Lesson 26: Percent Rate of Change

Student Outcomes

- Students develop a *general growth/decay rate formula* in the context of compound interest.
- Students compute *future values* of investments with continually compounding interest rates.

Lesson Notes

In this lesson, we develop a general growth/decay rate formula by investigating the compound interest formula. In Algebra I, the compound interest formula was described via sequences or functions whose domain is a subset of the integers. We start from this point (**F-IFA.3**) and extend the function to a domain of all real numbers. The function for compound interest is developed first using a recursive process to generate a geometric sequence, which is then rewritten in its explicit form (**F-BF.A.1a**, **F-BF.A.2**). Many of the situations and problems presented here were first encountered in Module 3 of Algebra I, but now students are able to use logarithms to find solutions, using technology appropriately to evaluate the logarithms (MP.5). Students also work on converting between different growth rates and time units (**A-SSE.B.3c**). Students continue to create equations in one variable from the exponential models to solve problems (**A-CED.A.3**).

Note: In this lesson, the letter r stands for *the percent rate of change,* which is different from how the letter r was used in the Lesson 25 where it denoted the common ratio. These two concepts are slightly different (in this lesson, $1+r$ is *the common ratio*), and this difference might cause confusion for your students. We use the letter r to refer to both, due to historical reasons and because r is the notation most commonly used by adults in both situations. You will need to help your students understand how the context dictates whether r stands for the common ratio or the percent rate of change.

Classwork

Example 1 (8 minutes)

MP.4

Present the following situation, which was first seen in Algebra I, to the students. Some trigger questions are presented to help progress student understanding. A general exponential model is presented of the form $F = P(1+r)^t$, which is appropriate in most applications that can be modeled using exponential functions and was introduced in Module 3, Lesson 4, of Algebra I. It has been a while since the students have seen this formula, so it is developed slowly through this example first using a recursive process before giving the explicit translation (**F-BF.A.1a**, **F-BF.A.2**).

- A youth group has a yard sale to raise money for charity. The group earns $\$800$ but decides to put the money in the bank for a while. Their local bank pays an interest rate of 3% per year, and the group decides to put all of the interest they earn back into the account to earn even more interest.

Scaffolding:

- Either present the following information explicitly or encourage students to write out the first few terms without evaluating to see the structure. Once they see that $P_2 = 800 \cdot 1.03^2$ and that $P_3 = 800 \cdot 1.03^3$, they should be able to see that $P_m = 800 \cdot 1.03^m$.
- Have advanced learners work on their own to develop the values for years 0–3 and year m.

- We will refer to the time at which the money was deposited into the bank as year 0. At the end of each year, how can we calculate how much money is in the bank if we know the previous year's balance?
  - *Each year, multiply the previous year's balance by* 1.03. *For example, since* 3% *can be written* 0.03, *the amount at the end of the first year is* $800 + 800(0.03) = 800(1 + 0.03) = 800(1.03)$.
- How much money is in the bank at the following times?

Year	Balance in terms of last year's balance	Balance in terms of the year, m
0	$\$800$	$\$800$
1	$\$824 = 800(1.03)$	$\$824 = 800(1.03)$
2	$\$848.72 = 824(1.03)$	$\$848.72 = 800(1.03)(1.03)$
3	$\$874.18 \approx 848.72(1.03)$	$\$874.18 \approx 800(1.03)(1.03)(1.03)$
m	$b_{m-1} \cdot (1.03)$	$800(1.03)^m$

- If instead of evaluating, we write these balances out as mathematical expressions, what pattern do you notice?
  - *For instance, the second year would be* $800(1.03)(1.03) = 800(1.03)^2$. *From there we can see that the balance in the m*th *year would be* $800(1.03)^m$.
- What kind of sequence do these numbers form? Explain how you know.
  - *They form a geometric sequence because each year's balance is* 1.03 *times the previous year's balance.*
- Write a recursive formula for the balance in the $(m+1)^{\text{st}}$ year, denoted by b_{m+1}, in terms of the balance of the n^{th} year, denoted by b_m.
  - $b_{m+1} = (1.03)b_m$

MP.8

- What is the explicit formula that gives the amount of money, F (i.e., future value), in the bank account after m years?
  - *The group started with* $\$800$ *and this increases* 3% *each year. After the first year, the group will have* $\$800 \cdot 1.03$ *in the account, and after* n *years, they should have* $800 \cdot 1.03^m$. *Thus, the formula for the amount they have could be represented by* $F = \$800(1.03)^m$.
- Let us examine the base of the exponent in the above problem a little more closely, and write it as $1.03 = 1 + 0.03$. Rewrite the formula for the amount they have in the bank after n years using $1 + 0.03$ instead of 1.03.
  - $F = 800(1 + 0.03)^m$

MP.2 & MP.7

- What does the 800 represent? What does the 1 represent? What does the 0.03 represent?
  - *The number* 800 *represents the starting amount. The* 1 *represents* 100% *of the previous balance that is maintained every year. The* 0.03 *represents the* 3% *of the previous balance that is added each year due to interest.*

Scaffolding:

For struggling classes, Example 2 may be omitted in lieu of developing fluency with the formula through practice exercises. The ending discussion questions in Example 2 should be discussed throughout the practice. Some practice exercises are presented below:

- Evaluate $300(1 + 0.12)^3$
- Find the future value of an investment of $\$1000$ growing at a rate of 3% per year, compounded monthly.
- Find the growth rate and how many days it would take to grow $\$2$ into $\$2$ million if the amount doubles every day.
- Find the growth rate per year necessary to grow $\$450$ into $\$900$ after ten years.

- Let P be the present or starting value of 800, let r represent the interest rate of 3%, and t be the number of years. Write a formula for the future value F in terms of P, r, and t.
  - $F = P(1+r)^t$

Discussion (5 minutes)

MP.8 Make three important points during this discussion: (1) that the formula $F = P(1+r)^t$ can be used in situations far more general than just finance, (2) that r is the percent rate of change expressed as a unit rate, and (3) that the domain of the function given by the formula now includes all real numbers. Note: r is expressed as a unit rate for a unit of time; in finance, that unit of time is typically a year given by the yearly interest rate. In the next examples, we will investigate compounding interest problem with different compounding periods.

- This formula, $F = P(1+r)^t$, can be used in far more situations than just finance, including radioactive decay and population growth. Given a *percent rate of change*, i.e., the percentage increase or decrease of an amount over a unit of time, the number r is the unit rate of that percentage rate of change. For example, if a bank account grows by 2.5% a year, the unit rate is $\frac{2.5}{100}$, which means $r = 0.025$. What is the unit rate if the percent rate of change is a 12% increase? A 100% increase? A 0.2% increase? A 5% decrease?
  - $r = 0.12, r = 1, r = 0.002, r = -0.05.$
- Given the value P and the percent rate of change represented by the unit rate r, we can think of the formula as function of time t, that is, $F(t) = P(1+r)^t$. In Algebra I, t represented a positive integer, but now we can think of the function as having a domain of all real numbers. Why can we think of the domain of this function as being all real numbers?
  - *Earlier in this module, we learned how to define the value of an exponent when the power is a rational number, and we showed how to use that definition to evaluate exponents when the power is an irrational number. Thus, we can assume that the domain of the function F can be any real number.*

Students can now use the fact that the function has a domain of all real numbers and their knowledge of logarithms to solve equations involving the function.

- In Example 1, the group's goal is to save $\$1,000$ with the money they made from the yard sale. How many years will it take for the amount in the bank to be at least $\$1,000$?
  - *Substitute 1000 for F and solve for t using logarithms.*

$$\begin{aligned} 1000 &= 800 \cdot 1.03^t \\ \frac{1000}{800} &= 1.03^t \\ 1.25 &= 1.03^t \\ \ln(1.25) &= t \cdot \ln(1.03) \\ t &= \frac{\ln(1.25)}{\ln(1.03)} \approx 7.5 \end{aligned}$$

  Since they earn interest every year, it will take them 8 years to save more than $\$1,000$ with this money.

- What does the approximation 7.5 mean?
  - *The amount in the bank will reach $\$1000$ after roughly 7 years, 6 months.*

The percent rate of change can also be negative, which usually corresponds to a negative unit rate r, with $-1 < r < 0$.

- Can you give an example of percent rate of change that we have studied before that has a negative rate of change?
  - *Radioactive decay, populations that are shrinking, etc. An interesting example is the bean counting experiment where they started with lots of beans and removed beans after each trial.*

At this point in the lesson, you may want to work out one problem from the non-financial Problem Set as an example, or have students work one as an exercise.

Example 2 (8 minutes)

In the function, $F(t) = P(1+r)^t$, the number r is the unit rate of the percent rate of change, and t is time. Frequently, the time units for the percent rate of change and the time unit for t do not agree and some calculation needs to be done so that they do. For instance, if the growth rate is an amount per hour and the time period is a number of days, the formula needs to be altered by factors of 24 and its inverse.

In this example, students learn about compounding periods and percent rates of change that are based upon different units (**A-SSE.B.3c**). Students explore these concepts through some exercises immediately following the example.

- In finance, the interest rates are almost always tied to a specific time period and only accumulate once this has elapsed (called *compounding*). In this context, we refer to the time periods as compounding periods.
- Interest rates for accounts are frequently given in terms of what is called the *nominal annual percentage rate of change* or *nominal APR.* Specifically, the nominal APR is the percent rate of change per compounding period times the number of compounding periods per year. For example, if the percent rate of change is 0.5% per month, then the nominal APR is 6% since there are 12 months in a year. The nominal APR is an easy way of discussing a monthly or daily percent rate of change in terms of a yearly rate, but as we will see in the examples below, it does not necessarily reflect actual or effective percent rate of change per year.

Note about language: In this lesson and later lessons, we will often use the phrase "an interest rate of 3% per year compounded monthly" to mean, a nominal APR of 3% compounded monthly. Both phrases refer to nominal APR.

- Frequently in financial problems and real-life situations, the nominal APR is given and the percent rate of change per compounding period is deducted from it. The following example shows how to deduce the future value function in this context.
- If the nominal APR is 6% and is compounded monthly, then monthly percent rate of change is $\frac{6\%}{12}$ or 0.5% per month. That means that, if a starting value of $\$800$ was deposited in a bank, after one month there would be $\$800\left(1+\frac{0.06}{12}\right)^1$ in the account, after two months there would be $\$800\left(1+\frac{0.06}{12}\right)^2$, and after 12 months in the bank there would be $\$800\left(1+\frac{0.06}{12}\right)^{12}$ in the account. In fact, since it is compounding 12 times a year, it would compound 12 times over 1 year, 24 times over 2 years, 36 times over 3 years, and $12t$ times over t years. Hence, a function that describes the amount in the account after t years is
$$F(t) = 800\left(1+\frac{0.06}{12}\right)^{12t}.$$
- Describe a function F that describes the amount that would be in an account after t years if P was deposited in an account with a nominal APR given by the unit rate r that is compounded n times a year.
  - $F(t) = P\left(1+\frac{r}{n}\right)^{nt}$

- In this form, $\frac{r}{n}$ is the unit rate per compounding period, and nt is the total number of compounding periods over time t.
- However, time t can be any real number; it does not have to be integer valued. For example, if a savings account earns 1% interest per year, compounded monthly, then we would say that the account compounds at a rate of $\frac{0.01}{12}$ per month. How much money would be in the account after $2\frac{1}{2}$ years with an initial deposit of $\$200$?
  - $F(2.5) = 200\left(1+\frac{0.01}{12}\right)^{12(2.5)} \approx \$205.06.$

Exercise (8 minutes)

Have students work through the following problem to explore the consequences of having different compounding periods. After students finish, debrief them to ensure understanding.

Exercise

Answer the following questions.

The youth group from Example 1 is given the option of investing their money at 2.976% interest per year, compounded monthly.

a. After two years, how much would be in each account with an initial deposit of $\$800$?

The account from the beginning of the lesson would have $\$848.72$, and the new account would have $\$800\left(1+\frac{0.02976}{12}\right)^{12\cdot 2} \approx \$849.00$.

b. Compare the total amount from part (a) to how much they would have made using the interest rate of 3% compounded yearly for two years. Which account would you recommend the youth group invest its money in? Why?

The 3% compounded yearly yields $\$848.72$, while the 2.976% compounded monthly yields $\$849.00$ after two years. I would recommend either—the difference between both types of investments is only $\$0.28$, hardly an amount to worry over.

In part (b), the amount from both options is virtually the same: $\$848.72$ versus $\$849.00$. But point out that there is something strange about the numbers; even though the interest rate of 2.975% is less than the interest rate of 3%, the total amount is more. This is due to compounding every month versus every year.

To illustrate this, rewrite the expression $\left(1+\frac{0.02976}{12}\right)^{12t}$ as $((1+0.00248)^{12})^t$, and take the 12^{th} power of 1.00248 to get approximately $((1.00248)^{12})^t \approx (1.030169)^t$. This shows that when the nominal APR of 2.975% compounded monthly is written as a percent rate of change compounded yearly (the same compounding period as in the Example 1), then the interest rate is approximately 3.0169%, which is more than 3%. In other words, interest rates can be accurately compared when they are both converted to the same compounding period.

Example 3 (10 minutes)

MP.7 & MP.8

In this example, students develop the $F(t) = Pe^{rt}$ model using a numerical analysis approach (MP.7, MP.8). Have students perform the beginning calculations on their own as much as possible before transitioning into continuous compounding.

- Thus far, we have seen that the number of times a quantity compounds per year does have an effect on the future value. For instance, if someone tells you that one savings account gives you a nominal APR 3% per year compounded yearly, and another gives you a nominal APR 3% per year compounded monthly, which account will give you more interest at the end of the year?
  - *The account that compounds monthly gives more interest.*
- How much more interest though? Does it give twelve times as much? How can we find out how much money we will have at the end of the year if we deposit $\$100$?
  - *Calculate using the formula.*

$$F = 100(1 + 0.03)^1 = 103$$

$$F = 100\left(1 + \frac{0.03}{12}\right)^{12} = 103.04$$

The account compounding monthly earned 4 cents more.

- So, even though the second account compounded twelve times as much as the other, it only earned a fraction of a dollar more. Do we think that there is a limit to how much an account can earn through increasing the number of times compounding?
  - *Answers may vary. At this point, although the increase is very small, students have experience with logarithms that grow incredibly slowly but have no upper bound. This could be a situation with an upper bound or not.*
- Let's explore this idea of a limit using our calculators to do the work. Holding the principal, percent rate of change, and number of time units constant, is there a limit to how large the future value can become solely through increasing the number of compounding periods?
- We can simplify this question by setting $P = 1$, $r = 1$, and $t = 1$. What does the expression become for n compounding periods?
  - $F = 1\left(1 + \frac{1}{n}\right)^n$
- Then the question becomes, as $n \to \infty$, does F converge to a specific value, or does it also increase without bound?

- Let's rewrite this expression as something our calculators and computers can evaluate: $y = \left(1 + \frac{1}{x}\right)^x$. For now, go into the table feature of your graphing utilities, and let x start at 1, and go up by 1. Can we populate the following table as a class?

x	$y = \left(1 + \frac{1}{x}\right)^x$
1	2
2	2.25
3	2.3704
4	2.4414
5	2.4883
6	2.5216
7	2.5465

- This demonstrates that although the value of the function is continuing to increase as x increases, it is increasing at a decreasing rate. Still, does this function ever start decreasing? Let's set our table to start at 10,000 and increase by 10,000.

x	$y = \left(1 + \frac{1}{x}\right)^x$
10000	2.7181459
20000	2.7182139
30000	2.7182365
40000	2.7182478
50000	2.7182546
60000	2.7182592
70000	2.7182624

- It turns out that we are rapidly approaching the limit of what our calculators can reliably compute. Much past this point, the rounding that the calculator does to perform its calculations starts to insert horrible errors into the table. However, it is true that the value of the function will increase forever but at a slower and slower rate. In fact, as $x \to \infty$, y does approach a specific value. You may have started to recognize that value from earlier in the module: Euler's number, e.
- Unfortunately, a proof that the expression $\left(1 + \frac{1}{x}\right)^x$ approaching is e as $x \to \infty$ requires a lot more mathematics than we have available currently. Using calculus and other advanced mathematics, mathematicians have been able to show not only that as $x \to \infty$, $\left(1 + \frac{1}{x}\right)^x \to e$, but also they have been able to show that as $x \to \infty$, $\left(1 + \frac{r}{x}\right)^x \to e^r$!

Note: As an extension, you can hint at why the expression involving r converges to e^r: Rewrite the expression above as $\left(1 + \frac{r}{x}\right)^{\left(\frac{x}{r}\right)\cdot r}$ or $\left(\left(1 + \frac{r}{x}\right)^{\frac{x}{r}}\right)^r$, and substitute $u = \frac{x}{r}$. Then the expression in terms of u becomes $\left(\left(1 + \frac{1}{u}\right)^u\right)^r$. If $x \to \infty$, then u does also, but as $u \to \infty$, the expression $\left(1 + \frac{1}{u}\right)^u \to e$.

- Revisiting our earlier application, what does x represent in our original formula, and what could it mean that $x \to \infty$?
  - The number x *represents the number of compounding periods in a year. If* x *was large (e.g.,* 365*), it would imply that interest was compounding once a day. If it were very large, say* $365{,}000$*, it would imply that interest was compounding* $1{,}000$ *times a day. As* $x \to \infty$*, the interest would be compounding continuously.*
- Thus, we have a new formula for when interest is compounding continuously: $F = Pe^{rt}$. This is just another representation of the exponential function that we have been using throughout the module.
- The formula is often called the *pert* formula.

Closing (2 minutes)

Have students summarize the key points of the lesson in writing. A sample is included which you may want to share with the class, or you can guide students to these conclusions on their own.

Lesson Summary

- **For application problems involving a percent rate of change represented by the unit rate r, we can write $F(t) = P(1+r)^t$, where F is the future value (or ending amount), P is the present amount, and t is the number of time units. When the percent rate of change is negative, r is negative, and the quantity decreases with time.**
- **The nominal APR is the percent rate of change per compounding period times the number of compounding periods per year. If the nominal APR is given by the unit rate r and is compounded n times a year, then function $F(t) = P\left(1+\frac{r}{n}\right)^{nt}$ describes the future value at time t of an account given that is given nominal APR and an initial value of P.**
- **For continuous compounding, we can write $F = Pe^{rt}$, where e is Euler's number and r is the unit rate associated to the percent rate of change.**

Exit Ticket (4 minutes)

Name ______________________________ Date ____________________

Lesson 26: Percent Rate of Change

Exit Ticket

April would like to invest $\$200$ in the bank for one year. Three banks all have a nominal APR of 1.5%, but compound the interest differently.

a. Bank A computes interest just once at the end of the year. What would April's balance be after one year with this bank?

b. Bank B compounds interest at the end of each six-month period. What would April's balance be after one year with this bank?

c. Bank C compounds interest continuously. What would April's balance be after one year with this bank?

d. Each bank decides to double the nominal APR it offers for one year. That is, they offer a nominal APR of 3%. Each bank advertises, "DOUBLE THE AMOUNT YOU EARN!" For which of the three banks, if any, is this advertised claim correct?

Exit Ticket Sample Solutions

April would like to invest $\$200$ in the bank for one year. Three banks all have a nominal APR of 1.5%, but compound the interest differently.

a. **Bank A computes interest just once at the end of the year. What would April's balance be after one year with this bank?**

$$I = 200 \cdot 0.015 = 3$$

April would have $\$203$ at the end of the year.

b. **Bank B compounds interest at the end of each six-month period. What would April's balance be after one year with this bank?**

$$F = 200\left(1 + \frac{0.015}{2}\right)^2$$
$$\approx 203.01$$

April would have $\$203.01$ at the end of the year.

c. **Bank C compounds interest continuously. What would April's balance be after one year with this bank?**

$$F = 200e^{0.015}$$
$$\approx 203.02$$

April would have $\$203.02$ at the end of the year.

d. **Each bank decides to double the nominal APR it offers for one year. That is, they offer a nominal APR of 3%. Each bank advertises, "DOUBLE THE AMOUNT YOU EARN!" For which of the three banks, if any, is this advertised claim correct?**

Bank A:

$$I = 200 \cdot 0.03 = 6$$

Bank B:

$$F = 200\left(1 + \frac{0.03}{2}\right)^2$$
$$\approx 206.045$$

Bank C:

$$F = 200e^{0.015}$$
$$\approx 206.09$$

All three banks earn at least twice as much with a double interest rate. Bank A earns exactly twice as much, Bank B earns 2 cents more than twice as much, and Bank C earns 5 cents more than twice as much.

Problem Set Sample Solutions

1. **Write each recursive sequence in explicit form. Identify each sequence as arithmetic, geometric, or neither.**

   a. $a_1 = 3$, $a_{n+1} = a_n + 5$

      $a_n = 3 + 5(n-1)$, *arithmetic*

   b. $a_1 = -1$, $a_{n+1} = -2a_n$

      $a_n = -(-2)^{n-1}$, *geometric*

   c. $a_1 = 30$, $a_{n+1} = a_n - 3$

      $a_n = 30 - 3(n-1)$, *arithmetic*

   d. $a_1 = \sqrt{2}$, $a_{n+1} = \frac{a_n}{\sqrt{2}}$

      $a_n = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{n-1}$, *geometric*

   e. $a_1 = 1$, $a_{n+1} = \cos(\pi a_n)$

      $a_1 = 1, a_n = -1$ *for* $n > 1$, *neither.*

2. **Write each sequence in recursive form. Assume the first term is when $n = 1$.**

   a. $a_n = \frac{3}{2}n + 3$

      $a_1 = \frac{9}{2}, a_{n+1} = a_n + \frac{3}{2}$

   b. $a_n = 3\left(\frac{3}{2}\right)^n$

      $a_1 = \frac{9}{2}, a_{n+1} = \frac{3}{2} \cdot a_n$

   c. $a_n = n^2$

      $a_1 = 1, a_{n+1} = a_n + 2n + 1$

   d. $a_n = \cos(2\pi n)$

      $a_1 = 1, a_{n+1} = a_n$

3. Consider two bank accounts. Bank A gives simple interest on an initial investment in savings accounts at a rate of 3% per year. Bank B gives compound interest on savings accounts at a rate of 2.5% per year. Fill out the following table.

Number of Years, n	Bank A Balance, a_n	Bank B Balance, b_n
0	$\$1000.00$	$\$1000.00$
1	$\$1030.00$	$\$1025.00$
2	$\$1060.00$	$\$1050.63$
3	$\$1090.00$	$\$1076.89$
4	$\$1120.00$	$\$1103.81$
5	$\$1150.00$	$\$1131.41$

a. What type of sequence do the Bank A balances represent?

Balances from Bank A represent an arithmetic sequence with constant difference $\$30$.

b. Give both a recursive and an explicit formula for the Bank A balances.

Recursive: $a_1 = 1000$, $a_n = a_{n-1} + 30$

Explicit: $a_n = 1000 + 30n$ *or* $f(n) = 1000 + 30n$

c. What type of sequence do the Bank B balances represent?

Balances from Bank B represent a geometric sequence with common ratio 1.025.

d. Give both a recursive and an explicit formula for the Bank B balances.

Recursive: $b_1 = 1000$, $b_n = b_{n-1} \cdot 1.025$

Explicit: $b_n = 1000 \cdot 1.025^n$ *or* $f(n) = 1000 \cdot 1.025^n$

e. Which bank account balance is increasing faster in the first five years?

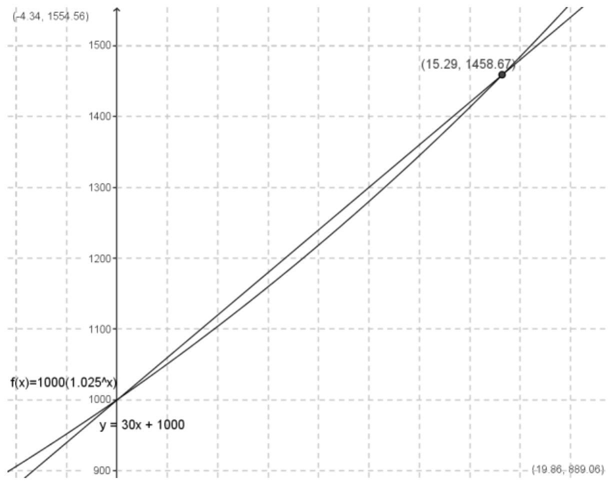
During the first five years, the balance at Bank A is increasing faster at a constant rate of $\$30$ *per year.*

f. If you were to recommend a bank account for a long-term investment, which would you recommend?

The balance at Bank B would eventually outpace the balance at Bank A since the balance at Bank B is increasing geometrically.

g. **At what point is the balance in Bank B larger than the balance in Bank A?**

Once the balance in Bank B overtakes the balance in Bank A, it will always be larger, so we just have to find when they are equal. Because of the complication of solving when a linear function is equal to an exponential function, it is probably easiest to graph the two functions and see where they intersect.



It appears that the balance in Bank B will overtake the balance in Bank A in the 16^{th} year and be larger from then on. Any investment made for 0 to 15 years would be better in Bank A than Bank B.

4. **You decide to invest your money in a bank that uses continuous compounding at 5.5% interest per year. You have $\$500$.**

a. **Ja'mie decides to invest $\$1,000$ in the same bank for one year. She predicts she will have double the amount in her account than you will have. Is this prediction correct? Explain.**

$$F = 1000 \cdot e^{0.055}$$
$$\approx 1056.54$$

$$F = 500 \cdot e^{0.055}$$
$$\approx 528.27$$

Her prediction was correct. Evaluating the formula with $1,000$, we can see that $1000e^{0.055} = 2 \cdot 500 \cdot e^{0.055}$.

b. **Jonas decides to invest $\$500$ in the same bank as well, but for two years. He predicts that after two years he will have double the amount of cash that you will after one year. Is this prediction correct? Explain.**

Jonas will earn more than double the amount of interest since the value increasing is in the exponent but will not have more than double the amount of cash.

5. **Use the properties of exponents to identify the percent rate of change of the functions below, and classify them as representing exponential growth or decay. (The first two problems are done for you.)**

a. $f(t) = (1.02)^t$

The percent rate of change is 2% and represents exponential growth.

b. $f(t) = (1.01)^{12t}$

Since $(1.01)^{12t} = ((1.01)^{12})^t \approx (1.1268)^t$, the percent rate of change is 12.68% and represents exponential growth.

c. $f(t) = (0.97)^t$

Since $(0.97)^t = (1 - 0.03)^t$, the percent rate of change is -3% and represents exponential decay.

d. $f(t) = 1000(1.2)^t$

The percent rate of change is 20% and represents exponential growth.

e. $f(t) = \frac{(1.07)^t}{1000}$

The percent rate of change is 7% and represents exponential growth.

f. $f(t) = 100 \cdot 3^t$

Since $3^t = (1+2)^t$, the percent rate of change is 200% and represents exponential growth.

g. $f(t) = 1.05 \cdot \left(\frac{1}{2}\right)^t$

Since $\left(\frac{1}{2}\right)^t = (0.5)^t = (1 - 0.5)^t$, the percent rate of change is -50% and represents exponential decay.

h. $f(t) = 80 \cdot \left(\frac{49}{64}\right)^{\frac{1}{2}t}$

Since $\left(\frac{49}{64}\right)^{\frac{1}{2}t} = \left(\left(\frac{49}{64}\right)^{\frac{1}{2}}\right)^t = \left(\frac{7}{8}\right)^t = \left(1 - \frac{1}{8}\right)^t = (1 - 0.125)^t$, the percent rate of change is -12.5% and represents exponential decay.

i. $f(t) = 1.02 \cdot (1.13)^{\pi t}$

Since $(1.13)^{\pi t} = ((1.13)^{\pi})^t \approx (1.468)^t$, the percent rate of change is 46.8% and represents exponential growth.

6. The effective rate of an investment is the percent rate of change per year associated with the nominal APR. The effective rate is very useful in comparing accounts with different interest rates and compounding periods. In general, the effective rate can be found with the following formula: $r_E = \left(1 + \frac{r}{k}\right)^k - 1$. The effective rate presented here is the interest rate needed for annual compounding to be equal to compounding n times per year.

a. For investing, which account is better: an account earning a nominal APR of 7% compounded monthly or an account earning a nominal APR of 6.875% compounded daily? Why?

The 7% account is better. The effective rate for the 7% account is $\left(1 + \frac{0.07}{12}\right)^{12} - 1 \approx 0.07229$ compared to the effective rate for the 6.875% account, which is 0.07116.

b. The effective rate formula for an account compounded continuously is $r_E = e^r - 1$. Would an account earning 6.875% interest compounded continuously be better than the accounts in part (a)?

The effective rate of the account continuously compounded at 6.75% is $e^{0.06875} - 1 \approx 0.07117$, which is less than the effective rate of the 7% account, so the 7% account is the best.

7. Radioactive decay is the process in which radioactive elements decay into more stable elements. A half-life is the time it takes for half of an amount of an element to decay into a more stable element. For instance, the half-life for half of an amount of uranium-235 to transform into lead-207 is 704 million years. Thus, after 704 million years, only half of any sample of uranium-235 will remain, and the rest will have changed into lead-207. We will assume that radioactive decay is modeled by exponential decay with a constant decay rate.

   a. Suppose we have a sample of A g of uranium-235. Write an exponential formula that gives the amount of uranium-235 remaining after m half-lives.

   The decay rate is constant on average and is 0.5. If the present value is A, then we have $F = A(1 + (-0.50))^m$, which simplifies to $F = A\left(\frac{1}{2}\right)^m$.

   b. Does the formula that you wrote in part (a) work for any radioactive element? Why?

   Since m represents the number of half-lives, this should be an appropriate formula for any decaying element.

   c. Suppose we have a sample of A g of uranium-235. What is the decay rate per million years? Write an exponential formula that gives the amount of uranium-235 remaining after t million years.

   The decay rate will be 0.5 every 704 million years. If the present value is A, then we have $F = A\big(1 + (-0.5)\big)^{\frac{t}{704}} = A(0.5)^{\frac{t}{704}}$. This tells us that the growth rate per million years is $(0.5)^{\frac{1}{704}} \approx 0.9990159005$, and the decay rate is 0.0009840995 per million years. Written with this decay rate, the formula becomes $F = A(0.9990159005)^t$.

   d. How would you calculate the number of years it takes to get to a specific percentage of the original amount of material? For example, how many years will it take there to be 80% of the original amount of uranium-235 remaining?

   Set $F = 0.80A$ in our formula and solve for t. For this example, this gives

$$0.80A = A\left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$0.80 = \left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$\ln(0.80) = \frac{t}{704}\left(\ln\left(\frac{1}{2}\right)\right)$$
$$t = 704\frac{\ln(0.80)}{\ln(0.5)}$$
$$t \approx 226.637$$

   Remember that t represents the number of millions of years. So, it takes approximately $227{,}000{,}000$ years.

e. How many millions of years would it take 2.35 kg of uranium-235 to decay to 1 kg of uranium?

For our formula, the future value is 1 kg, and the present value is 2.35.

$$1 = 2.35\left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$\frac{1}{2.35} = \left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$\ln\left(\frac{1}{2.35}\right) = \frac{t}{704}\cdot\ln\left(\frac{1}{2}\right)$$
$$t = 704\frac{\ln\left(\frac{1}{2.35}\right)}{\ln\left(\frac{1}{2}\right)}$$
$$t \approx 867.793$$

Since t is the number of millions of years, it would take approximately 868 million years for 2.35 kg of uranium-235 to decay to 1 kg.

MP.4

8. Doug drank a cup of tea with 130 mg of caffeine. Each hour, the caffeine in Doug's body diminishes by about 12%. (This rate varies between 6% and 14% depending on the person.)

a. Write a formula to model the amount of caffeine remaining in Doug's system after each hour.

$$c(t) = 130\cdot(1-0.12)^t$$
$$c(t) = 130\cdot(0.88)^t$$

b. About how long will it take for the level of caffeine in Doug's system to drop below 30 mg?

$$30 = 130\cdot(0.88)^t$$
$$\frac{3}{13} = 0.88^t$$
$$\ln\left(\frac{3}{13}\right) = t\cdot\ln(0.88)$$
$$t = \frac{\ln\left(\frac{3}{13}\right)}{\ln(0.88)}$$
$$t \approx 11.471$$

The caffeine level is below 30 mg after about 11 hours and 28 minutes.

c. The time it takes for the body to metabolize half of a substance is called a *half-life*. To the nearest 5 minutes, how long is the half-life for Doug to metabolize caffeine?

$$65 = 130\cdot(0.88)^t$$
$$\frac{1}{2} = 0.88^t$$
$$ln\left(\frac{1}{2}\right) = t\cdot ln(0.88)$$
$$t = \frac{ln\left(\frac{1}{2}\right)}{ln(0.88)}$$
$$t \approx 5.422$$

The half-life of caffeine in Doug's system is about 5 hours and 25 minutes.

d. Write a formula to model the amount of caffeine remaining in Doug's system after m half-lives.

$$c = 130 \cdot \left(\frac{1}{2}\right)^m$$

MP.4

9. A study done from 1950 through 2000 estimated that the world population increased on average by 1.77% each year. In 1950, the world population was 2.519 billion.

a. Write a function p for the world population t years after 1950.

$$p(t) = 2.519 \cdot (1 + 0.0177)^t$$
$$p(t) = 2.519 \cdot (1.0177)^t$$

b. If this trend continued, when should the world population have reached 7 billion?

$$7 = 2.519 \cdot (1.0177)^t$$
$$\frac{7}{2.519} = 1.0177^t$$
$$\ln\left(\frac{7}{2.519}\right) = t \cdot ln(1.0177)$$
$$t = \frac{\ln\left(\frac{7}{2.519}\right)}{\ln(1.0177)}$$
$$t \approx 58.252$$

The model says that the population should reach 7 billion sometime roughly $58\frac{1}{4}$ *years after* 1950. *This would be around April* 2008.

c. The world population reached 7 billion October $31, 2011$, according to the United Nations. Is the model reasonably accurate?

Student responses will vary. The model was accurate to within three years, so, yes, it is reasonably accurate.

d. According to the model, when will the world population be greater than 12 billion people?

$$12 = 2.519 \cdot (1.0177)^t$$
$$\frac{12}{2.519} = 1.0177^t$$
$$\ln\left(\frac{12}{2.519}\right) = t \cdot \ln(1.0177)$$
$$t = \frac{\ln\left(\frac{12}{2.519}\right)}{\ln(1.0177)}$$
$$t \approx 88.973$$

According to the model, it will take a little less than 89 *years from* 1950 *to get a world population of* 12 *billion. This would be the year* 2039.

10. A particular mutual fund offers 4.5% nominal APR compounded monthly. Trevor wishes to deposit $\$1,000$.

a. What is the percent rate of change per month for this account?

There are twelve months in a year, so $\frac{4.5\%}{12} = 0.375\% = 0.00375$.

b. Write a formula for the amount Trevor will have in the account after m months.

$$A = 1000 \cdot (1 + 0.00375)^m$$
$$A = 1000 \cdot (1.00375)^m$$

c. *Doubling time* is the amount of time it takes for an investment to double. What is the doubling time of Trevor's investment?

$$2000 = 1000 \cdot (1.00375)^m$$
$$2 = 1.00375^m$$
$$\ln(2) = m \cdot \ln(1.00375)$$
$$m = \frac{\ln(2)}{\ln(1.00375)}$$
$$m \approx 185.186$$

It will take 186 months for Trevor's investment to double. This is 15 years and 6 months.

11. When paying off loans, the monthly payment first goes to any interest owed before being applied to the remaining balance. Accountants and bankers use tables to help organize their work.

a. Consider the situation that Fred is paying off a loan of $\$125,000$ with an interest rate of 6% per year compounded monthly. Fred pays $\$749.44$ every month. Complete the following table:

Payment	Interest Paid	Principal Paid	Remaining Principal
$\$749.44$	$\$625.00$	$\$124.44$	$\$124,875.56$
$\$749.44$	$\$624.38$	$\$125.06$	$\$124,750.50$
$\$749.44$	$\$623.75$	$\$125.69$	$\$124,624.82$

b. Fred's loan is supposed to last for 30 years. How much will Fred end up paying if he pays $\$749.44$ every month for 30 years? How much of this is interest if his loan was originally for $\$125,000$?

$$\$749.44(30)(12) = \$269,798.40$$

Fred will pay $\$269,798.40$ for his loan, paying $\$269,798.40 - \$125,000.00 = \$144,793.40$ in interest.