payment:

This, of course, is a geometric series with terms, , and , which after substituting into the formula for a geometric series and rearranging is

Exercises 4–5

1. Write the sum without using summation notation, and find the sum.

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1. Write each sum using summation notation.

Lesson Summary

* **Series**: Let be a sequence of numbers. A sum of the form

for some positive integer is called a *series* (or *finite series*) and is denoted *.* The ’s are called the *terms* of the series. The number that the series adds to is called the *sum* of the series.

* **Geometric series:** A *geometric series* is a series whose terms form a geometric sequence.
* **Sum of a finite geometric series**: The sum of the first terms of the geometric series   
   (when ) is given by

The sum of a finite geometric series can be written in summation notation as

* The generic formula for calculating the future value of an annuity in terms of the recurring payment , interest rate , and number of periods is given by

Problem Set

1. A car loan is one of the first secured loans most Americans obtain. Research used car prices and specifications in your area to find a reasonable used car that you would like to own (under ). If possible, print out a picture of the car you selected.
   1. What is the year, make, and model of your vehicle?
   2. What is the selling price for your vehicle?
   3. The following table gives the monthly cost per financed on a -year auto loan. Assume you can get a annual interest rate. What is the monthly cost of financing the vehicle you selected? (A formula will be developed to find the monthly payment of a loan in Lesson 30.)

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| Five-Year (-month) Loan | |
| Interest Rate | Amount per Financed |
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* 1. What is the gas mileage for your vehicle?
  2. If you drive miles per week and gas is per gallon, then how much will gas cost per month?

1. Write the sum without using summation notation, and find the sum.

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|  |  | e. |  | j. |  | n. |  |
|  |  | f. |  | k. |  |  | (Hint: You do not need a calculator to find the sum.) |
|  |  | g. |  | l. |  |  |  |
|  |  | h. |  | m. |  |  |  |

1. Write the sum without using sigma notation (you do not need to find the sum).

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1. Write each sum using summation notation.
2. Find the sum of the geometric series.
   1. in terms of
3. Let represent the sequence of even natural numbers and evaluate the following expressions.

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1. Let represent the sequence of integers giving the yardage gained per rush in a high school football game .
   1. Evaluate . What does this sum represent in the context of the situation?
   2. Evaluate . What does this expression represent in the context of the situation?
   3. In general, if describes any sequence of numbers, what does represent?
2. Let represent the sequence given by the following recursive formula: , .
   1. Write the first terms of this sequence.
   2. Expand the sum . Is it easier to add this series, or is it easier to use the formula for the sum of a finite geometric sequence? Explain your answer. Evaluate .
   3. Write an explicit form for .
   4. Evaluate .
3. Consider the sequence given by , .
   1. Evaluate , , and .
   2. What value does it appear this series is approaching as continues to increase? Why might it seem like the series is bounded?
4. The sum of a geometric series with four terms is , and the common ratio is . Find the first term.
5. The sum of the first four terms of a geometric series is , and the common ratio is . Find the first term.
6. The third term in a geometric series is , and the sixth term is . Find the common ratio.
7. The second term in a geometric series is , and the seventh term is . Find the sum of the first six terms.
8. Find the interest earned and the future value of an annuity with monthly payments of for two years into an account that pays interest per year compounded monthly.
9. Find the interest earned and the future value of an annuity with annual payments of for years into an account that pays interest per year.
10. Find the interest earned and the future value of an annuity with semiannual payments of for years into an account that pays interest per year compounded semiannually.
11. Find the interest earned and the future value of an annuity with weekly payments of for three years into an account that pays interest per year compounded weekly.
12. Find the interest earned and the future value of an annuity with quarterly payments of for years into an account that pays interest per year compounded quarterly.
13. How much money should be invested every month with interest per year compounded monthly in order to save up in months?
14. How much money should be invested every year with interest per year in order to save up in years?
15. Julian wants to save up to buy a car. He is told that a loan for a car will cost a month for five years, but Julian does not need a car presently. He decides to invest in a structured savings plan for the next three years. Every month Julian invests at an annual interest rate of compounded monthly.
    1. How much will Julian have at the end of three years?
    2. What are the benefits of investing in a structured savings plan instead of taking a loan out? What are the drawbacks?
16. An *arithmetic series* is a series whose terms form an arithmetic sequence. For example, is an arithmetic series since is an arithmetic sequence with constant difference .

The most famous arithmetic series is for some positive integer . We studied this series in Algebra I and showed that its sum is . It can be shown that the general formula for the sum of an arithmetic series is

where is the first term and is the constant difference.

* 1. Use the general formula to show that the sum of is .
  2. Use the general formula to find the sum of .

1. The sum of the first five terms of an arithmetic series is and the first term is . Find the constant difference.
2. The sum of the first nine terms of an arithmetic series is and the first term is . Find the ninth term.
3. The sum of the first and th terms of an arithmetic series is . Find the sum of the first terms.