Lesson 27: Modeling with Exponential Functions

Classwork

Opening Exercise

The following table contains U.S. population data for the two most recent census years, 2000 and 2010.

Census Year	U.S. Population (in millions)
2000	281.4
2010	308.7

a. Steve thinks the data should be modeled by a linear function.

   i. What is the average rate of change in population per year according to this data?

   ii. Write a formula for a linear function, L, that will estimate the population t years since the year 2000.

b. Phillip thinks the data should be modeled by an exponential function.

   i. What is the growth rate of the population per year according to this data?

   ii. Write a formula for an exponential function, E, that will estimate the population t years since the year 2000.

c.   Who has the correct model? How do you know?

Mathematical Modeling Exercise/Exercises 1–14

In this challenge, you will continue to examine U.S. census data to select and refine a model for the population of the United States over time.

1.   The following table contains additional U.S. census population data. Would it be more appropriate to model this data with a linear or an exponential function? Explain your reasoning.

Census Year	U.S. Population (in millions of people)
1900	76.2
1910	92.2
1920	106.0
1930	122.8
1940	132.2
1950	150.7
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4
2010	308.7

2.   Use a calculator's regression capability to find a function, f, that models the U.S. Census Bureau data from 1900 to 2010.

3. Find the growth factor for each 10-year period and record it in the table below. What do you observe about these growth factors?

Census Year	U.S. Population (in millions of people)	Growth Factor (10-year period)
1900	76.2	--
1910	92.2	
1920	106.0	
1930	122.8	
1940	132.2	
1950	150.7	
1960	179.3	
1970	203.3	
1980	226.5	
1990	248.7	
2000	281.4	
2010	308.7	

4. For which decade is the 10-year growth factor the lowest? What factors do you think caused that decrease?

5. Find an average 10-year growth factor for the population data in the table. What does that number represent? Use the average growth factor to find an exponential function, g, that can model this data.

6. You have now computed three potential models for the population of the United States over time: functions E, f, and g. Which one do you expect would be the most accurate model based on how they were created? Explain your reasoning.

7. Summarize the three formulas for exponential models that you have found so far: Write the formula, the initial populations, and the growth rates indicated by each function. What is different between the structures of these three functions?

8. Rewrite the functions E, f, and g as needed in terms of an annual growth rate.

9. Transform the functions as needed so that the time $t = 0$ represents the same year in functions E, f, and g. Then compare the values of the initial populations and annual growth rates indicated by each function.

10. Which of the three functions is the best model to use for the U.S. census data from 1900 to 2010? Explain your reasoning.

11. The U.S. Census Bureau website http://www.census.gov/popclock displays the current estimate of both the United States and world populations.

    a. What is today's current estimated population of the U.S.?

    b. If time $t = 0$ represents the year 1900, what is the value of t for today's date? Give your answer to two decimal places.

    c. Which of the functions E, f, and g gives the best estimate of today's population? Does that match what you expected? Justify your reasoning.

    d. With your group, discuss some possible reasons for the discrepancy between what you expected in Exercise 8 and the results of part (c) above.

12. Use the model that most accurately predicted today's population in Exercise 9, part (c) to predict when the U.S. population will reach half a billion.

13. Based on your work so far, do you think this is an accurate prediction? Justify your reasoning.

14. Here is a graph of the U.S. population since the census began in 1790. Which type of function would best model this data? Explain your reasoning.
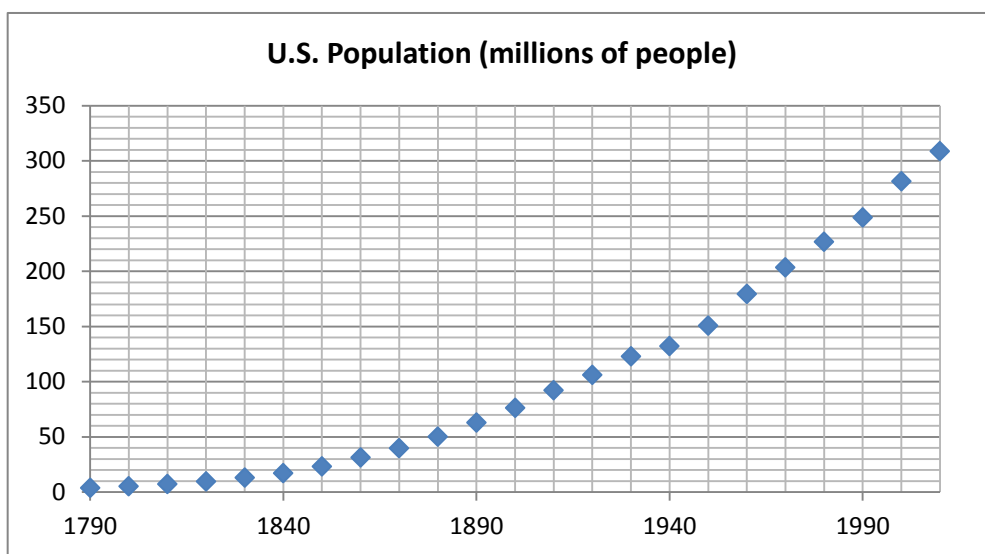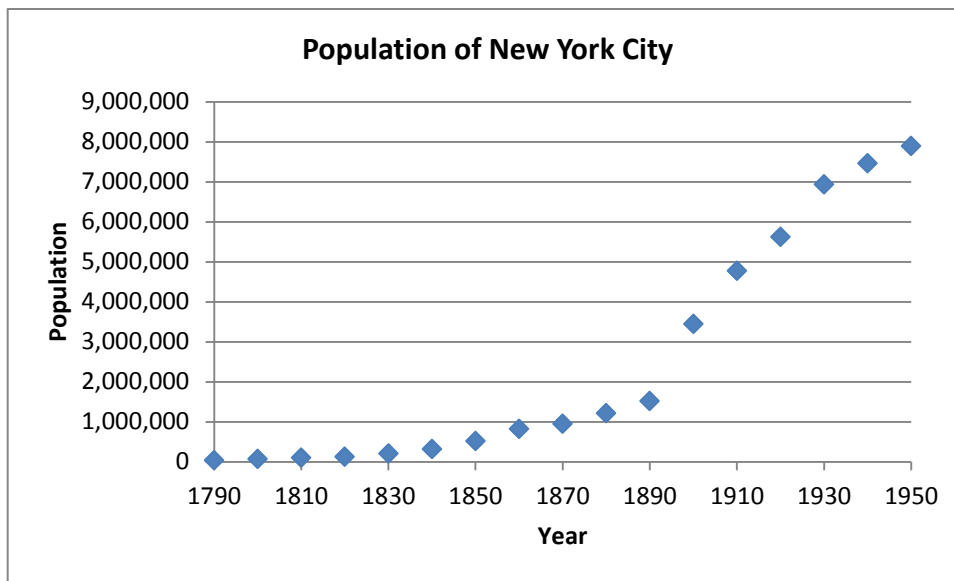
U.S. Population (millions of people)

Figure 1: Source U.S. Census Bureau

15. The graph below shows the population of New York City during a time of rapid population growth.

Population of New York City

Finn averaged the 10-year growth rates and wrote the function $f(t) = 33131(1.44)^{\frac{t}{10}}$, where t is the time in years since 1790.

Gwen used the regression features on a graphing calculator and got the function $g(t) = 48661(1.036)^t$, where t is the time in years since 1790.

a. Rewrite each function to determine the annual growth rate for Finn's model and Gwen's model.

b. What is the predicted population in the year 1790 for each model?

c. Lenny calculated an exponential regression using his graphing calculator and got the same growth rate as Gwen, but his initial population was very close to 0. Explain what data Lenny may have used to find his function.

d. When does Gwen's function predict the population will reach 1,000,000? How does this compare to the graph?

e. Based on the graph, do you think an exponential growth function would be useful for predicting the population of New York in the years after 1950?

16. Suppose each function below represents the population of a different U.S. city since the year 1900.

   a. Complete the table below. Use the properties of exponents to rewrite expressions as needed to help support your answers.

City Population Function (t is years since 1900)	Population in the Year 1900	Annual Growth/Decay Rate	Predicted in 2000	Between Which Years Did the Population Double?
$A(t) = 3000(1.1)^{\frac{t}{5}}$				
$B(t) = \dfrac{(1.5)^{2t}}{2.25}$				
$C(t) = 10000(1 - 0.01)^t$				
$D(t) = 900(1.02)^t$				

b. Could the function $(t) = 6520(1.219)^{\frac{t}{10}}$, where t is years since 2000 also represent the population of one of these cities? Use the properties of exponents to support your answer.

c. Which cities are growing in size and which are decreasing according to these models?

d. Which of these functions might realistically represent city population growth over an extended period of time?

Lesson Summary

To model data with an exponential function:

- Examine the data to see if there appears to be a constant growth or decay factor.

- Determine a growth factor and a point in time to correspond to $t = 0$.

- Create a function to model the situation $f(t) = a \cdot b^{ct}$, where b is the growth factor every $\frac{1}{c}$ years and a is the value of f when $t = 0$.

Logarithms can be used to solve for t when you know the value of $f(t)$ in an exponential function model.

Problem Set

1. Does each pair of formulas described below represent the same sequence? Justify your reasoning.

   a. $a_{n+1} = \frac{2}{3} a_n$, $a_0 = -1$ and $b_n = -\left(\frac{2}{3}\right)^n$ for $n \geq 0$.

   b. $a_n = 2a_{n-1} + 3$, $a_0 = 3$ and $b_n = 2(n-1)^3 + 4(n-1) + 3$ for $n \geq 1$.

   c. $a_n = \frac{1}{3}(3)^n$ for $n \geq 0$ and $b_n = 3^{n-2}$ for $n \geq 0$.

2. Tina is saving her babysitting money. She has $500 in the bank, and each month she deposits another $100. Her account earns 2% interest compounded monthly.

   a. Complete the table showing how much money she has in the bank for the first four months.

Month	Amount
1	
2	
3	
4	

   b. Write a recursive sequence for the amount of money she has in her account after n months.

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3. Assume each table represents values of an exponential function of the form $f(t) = a(b)^{ct}$, where b is a positive real number and a and c are real numbers. Use the information in each table to write a formula for f in terms of t for parts (a)–(d).

a.

t	$f(t)$
0	10
4	50

b.

t	$f(t)$
0	1000
5	750

c.

t	$f(t)$
6	25
8	45

d.

| t	$f(t)$
3	50
6	40

e. Rewrite the expressions for each function in parts (a)–(d) to determine the annual growth or decay rate.

f. For parts (a) and (c), determine when the value of the function is double its initial amount.

g. For parts (b) and (d), determine when the value of the function is half its initial amount.

4. When examining the data in Example 1, Juan noticed the population doubled every five years and wrote the formula $P(t) = 100(2)^{\frac{t}{5}}$. Use the properties of exponents to show that both functions grow at the same rate per year.

5. The growth of a tree seedling over a short period of time can be modeled by an exponential function. Suppose the tree starts out 3 ft. tall and its height increases by 15% per year. When will the tree be 25 ft. tall?

6. Loggerhead turtles reproduce every 2–4 years, laying approximately 120 eggs in a clutch. Studying the local population, a biologist records the following data in the second and fourth years of her study:

| Year	Population
2	50
4	1250

a. Find an exponential model that describes the loggerhead turtle population in year t.

b. According to your model, when will the population of loggerhead turtles be over 5,000? Give your answer in years and months.

7. The radioactive isotope seaborgium-266 has a half-life of 30 seconds, which means that if you have a sample of A g of seaborgium-266, then after 30 seconds half of the sample has decayed (meaning it has turned into another element) and only $\frac{A}{2}$ g of seaborgium-266 remain. This decay happens continuously.

a. Define a sequence $a_0, a_1, a_2, \dots$ so that a_n represents the amount of a 100 g sample that remains after n minutes.

b. Define a function $a(t)$ that describes the amount of seaborgium-266 that remains of a 100 g sample after t minutes.

c. Does your sequence from part (a) and your function from part (b) model the same thing? Explain how you know.

d. How many minutes does it take for less than 1 g of seaborgium-266 to remain from the original 100 g sample? Give your answer to the nearest minute.

8. Compare the data for the amount of substance remaining for each element: strontium-90, magnesium-28, and bismuth.

Strontium-90 (grams) vs. time (hours)
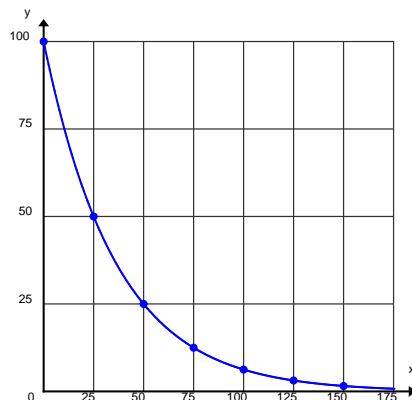
Radioactive Decay of Magnesium-28	
R	t hours
1	0
0.5	21
0.25	42
0.125	63
0.0625	84
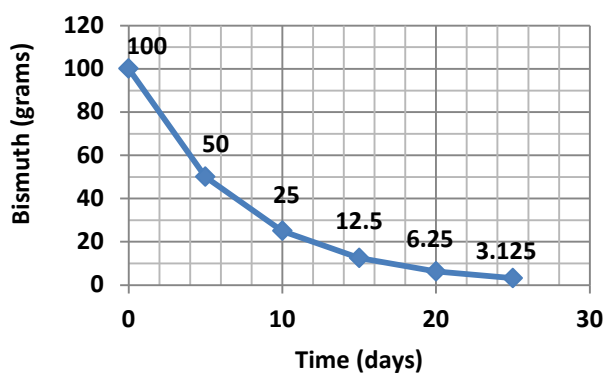
a. Which element decays most rapidly? How do you know?

b. Write an exponential function for each element that shows how much of a 100 g sample will remain after t days. Rewrite each expression to show precisely how their exponential decay rates compare to confirm your answer to part (a).

9. The growth of two different species of fish in a lake can be modeled by the functions shown below where t is time in months since January 2000. Assume these models will be valid for at least 5 years.

    Fish A: $f(t) = 5000(1.3)^t$

    Fish B: $g(t) = 10{,}000(1.1)^t$

    According to these models, explain why the fish population modeled by function f will eventually catch up to the fish population modeled by function g. Determine precisely when this will occur.

10. When looking at U.S. minimum wage data, you can consider the nominal minimum wage, which is the amount paid in dollars for an hour of work in the given year. You can also consider the minimum wage adjusted for inflation. Below is a table showing the nominal minimum wage and a graph of the data when the minimum wage is adjusted for inflation. Do you think an exponential function would be an appropriate model for either situation? Explain your reasoning.

Year	Nominal Minimum Wage
1940	$0.30
1945	$0.40
1950	$0.75
1955	$0.75
1960	$1.00
1965	$1.25
1970	$1.60
1975	$2.10
1980	$3.10
1985	$3.35
1990	$3.80
1995	$4.25
2000	$5.15
2005	$5.15
2010	$7.25

U.S. Minimum Wage Adjusted for Inflation
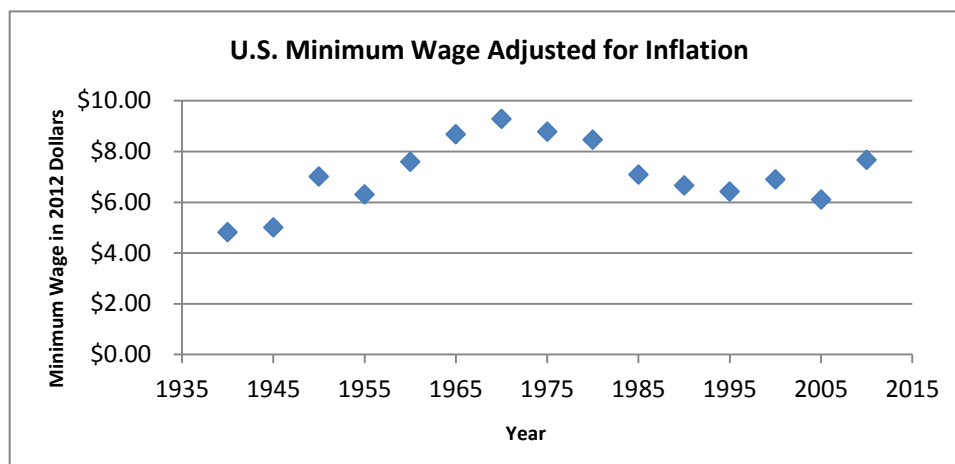
11. A dangerous bacterial compound forms in a closed environment but is immediately detected. An initial detection reading suggests the concentration of bacteria in the closed environment is one percent of the fatal exposure level. Two hours later, the concentration has increased to four percent of the fatal exposure level.

   a. Develop an exponential model that gives the percent of fatal exposure level in terms of the number of hours passed.

   b. Doctors and toxicology professionals estimate that exposure to two-thirds of the bacteria's fatal concentration level will begin to cause sickness. Provide a rough time limit (to the nearest 15 minutes) for the inhabitants of the infected environment to evacuate in order to avoid sickness.

   c. A prudent and more conservative approach is to evacuate the infected environment before bacteria concentration levels reach 45% of the fatal level. Provide a rough time limit (to the nearest 15 minutes) for evacuation in this circumstance.

   d. When will the infected environment reach 100% of the fatal level of bacteria concentration (to the nearest minute)?

12. Data for the number of users at two different social media companies is given below. Assuming an exponential growth rate, which company is adding users at a faster annual rate? Explain how you know.

Social Media Company A	
Year	Number of Users (Millions)
2010	54
2012	185

Social Media Company B	
Year	Number of Users (Millions)
2009	360
2012	1,056